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Connectivity properties of classes of linear systems

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The number of pathwise-connected components of various classes of linear systems is determined. The proofs are based on the representation of these classes of systems in terms of balanced realizations. This provides a unified way of deriving well-known results as well as the new results presented here.

1. Introduction

Brockett (1976) was one of the first to point out the importance of topological investigations for system identification. He examined the space of scalar rational transfer functions of degree n and proved that this space has $n+1$ connected components. Glover (1975) showed that in the case of multivariable systems, however, there is only one connected component. These papers were at the beginning of a series of investigations into the topology of spaces of linear systems (see e.g. Segal 1979, Helmke 1982, Delchamps 1982).

To define a topology on the set of linear systems of given order n consider the set $L_n^{p,m} \subseteq \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ of all minimal n-dimensional systems. We can put a topology on $L_n^{p,m}$ by embedding this set in $\mathbb{R}^{n(p+n+m)+pm}$ with the natural topology. We now take the set of equivalence classes $L_n^{p,m}/\sim$ in $L_n^{p,m}$ with respect to the system equivalence, i.e. $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2)$ if and only if $(A_1, B_1, C_1, D_1) = (TAT^{-1}, TB, CT^{-1}, D)$ for some invertible T. The space $L_n^{p,m}/\sim$ is endowed with the quotient topology.

The number of connected components of $L_n^{1,1}/\sim$ was determined by Brockett as $n+1$. Glover (1975) showed that if $\max(m,p) > 1$ then $L_n^{p,m}/\sim$ has only one connected component. The same results were obtained independently by Hanzon (1986) and Ober (1987 a) for the subsets of asymptotically stable systems. An important entity in the study of scalar systems is the Cauchy index of a transfer function.

Definition 1.1

Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients. The *Cauchy index* $C_{\text{ind}}(g(x))$ of $g(x) = p(x)/q(x)$ is defined as the number of jumps from $-\infty$ to $+\infty$ minus the number of jumps from $+\infty$ to $-\infty$ of $g(x)$ as x varies from $-\infty$ to $+\infty$.

Brockett (1976) showed that the connected components in $L_n^{1,1}/\sim$ are characterized by the Cauchy indices of the corresponding transfer functions.

Whereas there has been a considerable amount of work done on the space of minimal systems, comparatively little attention has been given to other classes of linear systems that are of equal importance in linear systems theory. In this paper we shall determine the number of connected components for several classes of such

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systems like non-strictly-proper scalar positive real and minimum-phase systems, bounded real and allpass systems. The case of strictly proper scalar minimum-phase and positive real systems has been studied by Krishnaprasad (1980) and Helmke (1989). We take an approach to the proof of these results that allows us to treat the different problems in a unified way. This is done by using balanced parametrizations of the various classes of systems given by Ober (1988). These parametrizations have a common structure, which can be exploited to prove our results in a unified and elementary way. We therefore have a way of rederiving the results of Brockett (1976), Glover (1975), Hanzon (1986) and Ober (1987 a) using virtually the same proof that will yield our new results on the other classes of systems under consideration. The results for asymptotically stable systems in Ober (1987 a) were proved using similar techniques.

The paper is organized as follows. Section 2 contains a review of the canonical forms derived by Ober (1988). It is seen that allpass systems are in some sense the building blocks of general systems. Hence we first derive results for this class of systems in § 3. General systems are considered in § 4. In § 5 we show that similar results also hold for discrete-time systems.

2. Canonical forms for multivariable systems

In this section we shall introduce several classes of multivariable systems and review canonical forms for these systems. We first define the classes of systems that we shall consider.

Definition 2.1

Let $(A, B, C, D) \in L_n^{p,m}$ be a system with transfer function $G(s) = C(sI - A)^{-1}B + D \in TL_n^{p,m}$.

(1) *Asymptotically stable systems:* the set of asymptotically stable continuous-time systems in $L_n^{p,m}$ is denoted by $C_n^{p,m}$ with $TC_n^{p,m}$ the corresponding set of transfer functions.

(2) *Allpass systems:* the set of asymptotically stable allpass transfer functions, i.e. $G(s) \in TC_n^{m,m}$ such that

$$G(s)G(-s)^{\mathrm{T}} = \sigma^2 I \quad (s \in \mathbb{C})$$

for some $\sigma > 0$ is denoted by TA_n^m, with A_n^m the corresponding set of minimal state-space systems.

(3) *Bounded real systems:* the set of bounded real systems, i.e. $(A, B, C, D) \in C_n^{p,m}$ with $I - D^{\mathrm{T}}D > 0$ such that

$$I - G(-iw)^{\mathrm{T}}G(iw) > 0 \quad (w \in \mathbb{R})$$

is denoted by $B_n^{p,m}$, with $TB_n^{p,m}$ the corresponding set of transfer functions.

(4) *Positive real systems:* the set of positive real systems, i.e. $(A, B, C, D) \in C_n^{p,m}$ with $D + D^{\mathrm{T}} > 0$ such that

$$G(iw) + G(-iw)^{\mathrm{T}} > 0 \quad (w \in \mathbb{R})$$

is denoted by P_n^m, with TP_n^m the corresponding set of transfer functions.

(5) *Minimum-phase systems:* the set of minimum-phase systems, i.e. $(A, B, C, D) \in C_n^{p,m}$, with D invertible, such that $A - BD^{-1}C$ has its eigenvalues in the open left half-plane, is denoted by M_n^m, with TM_n^m its corresponding set of transfer functions.

Ober (1988) derived canonical forms for these classes of linear systems. These canonical forms will be the main tools in the later analysis of these sets. Before reviewing the canonical forms, we need to introduce some notation and definitions.

Definitions 2.2

(*a*) A matrix $B = (b_{i,j})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l}$ is called *positive upper-triangular* if there exist indices

$$1 \leqslant t_1 < \ldots < t_j < \ldots < t_k \leqslant l$$

such that

$$\begin{aligned} & b_{i,t_i} > 0 \quad \text{for } 1 \leqslant i \leqslant k \\ & b_{i,j} = 0 \quad \text{for } 1 \leqslant j < t_i \text{ and } 1 \leqslant i \leqslant k \\ & b_{i,j} \in \mathbb{R} \quad \text{otherwise} \end{aligned}$$

i.e.

$$B = \begin{bmatrix} 0 & \ldots & 0 & b_{1,t_1} & b_{1,t_1+1} & \ldots & . & . & . & \ldots & & & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & b_{2,t_2} & b_{2,t_2+1} & \ldots & & & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & & & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & b_{k,t_k} & b_{k,t_k+1} & \ldots \end{bmatrix}$$

(*b*) A matrix A is said to be in *r-balanced form*, $1 \leqslant r \leqslant n$, if for

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{r \times r}$$

we have the following properties:

(1) A_{11} is skew-symmetric;

(2) A_{12} and A_{22} are given by the set of indices

$$\begin{aligned} & 1 = h_1 < \ldots < h_i < h_{i+1} < \ldots < h_g \leqslant n - r \\ & 1 \leqslant g_q < \ldots < g_{i+1} < g_i < \ldots < g_1 \leqslant r \end{aligned}$$

in the following way:

(i) for $A_{12} = (a_{st})_{1 \leqslant s \leqslant r, 1 \leqslant t \leqslant n-r}$

$$\begin{aligned} & a_{g_i,h_i} > 0 \quad \text{for } 1 \leqslant i \leqslant q \\ & a_{g_i,t} = 0 \quad \text{for } t > h_i, \text{ where } 1 \leqslant i \leqslant q \\ & a_{s,t} = 0 \quad \text{for } t \geqslant h_i \text{ and } s > g_i, \text{ where } 1 \leqslant i \leqslant q \end{aligned}$$

i.e.

$$A_{12}=\begin{bmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots & \\ \cdot & \cdot & \cdots & a_{g_2-1,h_2-1} & a_{g_2-1,h_2} & a_{g_2-1,h_2+1} & \cdots \\ \cdot & \cdot & \cdots & a_{g_2,h_2-1} & a_{g_2,h_2} & 0 & \cdots \\ \cdot & \cdot & \cdots & a_{g_2+1,h_2-1} & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ a_{g_1-1,h_1} & a_{g_1-1,h_1+1} & \cdots & a_{g_1-1,h_2-1} & 0 & 0 & \cdots \\ a_{g_1,h_1} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \end{bmatrix}$$

(ii) A_{22} is given by

$$A_{22}=\begin{bmatrix} 0 & \alpha_2 & & & & \\ -\alpha_2 & 0 & \alpha_3 & & & \\ & -\alpha_3 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha_n-r \\ & & & & -\alpha_n-r & 0 \end{bmatrix}$$

where for $2 \leqslant i \leqslant n-r$

$$\alpha_i \begin{cases} =0 & \text{if } i=h_s \text{ for some } 1 \leqslant s \leqslant q \\ >0 & \text{otherwise} \end{cases}$$

(3) $A_{21}=-A_{12}^{\mathrm{T}}$.

