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On the Gap Metric and Coprime Factor Perturbations*

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The normalized coprime factor model is compared with the uncertainty model based on the gap metric. The connection between these two models is fully clarified. This result is used to give a criterion for robust stabilization.

Key Words—Control theory; frequency domain; multivariable systems; stability criteria; robust control.

Abstract—New conditions are derived for when the distance between two linear systems in the gap metric is less than one. By including a coprimeness assumption in the coprime factor uncertainty description it is shown that an open ball in the gap metric is equivalent to an open ball phrased in terms of coprime factor perturbations. A new criterion is given for robust stabilization.

1. INTRODUCTION

THE GENERAL PROBLEM in the area of robust control is to find a stabilizing controller that not only stabilizes the nominal plant but also stabilizes perturbed plants if the perturbation is not too large. In order to be able to treat this problem from a mathematical point of view it is necessary to have a formal description of the plants that are considered to result from such perturbations of the nominal plant. A great number of such descriptions of plant uncertainties have been proposed. Depending on the chosen uncertainty description typically different controllers will produce robust control designs.

Two forms of uncertainty have received considerable attention. El-Sakkary (1985) introduced an uncertainty description that was based on measuring the distance between the graph of the plant and the graph of the perturbed plant in the gap metric. Vidyasagar and Kimura (1986) independently introduced an uncertainty model that is based on perturbations to the coprime factors. Georgiou and Smith (1990) studied the connection between these two uncertainty descriptions in some detail. In particular it was shown that an uncertainty ball in the gap metric coincides with a ball of identical radius in the coprime factor description if the radius is small enough.

One of the main results of this paper is that uncertainty balls in the gap metric can in fact be completely described in terms of coprime factor uncertainty balls if the coprime factor uncertainty description is restricted to only contain plants that are described by a factorization that is coprime. This result is based on a new condition for which the gap between two plants is less than one. A consequence of this condition is also that the gap metric can be computed by performing just one H_∞ optimization rather than two (Georgiou, 1988). These results are illustrated by an example that shows that the gap between a plant and a perturbed plant can in fact be discontinuous if the coprime factor uncertainty is changed linearly.

Ober and Sefton (1990, 1991) showed that if the cosine of the minimum angle α between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of the controller is less than one, then the control system is internally stable. In the final section of this paper it is shown that the control system can tolerate perturbations of the plant in the form of a gap ball, if and only if the size of the gap ball is at most $\sin \alpha$. This result can in fact be seen to give a geometric interpretation to the robustness result by McFarlane and Glover [1989]. A related geometric result can be found in the paper by Foias *et al.* (1990). After submission of this paper, the paper by Qui and Davison (1992) appeared which also contains results related to those presented in the last section of this paper.

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This paper is based on the report by Sefton and Ober (1991).

2. NOTATION

The notation used throughout this paper is standard in the control literature (Francis (1987)). For a matrix $M \in \Re^{p \times m}$ or $\mathscr{C}^{p \times m}$, M^* denotes its conjugate transposed, $\sigma_{\max}(M)$ denotes its maximum singular value, and $\sigma_{\min}(M)$ its minimum singular value.

The Hardy spaces $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$, contain all rational p vector valued functions square-integrable on the imaginary axis with analytic continuation into the right and left half-plane respectively. The Hilbert space $\mathscr{L}_2^p$ is given by $\mathscr{L}_2^p = \mathscr{H}_2^p \oplus (\mathscr{H}_2^p)^\perp$, and the orthogonal projections P_+ and P_- map $\mathscr{L}_2^p$ onto $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$, respectively. The norm of a function $f \in \mathscr{H}_2^p$ is denoted $\|f\|_2$. The Hardy space $\mathscr{H}_\infty^{p \times m}$ contains all $p \times m$ bounded rational functions on the imaginary axis with analytic continuation in the right-half plane and is a subspace of $\mathscr{L}_\infty^{p \times m}$ of all $p \times m$ bounded functions on the imaginary axis. Clearly these functions all have finite $\mathscr{L}_\infty$-norm defined by $\|G\|_\infty := ess \sup_{\omega \in \Re} \sigma_{\max}[G(j\omega)]$. For a system G, G^* denotes its complex conjugate transposed, i.e. $G(s)^* = \overline{G(-\bar{s})^T}$. The symbol $\mathscr{R}\mathscr{H}_2^p$ denotes the subspace of $\mathscr{H}_2^p$ containing the real rational functions, similar definitions apply to the other spaces.