Let $A=(a_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$; then we denote by

(1) $[A]_l=(\bar{a}_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$ the *lower-triangular part* of A, i.e.

$$\bar{a}_{ij}=\begin{cases} 0 & (j \geqslant i) \\ a_{ij} & (j<i) \end{cases}$$

(2) $[A]_{\mathrm{d}}=(\tilde{a}_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$ the *diagonal part* of A, i.e.

$$\tilde{a}_{ij}=\begin{cases} 0 & (j \neq i) \\ a_{ij} & (j=i) \end{cases}$$

Ober (1988) showed that a transfer function $G(s)$ in one of the classes of systems discussed above can be realized by a system (A, B, C, D) in *standard form* that is parametrized by a set of *standard parameters*:

$$\sigma_1 > \ldots > \sigma_j > \ldots > \sigma_k > 0$$

$$\begin{array}{ll} n_1, \ldots, n_j, \ldots, n_k & n_j \in \mathbb{N}, \quad \sum_{j=1}^{k} n_j = n \\ r_1, \ldots, r_j, \ldots, r_k & r_j \in \mathbb{N}, \quad 1 \leqslant r_j \leqslant \min(n_j, m, p) \\ U_1, \ldots, U_j, \ldots, U_k & U_j \in \mathbb{R}^{p \times r_j}, \quad U_j^{\mathrm{T}} U_j = I_{r_j} \\ \tilde{B}_1, \ldots, \tilde{B}_j, \ldots, \tilde{B}_k & \tilde{B}_j \in \mathbb{R}^{r_j \times m} \text{ positive upper-triangular} \\ \tilde{A}_1, \ldots, \tilde{A}_j, \ldots, \tilde{A}_k & \tilde{A}_j \in \mathbb{R}^{n_j \times n_j} \text{ in } r_j\text{-balanced form} \\ \tilde{D}, & \tilde{D} \in \mathbb{R}^{p \times m} \end{array}$$

in the following way: If (A, B, C, D) is partitioned according to $n_1, \ldots, n_j, \ldots, n_k$, i.e.

$$A = (A_{ij})_{1 \leqslant i,j \leqslant k}, \qquad A_{ij} \in \mathbb{R}^{n_i \times n_j}$$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_j \\ \vdots \\ B_k \end{bmatrix}, \qquad B_j \in \mathbb{R}^{n_j \times m}$$

$$C = (C_1, \ldots, C_j, \ldots, C_k), \quad C_j \in \mathbb{R}^{p \times n_j}$$

then

(i) $B_j = \begin{pmatrix} \bar{B}_j \\ 0 \end{pmatrix}$, where $\bar{B}_j \in \mathbb{R}^{r_j \times m}$ is a function of $\tilde{B}_j$ and $\tilde{D}$;

(ii) $C_j = (\bar{C}_j, 0)$, where $\bar{C}_j \in \mathbb{R}^{p \times r_j}$ is a function of $\tilde{B}_j$, U_j, σ_j and $\tilde{D}$;

(iii)
$$A_{ij} = \begin{pmatrix} \bar{A}_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

$i \neq j$, where $\bar{A}_{ij} \in \mathbb{R}^{r_i \times r_j}$ is a function of σ_i, σ_j, $\tilde{B}_i$, $\tilde{B}_j$, U_i, U_j and $\tilde{D}$;

(iv)
$$A_{jj} = \tilde{A}_j + \begin{pmatrix} \bar{A}_{jj} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\bar{A}_{jj} \in \mathbb{R}^{r_j \times r_j}$ is a function of σ_j, $\tilde{B}_j$, U_j and $\tilde{D}$;

(v) D is a function of $\tilde{D}$.

The particular way in which the parameters enter the system matrices determines the class of systems that is parametrized. We can now state the canonical forms for the various classes of systems (Ober 1988).

Theorem 2.1

(1) *Minimal systems:* the following two statements are equivalent:

(i) $G(s) \in TL_n^{p,m}$;

(ii) $G(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ given by a standard set of parameters such that

$$D = \tilde{D}$$

$$\bar{B}_j = \tilde{B}_j S_r^{1/2}, \quad \text{where } S_r = I + D^{\mathrm{T}} D$$

$$\bar{C}_j = R_r^{1/2} U_j \Delta_j, \quad \text{where } \Delta_j = (\tilde{B}_j \tilde{B}_j^{\mathrm{T}})^{1/2}, \quad R_r = I + DD^{\mathrm{T}}$$

$$\bar{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}[\sigma_j(1+\sigma_i^2)\tilde{B}_i \tilde{B}_j^{\mathrm{T}} - \sigma_i(1+\sigma_j^2)\Delta_i U_i^{\mathrm{T}} U_j \Delta_j] + B_i D^{\mathrm{T}} U_j \Delta_j \quad (i \neq j)$$

$$\bar{A}_{jj} = -\frac{1-\sigma_j^2}{\sigma_j}[\Delta_j^2]_l - \frac{1-\sigma_j^2}{2\sigma_j}[\Delta_j^2]_{\mathrm{d}} + \tilde{B}_j D^{\mathrm{T}} U_j \Delta_j$$

(2) *Asymptotically stable systems:* the following two statements are equivalent:

(i) $G(s) \in TC_n^{p,m}$;

(ii) $G(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ given by a standard set of parameters such that

$$D = \tilde{D}$$

$$\bar{B}_j = \tilde{B}_j$$

$$\bar{C}_j = U_j \Delta_j, \quad \text{where } \Delta_j = (\tilde{B}_j \tilde{B}_j^{\mathrm{T}})^{1/2}$$

$$\bar{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j \tilde{B}_i \tilde{B}_j^{\mathrm{T}} - \sigma_i \Delta_i U_i^{\mathrm{T}} U_j \Delta_j) \quad (i \neq j)$$

$$\bar{A}_{jj} = -\frac{1}{\sigma_j}[\Delta_j^2]_l - \frac{1}{2\sigma_j}[\Delta_j^2]_{\mathrm{d}}$$

(3) *Allpass systems:* the following two statements are equivalent:

(i) $A(s) \in TA_n^m$;

(ii) $A(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$ given by a standard set of parameters with $k = 1$, $\tilde{D}$ with $\tilde{D}\tilde{D}^{\mathrm{T}} = \sigma^2 I$ and $U = -\sigma^{-1}\tilde{D}\tilde{B}^{\mathrm{T}}\Delta^{-1}$, $\Delta = (\tilde{B}\tilde{B}^{\mathrm{T}})^{1/2}$, such that

$$D = \tilde{D}$$

$$\bar{B} = \tilde{B}$$

$$\bar{C} = U\Delta, \quad \text{where } \Delta = (\tilde{B}\tilde{B}^{\mathrm{T}})^{1/2}$$

$$\bar{A} = -\frac{1}{\sigma}[\Delta^2]_l - \frac{1}{2\sigma}[\Delta^2]_{\mathrm{d}}$$

(4) *Bounded real systems:* the following two statements are equivalent:

(i) $B(s) \in TB_n^{p,m}$;

(ii) $B(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ given by a standard set of parameters with $\sigma_j < 1$ for $1 \leqslant j \leqslant k$ and $I - \tilde{D}^T\tilde{D} > 0$, such that

$$D = \tilde{D}$$

$$\bar{B}_j = \tilde{B}_j S^{1/2}, \quad \text{where } S = I - D^T D$$

$$\bar{C}_j = R^{1/2} U_j \Delta_j, \quad \text{where } \Delta_j = (\tilde{B}_j \tilde{B}_j^T)^{1/2}, \quad R = I - DD^T$$

$$\bar{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2} [\sigma_j(1-\sigma_i^2)\tilde{B}_i\tilde{B}_j^T - \sigma_i(1-\sigma_j^2)\Delta_i U_i^T U_j \Delta_j] - \tilde{B}_i D^T U_j \Delta_j \quad (i \neq j)$$

$$\bar{A}_{jj} = -\frac{1+\sigma_j^2}{\sigma_j}[\Delta_j^2]_l - \frac{1+\sigma_j^2}{2\sigma_j}[\Delta_j^2]_d - \tilde{B}_j D^T U_j \Delta_j$$

(5) *Positive real systems:* the following two statements are equivalent:

(i) $P(s) \in TP_n^m$;

(ii) $P(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$ given by a standard set of parameters with $\sigma_j < 1$ for $1 \leqslant j \leqslant k$ and $I - \tilde{D}^T\tilde{D} > 0$, such that

$$D = (I - \tilde{D})^{-1}(I + \tilde{D})$$

$$\bar{B}_j = \sqrt{2}\, \tilde{B}_j S^{1/2}(I - \tilde{D})^{-1}, \quad \text{where } S = I - \tilde{D}^T\tilde{D}$$

$$\bar{C}_j = \sqrt{2}\,(I - \tilde{D})^{-1} R^{1/2} U_j \Delta_j, \quad \text{where } \Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2}, \quad R = I - \tilde{D}\tilde{D}^T$$

$$\bar{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}[\sigma_j(1-\sigma_i^2)\tilde{B}_i\tilde{B}_j^T - \sigma_i(1-\sigma_j^2)\Delta_i U_i^T U_j\Delta_j] + \tilde{B}_i S^{-1/2}(1 - \tilde{D}^T)(I - \tilde{D})^{-1} R^{1/2} U_j \Delta_j \quad (i \neq j)$$

$$\bar{A}_{jj} = -\frac{1+\sigma_j^2}{\sigma_j}[\Delta_j^2]_l - \frac{1+\sigma_j^2}{2\sigma_j}[\Delta_j^2]_d + \tilde{B}_j S^{-1/2}(I - \tilde{D}^T)(I - \tilde{D})^{-1} R^{1/2} U_j \Delta_j$$

(6) *Minimum-phase systems:* the following two statements are equivalent:

(i) $M(s) \in TM_n^m$;

(ii) $M(s)$ has a standard realization $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$ given by a standard set of parameters with $\sigma_j < 1$ for $1 \leqslant j \leqslant k$ and $\tilde{D}$ invertible, such that

$$D = \tilde{D}$$

$$\bar{B}_j = \tilde{B}_j (D^T D)^{1/2}$$

$$\bar{C}_j = D^{-T}(D^T D)^{1/2}(U_j \Delta_j - \sigma_j \tilde{B}_j^T), \quad \text{where } \Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2}$$

$$\bar{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}[\sigma_j(1-\sigma_i^2)\tilde{B}_i\tilde{B}_j^T - \sigma_i(1-\sigma_j^2)\Delta_i U_i^T U_j \Delta_j]$$

$$+ \tilde{B}_i U_j \Delta_j \quad (i \neq j)$$

$$\bar{A}_{jj} = -\frac{1+\sigma_j^2}{\sigma_j}[\Delta_j^2]_l - \frac{1+\sigma_j^2}{2\sigma_j}[\Delta_j^2]_d + \tilde{B}_j U_j \Delta_j$$

Having reviewed the canonical forms for multivariable systems, we can now state those for scalar systems, which have a much simpler structure. Ober (1988) showed that each transfer function in one of the classes of systems we are discussing has a realization (A, b, c, d) that can be parametrized by a set of *standard parameters*:

$$\begin{array}{ll}
\sigma_1 > \ldots > \sigma_j > \ldots > \sigma_k > 0 & \\
n_1, \ldots, n_j, \ldots, n_k & n_j \in \mathbb{N}, \quad \sum_{j=1}^{k} n_j = n \\
s_1, \ldots, s_j, \ldots, s_k & s_j = \pm 1, \quad 1 \leqslant j \leqslant k \\
b_1, \alpha(1)_1, \ldots, \alpha(1)_j, \ldots, \alpha(1)_{n_1-1} & b_1 > 0, \quad \alpha(1)_j > 0, \quad 1 \leqslant j \leqslant n_1 - 1 \\
\vdots & \\
b_i, \alpha(i)_1, \ldots, \alpha(i)_j, \ldots, \alpha(i)_{n_i-1} & b_i > 0, \quad \alpha(i)_j > 0, \quad 1 \leqslant j \leqslant n_i - 1 \\
\vdots & \\
b_k, \alpha(k)_1, \ldots, \alpha(k)_j, \ldots, \alpha(k)_{n_k-1} & b_k > 0, \quad \alpha(k)_j > 0, \quad 1 \leqslant j \leqslant n_k - 1 \\
d & d \in \mathbb{R}
\end{array}$$

The *standard system* (A, b, c, d) is then given by

(1) $b = (\underbrace{b_1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{b_j, 0, \ldots, 0}_{n_j}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n_k})^{\mathrm{T}}$

(2) $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{s_j b_j, 0, \ldots, 0}_{n_j}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n_k})$

(3) For $A =: (A_{ij})_{1 \leqslant i,j \leqslant k}$ we have

(*a*) block-diagonal entries A_{jj}, $1 \leqslant j \leqslant k$:

$$A_{jj} = \begin{bmatrix}
a_{jj} & \alpha(j)_1 & & & & \\
-\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\
 & -\alpha(j)_2 & 0 & & & 0 \\
 & & \ddots & \ddots & \ddots & \\
 & 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\
 & & & & -\alpha(j)_{n_j-1} & 0
\end{bmatrix}$$

with a_{jj} a function of b_j, σ_j and d.

(*b*) off-diagonal blocks A_{ij}, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A_{ij} = \begin{bmatrix} a_{ij} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

with a_{ij} a function of b_i, b_j, s_i, s_j, σ_i, σ_j and d.

We now state the canonical forms for scalar systems.

Theorem 2.2

(1) *Minimal systems:* the following two statements are equivalent:

(i) $g(s) \in TL_n^{1,1}$;

(ii) $g(s)$ has a standard realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$ given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j}{1 + d^2} \left(\frac{1 - s_i s_j q_i q_j}{s_i s_j q_i + q_j} - s_j d \right)$$

(2) *Asymptotically stable systems:* the following two statements are equivalent:

(i) $g(s) \in TC_n^{1,1}$;

(ii) $g(s)$ has a standard realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$ given by a standard set of parameters such that

$$a_{ij} = \frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}$$

(3) *Allpass systems:* the following two statements are equivalent:

(i) $a(s) \in TA_n^{1,1}$;

(ii) $a(s)$ has a standard realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$ given by a standard set of parameters with $k = 1$ and $d = -s_1 \sigma_1$, such that

$$a_{11} = \frac{-b_1^2}{2\sigma_1}$$

(4) *Bounded real systems:* the following two statements are equivalent:

(i) $b(s) \in TB_n^{1,1}$;

(ii) $b(s)$ has a standard realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$ given by a standard set of parameters with $\sigma_j < 1$, $1 \leqslant j \leqslant k$, and $|d| < 1$, such that

$$a_{ij} = \frac{-b_i b_j}{1 - d^2} \left(\frac{1 + s_i s_j p_i p_j}{s_i s_j p_i + p_j} + s_j d \right)$$