The domain and range of an operator Z is denoted by $\mathscr{D}(Z)$ and $\mathscr{R}(Z)$, respectively. The orthogonal projection operator onto a closed space, $\mathscr{A}$ of $\mathscr{L}_2^p$ is denoted by $P_{\mathscr{A}}$. Given a $p \times m$ symbol G the multiplication operator $M_G: \mathscr{D}(M_G) \to \mathscr{H}_2^m$ is defined $f \mapsto Gf$. If $G \in \mathscr{L}_\infty^{p \times m}$ the Laurent operator $L_G: \mathscr{L}_2^m \to \mathscr{L}_2^p$, the Hankel operator $H_G: \mathscr{H}_2^m \to (\mathscr{H}_2^p)^\perp$ and the Toeplitz operator $T_G: \mathscr{H}_2^m \to \mathscr{H}_2^p$ with symbol G are defined by $f \mapsto G_f$, $f \mapsto P_{(\mathscr{H}_2^p)^\perp} Gf$ and $f \mapsto P_{\mathscr{H}_2^p} Gf$, respectively.

3. PRELIMINARIES

In this section a number of basic definitions and results are collected that will be needed in this paper. First the notion of the graph of an operator acting on a Hilbert space is introduced.

Consider two Hilbert spaces X, Y and a closed linear operator $A: X \to Y$ then,

Definition 3.1. The graph $\mathscr{G}(A)$ of the operator $A: X \to Y$ with domain $\mathscr{D}(A)$ is the totality of all ordered pairs $\{(Ax, x); x \in \mathscr{D}(A)\}$ considered as a linear subspace of the Hilbert space $Y \times X$ with the naturally defined inner product.

The rest of the discussion will be devoted to the multiplication operator with symbol G, a $p \times m$ matrix function; that is the operator $M_G: \mathscr{H}_2^m \to \mathscr{H}_2^p$, where $M_G: f \mapsto Gf$. Clearly the domain $\mathscr{D}(M_G)$ is not the whole space $\mathscr{H}_2^m$ if G is not in $\mathscr{H}_\infty$.

An important object in the study of the graph of the operator M_G will be the so-called coprime factorization of the function G.

Definition 3.2. The pair (M, N), where $M, N \in \mathscr{R}\mathscr{H}_\infty$ constitutes a right factorization (r.f.) of G (similarly, the pair $(\tilde{N}, \tilde{M})$ where $\tilde{N}, \tilde{M} \in \mathscr{R}\mathscr{H}_\infty$, is a left factorization (l.f.) of G) if
(1) M, $(\tilde{M})$, is square and $\det(M(\infty)) \neq 0$ $(\det(\tilde{M}(\infty)) \neq 0)$;
(2) $G = NM^{-1} (G = \tilde{M}^{-1}\tilde{N})$.
If moreover, N and M are right coprime, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathscr{R}\mathscr{H}_\infty$ such that $\tilde{X}N - \tilde{Y}M = I$ ($\tilde{N}$ and $\tilde{M}$ are left coprime, i.e. there exist $X, Y \in \mathscr{R}\mathscr{H}_\infty$ such that $\tilde{N}X - \tilde{M}Y = I$), then (N, M) $(\tilde{N}, \tilde{M})$ is called a right coprime factorization (r.c.f.) left coprime factorization (l.c.f.).

There are an infinite number of coprime factors of a transfer function. However throughout this paper a particular coprime factorization will be used called the normalized coprime factorization. A right (left) coprime factorization of $G = NM^{-1}$ $(G = \tilde{M}^{-1}\tilde{N})$ is called normalized if $N^*N + M^*M = I (\tilde{N}\tilde{N}^* + +\tilde{M}\tilde{M}^* = I)$. Note that normalized right (left) coprime factors are unique up to right (left) multiplication by a constant unitary matrix.

The next proposition expresses a number of results concerning the graph of the multiplication operator with symbol G and the orthogonal projection onto its graph space.

Proposition 3.3. For the transfer function G, with right coprime factorization (N, M) and left coprime factorization $(\tilde{N}, \tilde{M})$ we have

$$\mathscr{G}(M_G) = \mathscr{R}(T^*_{[N^* \;\; M^*]}), \quad \mathscr{G}(M_G)^\perp = \mathscr{R}(T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}).$$

Denote by $P_{\mathscr{G}(M_G)}: \mathscr{H}_2^p \times \mathscr{H}_2^m \to \mathscr{H}_2^p \times \mathscr{H}_2^m$ the orthogonal projection onto the closed subspace $\mathscr{G}(M_G)$, and by $P_{\mathscr{G}(M_G)^\perp}: \mathscr{H}_2^p \times \mathscr{H}_2^m \to \mathscr{H}_2^p \times \mathscr{H}_2^m$ the orthogonal projection onto its orthogonal complement, then

$$P_{\mathscr{G}(M_G)} = T^*_{[N^* \; M^*]}[T_{[N^* \; M^*]} T^*_{[N^* \; M^*]}]^{-1} T_{[N^* \; M^*]},$$
$$P_{\mathscr{G}(M_G)^\perp} = T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}[T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}]^{-1} T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}.$$

Proof. See Vidyasagar (1985) and Cordes and Labrousse (1963).

Some geometric notions are now introduced. A metric on the set of subspaces of a Hilbert space H is the so-called gap metric.

Definition 3.4. The gap between two closed subspaces A and B of a Hilbert space H is defined as

$$\text{gap}(A, B) := \|P_A - P_B\| = \|P_{A^\perp} - P_{B^\perp}\|.$$

The next theorem shows that the gap between two subspaces can be expressed as the maximum of two alternative expressions.

Theorem 3.5 (see e.g. Weidmann (1980)). Let A, B be closed subspaces of a Hilbert space H, then

$$\text{gap}(A, B) = \max\{\|P_A P_{B^\perp}\|, \|P_{A^\perp} P_B\|\}.$$

If the gap between two subspaces is less than one then we have the following simplified situation.

Proposition 3.6 (see e.g. Weidmann (1980)). Let A, B be closed subspaces of a Hilbert space H and assume that $\text{gap}(A, B) < 1$. Then $\text{gap}(A, B) = \|P_A P_{B^\perp}\| = \|P_{A^\perp} P_B\|$.

The following Lemma (see e.g. Nikolskii (1986)) summarizes a number of important facts concerning projections onto closed subspaces of a Hilbert space.

Lemma 3.7. Let H be a Hilbert space and let A, B be closed subspaces of H. Denote the orthogonal projection operator onto the space A as $P_A: H \to A$ and the orthogonal projection onto A restricted to the subspace B as $P_A \mid B: B \to A$, and use analogous notation for the similar operations onto the subspace B. Then the following of statements are equivalent:
(i) $P_B A = B$; (ii) $H = B^\perp + A$; (iii) $\|P_{A^\perp} P_B\| < 1$.
Also the following statements are equivalent:
(i) $P_B A = B$, $A \cap B^\perp = \{0\}$; (ii) $H = B^\perp + A$, $H = B + A^\perp$;
(iii) $\|P_{A^\perp} P_B\| < 1$, $\|P_A P_{B^\perp}\| < 1$.

Important tools in the study of uncertainty in the gap metric are connections to $\mathcal{H}_\infty$ problems. One of the identities we need to use frequently is the following (see e.g. Doyle *et al.* (1989)).

Proposition 3.8. Let $G_1, G_2 \in \mathcal{L}_\infty$. Then

$$\inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} G_1 - Q \\ G_2 \end{bmatrix} \right\|_\infty = \left\| \begin{matrix} H_{G_1} \\ L_{G_2} \mid \mathcal{H}_2 \end{matrix} \right\|.$$

4. THE GAP METRIC ON LINEAR SYSTEMS

The notion of the gap metric on subspaces of a Hilbert space can be used in a natural way to introduce a metric on the set of transfer functions by defining the distance between two transfer functions to be the gap between the graph spaces of the multiplication operators corresponding to the two transfer functions. This metric on the set of transfer functions was first introduced by El-Sakkary (1985) who used it to model uncertainty of a plant in the context of robust control.

In this section the gap metric on the set of transfer functions is being studied in some detail. A new proof is given that shows how the gap can be calculated as an $\mathcal{H}_\infty$-optimization problem. The main result of this section is however a new characterization of when the gap between two systems is less than one. The characterization will be central to the development in the later sections of this paper.

The gap metric between two systems is defined in the natural manner as the gap between their respective graphs (El-Sakkary (1985)).

Definition 4.1. Given two $p \times m$ systems with transfer functions G_1, G_2 then the gap metric between two systems, $\delta(G_1, G_2)$, is defined by,

$$\delta(G_1, G_2) := \text{gap}(\mathcal{G}(M_{G_1}), \mathcal{G}(M_{G_2})) = \|P_{\mathcal{G}(M_{G_1})} - P_{\mathcal{G}(M_{G_2})}\|.$$

This metric has received considerable attention (see El-Sakkary (1985); Georgiou (1988); Georgiou and Smith (1990); Vidyasagar (1985)). It was shown that this metric induces a topology on the class of linear systems such that closed-loop stability is a robust property. That is if (G_1, K) is internally stable then there exists an $\epsilon > 0$ such that if $\delta(G_1, G_2) < \epsilon$ then (G_2, K) is also internally stable.