(5) *Positive-real systems:* the following two statements are equivalent:

(i) $p(s) \in TP_n^1$;

(ii) $p(s)$ has a standard realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$ given by a standard set of parameters with $\sigma_j < 1$, $1 \leqslant j \leqslant k$, and $d > 0$, such that

$$a_{ij} = \frac{-b_i b_j}{2d(s_i s_j p_i + p_j)}(1 - s_i p_i)(1 - s_j p_j)$$

(6) *Minimum-phase systems:* the following two statements are equivalent:

(i) $m(s) \in TM_n^1$;

(ii) $m(s)$ has a realization $(A, b, c, d) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$, where A, b, d are as in the standard realization, given by a standard set of parameters with $\sigma_j < 1$, $1 \leqslant j \leqslant k$, and $d \neq 0$, such that

$$a_{ij} = \frac{-b_i b_j}{d^2(s_i s_j p_i + p_j)}(1 - s_i p_i)(1 - s_j p_j)$$

The c-vector is given by

$$c = \frac{1}{d}(\underbrace{(s_1 - \sigma_1)b_1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{(s_j - \sigma_j)b_j, 0, \ldots, 0}_{n_j}, \ldots, \underbrace{(s_k - \sigma_k)b_k, 0, \ldots, 0)}_{n_k}$$

It is interesting to note that the canonical forms for the various classes of linear systems have an almost identical structure. In particular, the parameter spaces differ only in a few details. In addition, the conditions on the parameters that guarantee that a system is in a certain class of systems are very explicit. This makes it possible to study some geometric properties of these classes of systems in an elementary and unified way.

With the exception of minimum-phase systems, all scalar systems that are given in one of the previously stated canonical forms have the so-called 'sign-symmetry property', i.e.

$$A^{\mathrm{T}} = SAS, \quad b = Sc^{\mathrm{T}}$$

where S is a diagonal matrix, whose diagonal terms are ± 1. In particular, if

$$n_1, \ldots, n_j, \ldots, n_k, \quad s_1, \ldots, s_j, \ldots, s_k$$

are the usual structural parameters of a scalar system given in one of the canonical forms of the previous theorem then the sign-symmetry matrix S is

$$S = \mathrm{diag}\,(s_1 \hat{I}_{n_1}, \ldots, s_j \hat{I}_{n_j}, \ldots, s_k \hat{I}_{n_k})$$

where

$$\hat{I}_{n_j} = \mathrm{diag}\,(+1, -1, +1, \ldots, (-1)^{n_j + 1}) \in \mathbb{R}^{n \times n}$$

Remark 2.1

An important property of the sign-symmetry matrix of a system is that it can be linked to the Cauchy index of its transfer function. A consequence of a result in Anderson (1972) is that if a system is sign-symmetric with respect to a sign symmetry matrix S then the Cauchy index of its transfer function $g(s)$ is given by

$$C_{\mathrm{ind}}(g(s)) = \mathrm{trace}\, S$$

3. Allpass systems

In the previous section we have seen that systems for which the structural parameter $k = 1$ are in some sense building blocks of general systems. Allpass systems were shown to be a particular class of such systems. In this section we are going to determine the number of connected components for such systems. The subclass of $L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ such that $k = 1$ is denoted by $L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^m, M_{n,1}^m)$.

The following lemma will be important in determining the number of connected components in the case of scalar systems. Part (1) is due to Brockett (1976).

Lemma 3.1

(1) The map

$$L_n^{1,1} \to \mathbb{Z}$$
$$(A, b, c, d) \mapsto C_{\text{ind}}(c(sI - A)^{-1}b + d)$$

is continuous.

(2) The map

$$L_n^{1,1} \to \mathbb{R}$$
$$(A, b, c, d) \mapsto c \exp(t_0 A)\, b$$

$t_0 > 0$, is continuous.

Proof

For Part (1) see Brockett (1976). Part (2) follows immediately by recalling the identity

$$\exp(t_0 A) = \sum_{i=0}^{n-1} \lambda_i(t_0) A^i$$

where $\lambda_i(t_0) \in \mathbb{R}$, $0 \leqslant i \leqslant n-1$. □

Since we are studying systems with non-zero D-terms, we have to consider the topological structure of the various sets of D matrices used to parametrize the different classes of systems. The following lemma summarizes several well-known connectivity results.

Lemma 3.2

(1) The set $\{D \in \mathbb{R}^{p \times m}\}$ is pathwise-connected.

(2) The set $\{D \in \mathbb{R}^{p \times m} \mid I - D^{\mathrm{T}}D > 0\}$ is pathwise-connected.

(3) The set $\{D \in \mathbb{R}^{m \times m} \mid D \text{ invertible}\}$ has two connected components.

(4) The set $\{D \in \mathbb{R}^{m \times m} \mid DD^{\mathrm{T}} = \sigma^2 I, \sigma > 0\}$ has two connected components.

We can now state the main theorem of the section. The proof of this theorem is based on the explicit construction of paths in the class of systems we are considering. Such a construction is possible using the canonical forms and parametrizations given in the previous section. Those parametrization results give sufficient conditions for a state-space system to be in a particular class of systems. Since these conditions are very explicit, it is straightforward to see that a constructed path connecting two elements of a class of systems is itself in this class.

Theorem 3.1

If $\max(p, m) > 1$ then the sets $L_{n,1}^{p,m}/\sim$, $C_{n,1}^{p,m}/\sim$, $B_{n,1}^{p,m}/\sim$ and $P_{n,1}^{m}/\sim$ have one connected component, whereas $M_{n,1}^{m}/\sim$ has two connected components. The sets $L_{n,1}^{1,1}/\sim$, $C_{n,1}^{1,1}/\sim$, $B_{n,1}^{1,1}/\sim$ and $P_{n,1}^{1}/\sim$ have two connected components whereas $M_{n,1}^{1}/\sim$ has four connected components.

Proof

The following lemma will be necessary for the proof.

Lemma 3.3

Let $(A, B, C, D) \in L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^{m}, M_{n,1}^{m})$ be given by the parameters σ, r, U, $\tilde{B}$, $\tilde{A}$ and $\tilde{D}$. There exists a continuous path in $L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^{m}, M_{n,1}^{m})$ connecting (A, B, C, D) with $(A_1, B_1, C_1, D_1) \in L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^{m}, M_{n,1}^{m})$ that is given by the parameters $\sigma_1 = \sigma$, $r_1 = 1$, $U_1 = (s_1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^{p \times 1}$, $s_1 = \pm 1$, $B_1 = (1, 0, \ldots, 0)$ and $\tilde{A}_1$, which is such that

$$A_1 = \begin{bmatrix} a_{11} & 1 & & & \\ -1 & 0 & \cdot & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \cdot & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

with a_{11} a function of σ_1 and $\tilde{D}$.

Proof of Lemma 3.3

First note that $\tilde{A}$ can be continuously perturbed such that the first super-diagonal of $\tilde{A}$ contains 1s and that the first subdiagonal contains -1s. Then the remaining entries $\tilde{A}$ can be brought continuously to zero. Now, the $\tilde{B}$ and the U parameters are continuously changed to be of the desired form. This can be done such that the path always stays in the parameter space. Hence we have constructed a continuous path in the respective class of systems connecting (A, B, C, D) with (A_1, B_1, C_1, D_1). □

We now consider the case where $\max(p, m) > 1$. The following lemma is formulated also to include the case $k > 1$, which will be useful in the following section.