Clearly by the results quoted in Section 3,

$$\delta(G_1, G_2) = \max\{\|P_{\mathcal{G}(M_{G_1})^\perp} P_{\mathcal{G}(M_{G_2})}\|, \|P_{\mathcal{G}(M_{G_2})^\perp} P_{\mathcal{G}(M_{G_1})}\|\},$$

where the two expressions are equal if $\delta(G_1, G_2) < 1$. Georgiou (1988) called the expression $\vec{\delta}(G_1, G_2) := \|P_{\mathcal{G}(M_{G_2})^\perp} P_{\mathcal{G}(M_{G_1})}\|$ the directed gap. Evidently we have that

$$\delta(G_1, G_2) = \max\{\vec{\delta}(G_1, G_2), \vec{\delta}(G_2, G_1)\}.$$

Georgiou (1988), using the commutant lifting theorem, proved that the directed gap can be calculated from an $\mathcal{H}_\infty$ optimization problem. In the following proposition we give a more elementary proof of this fact. We also prove the result for the case when the factorizations are not necessarily coprime. This result will allow an interesting extension of the theory later. We need the following lemma.

Lemma 4.2 (see e.g. Gohberg and Krein (1978)). Let $\mathcal{A}$ and $\mathcal{B}$ be two closed subspaces of a Hilbert space. Then

$$\|P_{\mathscr{A}^\perp}P_{\mathscr{B}}\| = \sup_{u\in\mathscr{B},\|u\|\le 1} \operatorname{dist}(u, \mathscr{A}),$$

where $\operatorname{dist}(u, \mathscr{A}) = \inf_{v\in\mathscr{A}} \|u - v\|$.

Proposition 4.3. Given two $p\times m$ systems G_1, G_2 with normalized right coprime factors (N_1, M_1) and (N_2, M_2) respectively, then

$$\|P_{([\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2\mathscr{H}_2^m)^\perp}P_{([\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1\mathscr{H}_2^m)}\| = \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_1\\M_1\end{bmatrix}\Theta_1 - \begin{bmatrix}N_2\\M_2\end{bmatrix}\Theta_2 Q\right\|_\infty,$$

for any inner functions $\Theta_1, \Theta_2 \in \mathscr{RH}_\infty^{m\times m}$.

Proof. First the norm of

$$\|P_{([\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2\mathscr{H}_2^m)^\perp}P_{([\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1\mathscr{H}_2^m)}\|$$

is expressed as an optimization problem using Lemma 4.2,

$$\begin{aligned}\|P_{([\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2\mathscr{H}_2^m)^\perp}P_{([\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1\mathscr{H}_2^m)}\| &= \sup_{\substack{\hat u\in([\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1\mathscr{H}_2^m)\\ \|\hat u\|_2\le 1}} \inf_{\hat v\in([\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2\mathscr{H}_2^m)} \|\hat u - \hat v\|_2\\ &= \sup_{u\in\mathscr{H}_2^m,\|u\|_2\le 1}\inf_{v\in\mathscr{H}_2^m}\|T_{[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1}u - T_{[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2}v\|_2.\end{aligned}$$

Let $(\tilde N_2, \tilde M_2)$ be normalized l.c.f. of G_2 and note that the function

$$\begin{bmatrix}\Theta_2^*N_2^* & \Theta_2^*M_2^*\\ \tilde M_2 & -\tilde N_2\end{bmatrix},$$

is all-pass. Now for any $u\in\mathscr{H}_2^m$,

$$\begin{aligned}&\inf_{v\in\mathscr{H}_2^m}\|T_{[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1}u - T_{[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2}v\|_2\\ &= \inf_{v\in\mathscr{H}_2^m}\|L_{[\begin{smallmatrix}\Theta_2^*N_2^* & \Theta_2^*M_2^*\\ \tilde M_2 & -\tilde N_2\end{smallmatrix}]}(T_{[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1}u - T_{[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2}v)\|_2\\ &= \inf_{v\in\mathscr{H}_2^m}\left\|\begin{bmatrix}L_{(\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1}u - v\\ L_{(\tilde M_2N_1-\tilde N_2M_1)}\Theta_1 u\end{bmatrix}\right\|_2\\ &= \left\|\begin{bmatrix}P_-L_{(\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1}u\\ L_{(\tilde M_2N_1-\tilde N_2M_1)}\Theta_1 u\end{bmatrix}\right\|_2\\ &= \left\|\begin{bmatrix}H_{(\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1}u\\ L_{(\tilde M_2N_1-\tilde N_2M_1)}\Theta_1 u\end{bmatrix}\right\|_2.\end{aligned}$$