Lemma 3.4

Let $\max(p, m) > 1$. If (A, B, C, D) is given by the parameters

$$\begin{aligned}
&\sigma_1, \ldots, \sigma_k \\
&n_1, \ldots, n_k \\
&r_1, \ldots, r_k, \quad r_j = 1 \\
&U_1, \ldots, U_k, \quad U_j = (s_j, 0, \ldots, 0)^{\mathrm{T}}, \quad s_j = \pm 1 \\
&\tilde{B}_1, \ldots, \tilde{B}_k, \quad \tilde{B}_j = (1, 0, \ldots, 0)
\end{aligned}$$

$$\tilde{A}_1, \ldots, \tilde{A}_k, \quad \tilde{A}_j = \begin{bmatrix} 0 & 1 & . & & \\ -1 & 0 & . & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & . & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

$$\tilde{D}$$

then (A, B, C, D) can be pathwise-connected in $L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ with the system $(A_1, B_1, C_1, D_1) \in L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ given by the parameters

$$\sigma_1, \ldots, \sigma_k$$

$$n_1, \ldots, n_k$$

$$r_1, \ldots, r_k, \quad r_j = 1$$

$$U_1, \ldots, U_k, \quad U_j = (1, 0, \ldots, 0)^{\mathrm{T}}$$

$$\tilde{B}_1, \ldots, \tilde{B}_k, \quad \tilde{B}_j = (1, 0, \ldots, 0)$$

$$\tilde{A}_1, \ldots, \tilde{A}_k, \quad \tilde{A}_j = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & . & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & . & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

$$\tilde{D}$$

Proof of Lemma 3.4

Case 1: $p \geq 2$. Since $p \geq 2$, it is straightforward to construct a path in $\mathbb{R}^{p \times 1}$ continuously connecting U_j with $\tilde{U}_j = (0, 1, 0, \ldots, 0)^{\mathrm{T}}$ such that the unit length of the vector is preserved. $\tilde{U}_j$ can now be connected with U_j by a continuous path that does not leave the parameter space.

Case 2: $p = 1$. In this case $U_j = s_j$, $s_j = \pm 1$. If $s_j = 1$, there is nothing to show. Therefore we can assume $s_j = -1$. Since, by assumption, $\max(p, m) > 1$, we have $m > 1$. So we can connect $\tilde{B}_j = (1, 0, \ldots, 0)$ with $\tilde{\tilde{B}}_j = (0, 1, 0, \ldots, 0)$. This in turn can be connected with $\tilde{\tilde{\tilde{B}}}_j = (-1, 0, \ldots, 0)$. A state-space transformation with $Q = \operatorname{diag}(I_{n_1 + \ldots + n_{j-1}}, -\hat{I}_{n_j}, I_{n_{j+1} + \ldots + n_k})$, $\hat{I} = \operatorname{diag}(1, -1, 1, -1, \ldots)$ will bring the system to the desired form. Note that the path can be constructed continuously. The path connecting the system with parameter $\tilde{B}_j$ to the system with parameter $\tilde{\tilde{B}}_j$ is clearly in the required class of systems. The same holds for the path connecting the system with parameter $\tilde{\tilde{B}}_j$ to the system with parameter $\tilde{\tilde{\tilde{B}}}_j$ since the systems along the path are in canonical form after a state-space transformation with Q. □

The parameter spaces for σ and for D are pathwise-connected for all the classes of systems that we consider, with the exception of $M_{n,1}^m/\sim$, in which case the parameter space of the D-matrices has two connected components. Hence we can summarize and conclude that for $\max(p, m) > 1$ each system in $L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^m)$ can be pathwise-connected with any other system in $L_{n,1}^{p,m}(C_{n,1}^{p,m}, B_{n,1}^{p,m}, P_{n,1}^{m})$. Since the parameter space of D-matrices for systems in $M_{n,1}^{m}/\sim$ has two connected components, we have $M_{n,1}^{m}/\sim$ has two connected components. The continuity of the natural projection then implies the result for the corresponding quotient sets.

We now consider the case of scalar systems, i.e. $p = m = 1$. First take the case of systems in $L_{n,1}^{1,1}/\sim$, $C_{n,1}^{1,1}/\sim$, $B_{n,1}^{1,1}/\sim$ and $P_{n,1}^{1}/\sim$. By Lemma 3.1, the function $(A, b, c, d) \mapsto cb$ is continuous on each subset of $L_n^{1,1}$. But $cb = s_1 b_1^2 \neq 0$ for the classes of systems we are considering. Therefore there are at least two connected components in the respective quotient sets. Lemma 3.3. together with the fact that the d-terms are parametrized by connected sets shows that $L_{n,1}^{1,1}/\sim$, $C_{n,1}^{1,1}/\sim$, $B_{n,1}^{1,1}/\sim$ and $P_{n,1}^{1}/\sim$ have two connected components.

In the case of systems in $M_{n,1}^{1}/\sim$ note that the condition $d \neq 0$ implies that the parameter set of the d-term has two connected components. Again examining the quantity $cb = d^{-1}(s_1 - \sigma) b_1^2$, we see that $M_{n,1}^{1}$ has at least four connected components. Lemma 3.3 implies that there are exactly four components. The result now follows by the continuity of the canonical projection. □

As a corollary, we obtain the following result on allpass systems.

Corollary 3.1

$A_n^m/\sim$ has two connected components, $m \geq 1$.

Proof

$A_n^m/\sim$ has at least two connected components since the set of D-terms has two connected components. We are going to show that a system (A, B, C, D) in A_n^m can be continuously connected with a system (A_1, B_1, C_1, D_1) as given in Lemma 3.3 with $D_1 = -s_1 \sigma I$, $s_1 = \pm 1$. First continuously perturb D to obtain D_1, for some s_1. Then continuously change $\tilde{A}$ to obtain the desired structure for $\tilde{A}_1$. Now $\tilde{B}$ can be connected with $\tilde{B}_1$. Since $U = -\sigma^{-1} D \tilde{B}^{\mathrm{T}} \Delta^{-1}$, this implies that U has been continuously changed to $U_1 = (s_1, 0, \ldots, 0)^{\mathrm{T}}$. Since σ can be continuously perturbed, we have therefore shown that $A_n^m/\sim$ has at most two connected components. □

4. General systems

Whereas in the previous section we determined the number of connected components with structural parameter $k = 1$, in this section we shall consider the same problem for systems without the imposition of such a constraint.

The main result is as follows.

Theorem 4.1

If $\max(p, m) > 1$ then the sets $L_n^{p,m}/\sim$, $C_n^{p,m}/\sim$, $B_n^{p,m}/\sim$, $P_n^m/\sim$ have one connected component, whereas the set $M_n^m/\sim$ has two connected components. If $p = m = 1$ then $L_n^{1,1}/\sim$, $C_n^{1,1}/\sim$, $B_n^{1,1}/\sim$ and $P_n^1/\sim$ have $n+1$ connected components whereas $M_n^1/\sim$ has $2(n+1)$ connected components.

Proof

We first prove two lemmas.

Lemma 4.1

Let $(A, B, C, D) \in L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ be given by the parameters

$$\begin{aligned}&\sigma_1, \ldots, \sigma_k\\ &n_1, \ldots, n_k\\ &r_1, \ldots, r_k\\ &U_1, \ldots, U_k\\ &\tilde{B}_1, \ldots, \tilde{B}_k\\ &\tilde{A}_1, \ldots, \tilde{A}_k\\ &\tilde{D}\end{aligned}$$

Then (A, B, C, D) can be pathwise-connected in $L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in the same class of systems where $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is given by the parameters

$$\begin{aligned}&\sigma_1, \ldots, \sigma_k\\ &n_1, \ldots, n_k\\ &\tilde{r}_1, \ldots, \tilde{r}_k, \quad \tilde{r}_j = 1\\ &\tilde{U}_1, \ldots, \tilde{U}_k, \quad \tilde{U}_j = (s_j, 0, \ldots, 0)^{\mathrm{T}}, \quad s_j = \pm 1\\ &\tilde{\tilde{B}}_1, \ldots, \tilde{\tilde{B}}_k, \quad \tilde{\tilde{B}}_j = (1, 0, \ldots, 0)\\ &\tilde{\tilde{A}}_1, \ldots, \tilde{\tilde{A}}_k, \quad \tilde{\tilde{A}}_j = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & . & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & . & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}\\ &\tilde{D}\end{aligned}$$

Proof of Lemma 4.1

The proof is analogous to the proof of Lemma 3.3. □

Lemma 4.2

Let $(A, B, C, D) \in L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ be given by the parameters

$$\begin{aligned}&\sigma_1, \ldots, \sigma_k\\ &n_1, \ldots, n_k\\ &r_1, \ldots, r_k, \quad r_j = 1\\ &U_1, \ldots, U_k, \quad U_j = (s_j, 0, \ldots, 0)^{\mathrm{T}}, \quad s_j = \pm 1\\ &\tilde{B}_1, \ldots, \tilde{B}_k, \quad \tilde{B}_j = (1, 0, \ldots, 0)\end{aligned}$$

$$\tilde{A}_1, \ldots, \tilde{A}_k, \quad \tilde{A}_j = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & . & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & . & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

$\tilde{D}$

Then (A, B, C, D) can be pathwise connected in $L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in the same class of systems where $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is given by the parameters

$\tilde{k} = n$

$$\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n, \quad \begin{aligned} &\tilde{\sigma}_1 = \sigma_1 > \tilde{\sigma}_2 > \ldots > \tilde{\sigma}_{n_1} > \\ &\tilde{\sigma}_{n_1+1} = \sigma_2 > \tilde{\sigma}_{n_1+2} > \cdots > \tilde{\sigma}_{n_1+n_2} > \\ &\tilde{\sigma}_{n_1+n_2+1} = \sigma_3 > \tilde{\sigma}_{n_1+n_2+2} > \cdots > \tilde{\sigma}_{n_1+n_2+n_3} > \\ &\vdots \\ &\tilde{\sigma}_{n_1+\ldots n_{k-1}+1} = \sigma_{n_k} > \tilde{\sigma}_{n_1+\ldots n_{k-1}+2} > \cdots > \tilde{\sigma}_n > 0 \end{aligned}$$

$\tilde{n}_1, \ldots, \tilde{n}_n, \quad \tilde{n}_j = 1$

$$\tilde{U}_1, \ldots, \tilde{U}_n, \quad \begin{aligned} &\tilde{U}_j = (\tilde{s}_j, 0, \ldots, 0)^{\mathrm{T}} \\ &\tilde{s}_1 = s_1, \quad \tilde{s}_2 = -s_1, \quad \tilde{s}_3 = s_1, \ldots, \tilde{s}_{n_1} = (-1)^{n_1+1} s_1 \\ &\tilde{s}_{n_1+1} = s_2, \quad \tilde{s}_{n_1+2} = -s_2, \quad \tilde{s}_{n_1+3} = s_2, \ldots, \tilde{s}_{n_1+n_2} = (-1)^{n_2+1} s_2 \\ &\tilde{s}_{n_1+n_2+1} = s_3, \quad \tilde{s}_{n_1+n_2+2} = -s_3, \quad \tilde{s}_{n_1+n_2+3} = s_3, \ldots, \tilde{s}_{n_1+n_2+n_3} = (-1)^{n_3+1} s_3 \\ &\vdots \\ &\tilde{s}_{n_1+\ldots+n_{k-1}+1} = s_k, \quad \tilde{s}_{n_1+\ldots+n_{k-1}+2} = -s_k, \ldots, \tilde{s}_n = (-1)^{n_k+1} s_k \end{aligned}$$

$\tilde{\tilde{B}}_1, \ldots, \tilde{\tilde{B}}_n, \quad \tilde{\tilde{B}}_j = (1, 0, \ldots, 0)$

$\tilde{D}$

Proof of Lemma 4.2

In order to avoid unnecessarily complex notation, we shall give an example that is sufficiently general to show all essential parts of a proof. Assume that the system (A, B, C, D) has three outputs $(p = 3)$ and two inputs $(m = 2)$ and is given by the following set of parameters:

$$n = 5$$

$$\sigma_1 > \sigma_2 > 0$$

$$n_1 = 3, \quad n_2 = 2$$

with the other parameters as specified above. Hence A is of the form

$$A = \begin{bmatrix} a_{11} & 1 & 0 & a_{12} & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & a_{22} & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

B is given as

$$B = \beta(\tilde{D}) \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $\beta(\tilde{D})$ is a function of $\tilde{D}$ depending on the class of systems considered, and for $(A, B, C, D) \in L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^m)$, C is given as

$$C = \gamma(\tilde{D}) \begin{bmatrix} s_1 & 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\gamma(\tilde{D})$ is a function of $\tilde{D}$ depending on the class of systems considered, whereas for $(A, B, C, D) \in M_n^m$, C is given as

$$C = \gamma(\tilde{D}) \begin{bmatrix} s_1 - \sigma_1 & 0 & 0 & s_2 - \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

First consider the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given by the parameters

$$n = 5$$
$$\tilde{k} = 5$$
$$\tilde{\sigma}_1 = \sigma_1 > \tilde{\sigma}_2 > \tilde{\sigma}_3 > \tilde{\sigma}_4 = \sigma_2 > \tilde{\sigma}_5 > 0$$
$$\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = \tilde{n}_4 = \tilde{n}_5 = 1$$
$$\tilde{U}_j = (\tilde{s}_j, 0, 0)^{\mathrm{T}}, \quad \tilde{s}_1 = s_1, \quad \tilde{s}_2 = -s_1, \quad \tilde{s}_3 = s_1$$
$$\tilde{s}_4 = s_2, \quad \tilde{s}_5 = -s_2$$
$$\tilde{B}_j = (b_j, 0), \quad b_j > 0$$

such that $\tilde{A}$ is given by

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

where

$$a_{ij} = \frac{-b_i b_j}{\tilde{s}_i \tilde{s}_j \tilde{\sigma}_i + \tilde{\sigma}_j} f_{ij}$$

with f_{ij} an expression depending on the class of systems under consideration. The B-matrix is given by

$$\tilde{B} = \tilde{\beta}(\tilde{D}) \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \\ b_4 & 0 \\ b_5 & 0 \end{bmatrix}$$

with $\tilde{\beta}(\tilde{D})$ a function of $\tilde{D}$. For $(A, B, C, D) \in L_n^{p,m}(C_n^{p,m}, B_n^{p,m}, P_n^p)$, $\tilde{C}$ is given by

$$\tilde{C} = \tilde{\gamma}(\tilde{D}) \begin{bmatrix} \tilde{s}_1 b_1 & \tilde{s}_2 b_2 & \tilde{s}_3 b_3 & \tilde{s}_4 b_4 & \tilde{s}_5 b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{\gamma}(\tilde{D})$ is a function of $\tilde{D}$ depending on the class of systems under consideration, whereas for $(A, B, C, D) \in M_n^m$, $\tilde{C}$ is given as

$$\tilde{C} = \tilde{\gamma}(\tilde{D}) \begin{bmatrix} (\tilde{s}_1 - \tilde{\sigma}_1) b_1 & (\tilde{s}_2 - \tilde{\sigma}_2) b_2 & (\tilde{s}_3 - \tilde{\sigma}_3) b_3 & (\tilde{s}_4 - \tilde{\sigma}_4) b_4 & (\tilde{s}_5 - \tilde{\sigma}_5) b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We proceed stepwise and first consider the first three states corresponding to the first (n_1 dimensional) subsystem of (A, B, C, D). Within this subsystem, we shall also proceed stepwise and consider the third state of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ corresponding to the parameter $\tilde{\sigma}_3$. From the above representation of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, it follows that we have constructed a path in the respective class of systems if we let

$$b_3 \rightarrow 0$$

$$\tilde{\sigma}_3 \rightarrow \tilde{\sigma}_2$$

at the same rate. Since for some constant $\delta_{ij} > 0$, $|f_{ij}| > \delta_{ij} > 0$ for all i, j along this path, we have

$$a_{13} \to 0$$
$$a_{31} \to 0$$
$$a_{34} \to 0$$
$$a_{43} \to 0$$
$$a_{35} \to 0$$
$$a_{53} \to 0$$
$$a_{23} \to \alpha_2$$
$$a_{32} \to -\alpha_2$$

for some $\alpha_2 > 0$. Thus the limiting system (A_1, B_1, C_1, D_1), which is again in the class of systems under consideration, is given by

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ a_{21} & a_{22} & \alpha_2 & a_{24} & a_{25} \\ 0 & -\alpha_2 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix}$$

$$B_1 = \beta_1(\tilde{D}) \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ 0 & 0 \\ b_4 & 0 \\ b_5 & 0 \end{bmatrix}$$

$$C_1 = \gamma_1(\tilde{D}) \begin{bmatrix} \tilde{s}_1 b_1 & \tilde{s}_2 b_2 & 0 & \tilde{s}_4 b_4 & \tilde{s}_5 b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$C_1 = \gamma_1(\tilde{D}) \begin{bmatrix} (\tilde{s}_1 - \tilde{\sigma}_1) b_1 & (\tilde{s}_2 - \tilde{\sigma}_2) b_2 & 0 & (\tilde{s}_4 - \tilde{\sigma}_4) b_4 & (\tilde{s}_5 - \tilde{\sigma}_5) b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in M_n^m$. The structural parameters of (A_1, B_1, C_1, D_1) are given by

$$k = 4$$

$$\tilde{\sigma}_1, \quad \tilde{\sigma}_2, \quad \tilde{\sigma}_4, \quad \tilde{\sigma}_5$$

$$n_1 = 1, \quad n_2 = 2, \quad n_3 = 1, \quad n_4 = 1,$$

As the next step, we let

$$b_2 \to 0$$
$$\tilde{\sigma}_2 \to \tilde{\sigma}_1$$

at the same rate. This implies similarly that (A_1, B_1, C_1, D_1) is continuously connected with the system (A_2, B_2, C_2, D_2) in the same class of systems, which is given by

$$A_2 = \begin{bmatrix} a_{11} & \alpha_1 & 0 & a_{14} & a_{15} \\ -\alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & a_{44} & a_{45} \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

with $\alpha_1 = \lim_{b_2 \to 0, \tilde{\sigma}_2 \to \tilde{\sigma}_1} a_{12}$,

$$B_2 = \beta_2(\tilde{D}) \begin{bmatrix} b_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ b_4 & 0 \\ b_5 & 0 \end{bmatrix}$$

$$C_2 = \gamma_2(\tilde{D}) \begin{bmatrix} \tilde{s}_1 b_1 & 0 & 0 & \tilde{s}_4 b_4 & \tilde{s}_5 b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$C_2 = \gamma_2(\tilde{D}) \begin{bmatrix} (\tilde{s}_1 - \tilde{\sigma}_1) b_1 & 0 & 0 & (\tilde{s}_4 - \tilde{\sigma}_4) b_4 & (\tilde{s}_5 - \tilde{\sigma}_5) b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in M_n^m$.

As a final step, it remains to consider the second subsystem of (A, B, C, D) corresponding to the parameter $n_2 = 2$. In the same way as shown above, it is possible to find a continuous path in the class of systems under consideration to connect (A_2, B_2, C_2, D_2) with (A, B, C, D) by letting

$$b_2 \to 0$$
$$\tilde{\sigma}_5 \to \tilde{\sigma}_4 = \sigma_2$$

at the same rate. Hence we have constructed a path connecting $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with (A, B, C, D) without leaving the class of systems in which these two systems lie. □

Combining these two lemmas, we have shown that each system in the set of systems under consideration can be pathwise-connected with a system in the same class whose structural parameter $k=n$ and whose parameters U_j are given by $U_j = (s_j, 0, \ldots, 0)$, $s_j = \pm 1$. In the last step the distinction between multivariable systems and scalar systems will become important.

Case 1: $\max(p, m) > 1$. Now consider the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given in Lemma 4.2. By Lemma 3.4, there is a continuous path in the class of systems connecting $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with a system (A, B, C, D) that has the same parameters with the exception of U_j, which are now given by $U_j = (1, 0, \ldots, 0)$. The only parameters in this system that are not fixed to certain constants are the σ_j- and the $\tilde{D}$-parameters. Since the σ_j can be continuously perturbed to any other possible values, the number of connected components for multivariable systems only depends on the number of connected components of the parameter space of the $\tilde{D}$-parameter. But this parameter space has one connected component for all classes of systems, with the exception of the class M_n^m, where there are two components. Therefore the sets $L_n^{p,m}/\sim$, $C_n^{p,m}/\sim$, $B_n^{p,m}/\sim$, $P_n^m/\sim$ have one connected component whereas $M_n^m/\sim$ has two connected components.

Case 2: $p = m = 1$. The following lemma is the main part in proving the last step for scalar systems.

Lemma 4.3

Let (A, b, c, d) be a system given by the parameters

$$k = n$$
$$\sigma_1 > \sigma_2 > \ldots > \sigma_n$$
$$s_1, \ldots, s_j, s_{j+1}, \ldots, s_n, \quad s_{j+1} = -s_j$$
$$b_1, \ldots, b_n$$

This system can be pathwise-connected with the system (A_1, b_1, c_1, d_1) in the same class of systems given by the parameters

$$k = n$$
$$\sigma_1 > \sigma_2 > \ldots > \sigma_n$$
$$\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1}, \ldots, \tilde{s}_n, \quad \tilde{s}_i = s_i \quad (i \neq j, j+1)$$
$$\tilde{s}_j = -s_j, \quad \tilde{s}_{j+1} = s_j$$
$$b_1, \ldots, b_n$$
$$d_1 = d$$

Proof of Lemma 4.3

Using the same approach as for Lemma 4.2, we construct a path in the class of systems under consideration connecting (A, b, c, d) with (A_2, b_2, c_2, d_2) that is given by the parameters

$$k = n - 1$$
$$n_1 = 1, \quad \ldots, \quad n_{j-1} = 1, \quad n_j = 2, \quad n_{j+1} = 1, \quad \ldots, \quad n_k = 1$$

$$\sigma_1 > \ldots, \sigma_{j-1} > \sigma_{j+1} > \ldots > \sigma_n$$

$$s_1, \ldots, s_j, s_{j+2}, \ldots, s_n$$

$$b_1, \ldots, b_j, b_{j+2}, \ldots, b_n$$

$$\alpha(j)_1$$

This is done by letting

$$b_{j+1} \to 0$$

$$\sigma_{j+1} \to \sigma_j$$

at the same rate such that $\alpha(j)_1 = \lim_{b_{j+1}\to 0, \sigma_{j+1}\to\sigma_j} a_{j,j+1}$.

Now consider the system (A_1, b_1, c_1, d_1). If we let

$$b_j \to 0$$

$$\tilde{\sigma}_{j+1} \to \tilde{\sigma}_j$$

at the same rate then we have constructed a path connecting (A_1, b_1, c_1, d_1) with the system (A_3, b_3, c_3, d_3) that is, however, no longer in canonical form. But it can be easily verified, by applying the state-space transformation

$$Q = \operatorname{diag}(I_{j-1}, \tilde{Q}, I_{n-j-1}),\ \tilde{Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

that (A_2, b_2, c_2, d_2) and (A_3, b_3, c_3, d_3) are equivalent systems. This implies that (A_3, b_3, c_3, d_3) is an element of the class of systems under consideration. Therefore we have constructed a continuous path in the respective class of systems connecting (A, b, c, d) with (A_1, b_1, c_1, d_1). □

Applying this lemma several times, we can now find a path connecting each system $(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d})$ to a system given by the following parameters:

$$k = n$$

$$\sigma_1 > \sigma_2 > \ldots > \sigma_n > 0$$

$$s_1, \ldots, s_{i_0}, s_{i_0+1}, \ldots, s_n, \quad s_1 = \ldots = s_{i_0} = -1. \quad s_{i_0+1} = \ldots = s_n = 1$$

$$b_1, \ldots, b_n$$

$$d$$

For given fixed i_0 it therefore follows that there is only one connected component for systems in $L_n^{1,1}$, $C_n^{1,1}$, $B_n^{1,1}$ and P_n^1, whereas for systems in M_n^1 we have two connected components since the set of d parameters has two connected components.

Since there are $n+1$ possible choices for i_0, there are at most $n+1$ connected components in $L_n^{1,1}$, $C_n^{1,1}$, $B_n^{1,1}$ and P_n^1. M_n^1 has at most $2(n+1)$ connected components. To show that these are the exact numbers of connected components, note that by Remark 2.1 the Cauchy index of the transfer function of the system $(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d})$ is given by $n - 2i_0$, provided $(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d})$ is sign-symmetric. Lemma 3.1 therefore implies that $L_n^{1,1}$, $C_n^{1,1}$, $B_n^{1,1}$ and P_n^1 have exactly $n+1$ connected components. Given a number $n-2i$, $i = 0, \ldots, n$, it is straightforward to construct a minimum-phase system having Cauchy index $n-2i$. This implies that M_n^1 has precisely $2(n+1)$ connected components. Therefore we have proved the result. □

The result on $L_n^{1,1}/\sim$ was first proved by Brockett (1970. Glover (1975) showed the multivariable version for $L_n^{p,m}/\sim$. Using the same approach as here, the results for asymptotically stable continuous-time systems were derived by Ober (1987 a). Results on the number of connected components for scalar positive real and minimum-phase transfer functions without the condition that the systems are nonstrictly proper were given by Krishnaprasad (1980).

5. Discrete-time systems

In the previous sections we determined the number of connected components for various classes of continuous-time systems. In this section we shall establish the corresponding results for discrete-time systems. This will be done by mapping continuous-time systems to discrete-time systems via a bilinear transformation. This is a well-known technique to map results of one class of systems to the other and was introduced to the study of topological results by Ober (1987 a).

We first define the classes of systems that will be considered.

Definition 5.1

Let $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ and $G(z) = C(zI-A)^{-1}B + D$.

(1) If all eigenvalues of A are in the open unit disk then (A, B, C, D) is called *discrete-time asymptotically stable*. The set of discrete-time asymptotically-stable systems in $L_n^{p,m}$ is denoted by $D_n^{p,m}$, with $TD_n^{p,m}$ the corresponding set of transfer functions.

(2) A system $(A, B, C, D) \in D_n^{p,m}$ is called *discrete-time bounded real* if

$$I - G(e^{-i\theta})^{\mathrm{T}} G(e^{i\theta}) > 0 \quad (\theta \in [0, 2\pi])$$

The set of discrete-time bounded real systems in $D_n^{m,m}$ is denoted by $DB_n^{p,m}$, with $TDB_n^{p,m}$ the corresponding set of transfer functions.

(3) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time positive real* if

$$G(e^{-i\theta})^{\mathrm{T}} + G(e^{i\theta}) > 0 \quad (\theta \in [0, 2\pi])$$

The set of discrete-time positive real systems in DP_n^m is denoted by $D_n^{p,m}$, with TDP_n^m the corresponding set of transfer functions.

(4) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time minimum phase* if

$$\tilde{G}(z) := G\left(\frac{1}{z}\right) \in TD_n^{m,m}$$

The set of discrete-time minimum-phase systems in $D_n^{m,m}$ is denoted by DM_n^m, with TDM_n^m the corresponding set of transfer functions.

(5) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time allpass* if for some $\sigma > 0$

$$G(e^{i\theta})G(e^{-i\theta})^{\mathrm{T}} = \sigma^2 I \quad (\theta \in [0, 2\pi])$$

The set of discrete-time allpass systems in $D_n^{m,m}$ is denoted by DA_n^m, with TDA_n^m the corresponding set of transfer functions.

The following proposition summarizes some basic results on the bilinear transformation.

Proposition 5.1

The transformation

$$TU_n^{p,m}: TC_n^{p,m} \to TD_n^{p,m}$$

$$G_c(s) \mapsto G_d(z) := G_c\left(\frac{z-1}{z+1}\right)$$

is a bijection with inverse

$$(TU_n^{p,m})^{-1}: TD_n^{p,m} \to TC_n^{p,m}$$

$$G_d(z) \mapsto G_c(s) := G_d\left(\frac{1+s}{1-s}\right)$$

that induces a bijection between $TB_n^{p,m}$ and $TDB_n^{p,m}$. If $p = m$ then $TU_n^{m,m}$ induces a bijection between TA_n^m and TDA_n^m, TP_n^m and TDP_n^m, as well as TM_n^m and TDM_n^m.

This mapping also has a formulation in terms of state-space systems that is given in the next proposition (Anderson *et al.* 1974, Glover 1984, Ober 1987 b, 1988).

Proposition 5.2

The transformation

$$SU_n^{p,m}: C_n^{p,m} \to D_n^{p,m}$$

$$(A_c, B_c, C_c, D_c) \mapsto (A_d, B_d, C_d, D_d)$$

$(A_d, B_d, C_d, D_d) := ((I - A_c)^{-1}(I + A_c),\ \sqrt{2}(I - A_c)^{-1}B_c,\ \sqrt{2}C_c(I - A_c)^{-1},\ D_c + C_c(I - A_c)^{-1}B_c)$ is a bijection with inverse

$$(SU_n^{p,m})^{-1}: D_n^{p,m} \to C_n^{p,m}$$

$$(A_d, B_d, C_d, D_d) \mapsto (A_c, B_c, C_c, D_c)$$

$(A_c, B_c, C_c, D_c) := ((I + A_d)^{-1}(A_d - I),\ \sqrt{2}(I + A_d)^{-1}B_d,\ \sqrt{2}C_d(I + A_d)^{-1},\ D_d - C_d(I + A_d)^{-1}B_d)$ that induces a bijection between $B_n^{p,m}$ and $DB_n^{p,m}$. If $p = m$ then $SU_n^{m,m}$ induces a bijection between A_n^m and DA_n^m, P_n^m and DP_n^m, as well as M_n^m and DM_n^m. The map $SU_n^{p,m}$ preserves system equivalence as well as sign symmetry of state-space realizations, i.e. for $(A_c, B_c, C_c, D_c) = (SU_n^{m,m})^{-1}((A_d, B_d, C_d, D_d))$, $(A_d, B_d, C_d, D_d) \in D_n^{m,m}$ we have

$$A_c = SA_c^{\mathrm{T}}S, \quad B_c = SC_c^{\mathrm{T}}$$

if and only if

$$A_d = SA_d^{\mathrm{T}}S, \quad B_d = SC_d^{\mathrm{T}}$$

for some $S = \mathrm{diag}(\pm 1, \ldots, \pm 1)$.

An important corollary to these results is that the map T is in fact a homeomorphism (Ober 1987 a).

Corollary 5.1

The map T is a homeomorphism.

Hence we can carry all topological results over from the continuous-time investigations.

Theorem 5

Multivariable systems ($\max(p, m) > 1$): $DA_n^m/\sim$ and $DM_n^m/\sim$ have two connected components. $DC_n^{p,m}/\sim$, $DB_n^{p,m}/\sim$ and $DP_n^m/\sim$ have one connected component.

Scalar systems ($p = m = 1$): $DA_n^1/\sim$ has two connected components. $DC_n^{1,1}/\sim$, $DB_n^{1,1}/\sim$ and $DP_n^1/\sim$ have $n+1$ connected components, whereas $DM_n^1/\sim$ has $2(n+1)$ connected components.

The results presented in the previous theorem are new, with the exception of those on asymptotically stable systems, which were first established by Ober (1987 a) using the same approach and independently by Hanzon (1986) using different methods. It is interesting to compare the results on non-strictly-proper scalar minimum-phase systems presented here with those by Krishnaprasad (1980) and Helmke (1989) on strictly proper scalar minimum-phase systems. Whereas in their case there are $n(n+1)$ connected components, in our case there are $n+1$ connected components.

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