This implies that,

$$\begin{aligned}&\|P_{[\begin{smallmatrix}N_2\\M_2\end{smallmatrix}]\Theta_2\mathscr{H}_2^m)^\perp}P_{[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]\Theta_1\mathscr{H}_2^m)}\|\\ &= \sup_{u\in\mathscr{H}_2^m,\|u\|_2\le 1}\left\|\begin{bmatrix}H_{(\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1)}u\\ L_{(\tilde M_2N_1-\tilde N_2M_1)\Theta_1}u\end{bmatrix}\right\|_2\\ &= \left\|\begin{bmatrix}H_{(\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1)}\\ L_{(\tilde M_2N_1-\tilde N_2M_1)\Theta_1|\mathscr{H}_2^m}\end{bmatrix}\right\|\\ &= \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}\Theta_2^*(N_2^*N_1+M_2^*M_1)\Theta_1 - Q\\ (\tilde M_2N_1-\tilde N_2M_1)\Theta_1\end{bmatrix}\right\|_\infty\\ &= \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}\Theta_2^*N_2^* & \Theta_2^*M_2^*\\ \tilde M_2 & -\tilde N_2\end{bmatrix}\times\left(\begin{bmatrix}N_1\\M_1\end{bmatrix}\Theta_1 - \begin{bmatrix}N_2\\M_2\end{bmatrix}\Theta_2Q\right)\right\|_\infty\\ &= \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_1\\M_1\end{bmatrix}\Theta_1 - \begin{bmatrix}N_2\\M_2\end{bmatrix}\Theta_2Q\right\|_\infty,\end{aligned}$$

where the third equality follows from Proposition 3.8.

As a corollary we obtain a result by Georgiou (1988) which shows how the gap between two systems can be calculated.

Corollary 4.4. Given two $p\times m$ systems G_1, G_2 with normalized right coprime factors (N_1, M_1) and (N_2, M_2), respectively, then

$$\vec\delta(G_1, G_2) = \|P_{\mathscr{G}(M_{G_2})^\perp}P_{\mathscr{G}(M_{G_1})}\| = \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_1\\M_1\end{bmatrix} - \begin{bmatrix}N_2\\M_2\end{bmatrix}Q\right\|,$$

$$\vec\delta(G_2, G_1) = \|P_{\mathscr{G}(M_{G_1})^\perp}P_{\mathscr{G}(M_{G_2})}\| = \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_2\\M_2\end{bmatrix} - \begin{bmatrix}N_1\\M_1\end{bmatrix}Q\right\|,$$

and therefore,

$$\delta(G_1, G_2) = \max\left\{\inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_1\\M_1\end{bmatrix} - \begin{bmatrix}N_2\\M_2\end{bmatrix}Q\right\|, \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_2\\M_2\end{bmatrix} - \begin{bmatrix}N_1\\M_1\end{bmatrix}Q\right\|\right\}.$$

The next lemma states a necessary and sufficient condition for the directed gap to be less than one. A very similar result was obtained by Zhu (1989).

Lemma 4.5. Given two $p\times m$ systems G_1, G_2 with normalized right coprime factors (N_1, M_1) and (N_2, M_2), respectively, then the following statements are equivalent,
(1) $\vec\delta(G_1, G_2) < 1$,
(2) $T_{(N_1^*N_2+M_1^*M_2)}(\mathscr{H}_2^m) = \mathscr{H}_2^m$.

Proof. Applying Lemma 3.7 to this situation gives, $\|P_{\mathscr{G}(M_{G_2})^\perp}P_{\mathscr{G}(M_{G_1})}\| < 1$ if and only if

$$P_{\mathscr{G}(M_{G_1})}\left(\begin{bmatrix}N_2\\M_2\end{bmatrix}\mathscr{H}_2^m\right) = \begin{bmatrix}N_1\\M_1\end{bmatrix}\mathscr{H}_2^m.$$

From the expression for the orthogonal projection operator onto the space $\begin{bmatrix}N_1\\M_1\end{bmatrix}\mathscr{H}_2^m$ in Proposition 3.3 we have that, $P_{\mathscr{G}(M_{G_1})}\left(\begin{bmatrix}N_2\\M_2\end{bmatrix}\mathscr{H}_2^m\right) = \begin{bmatrix}N_1\\M_1\end{bmatrix}\mathscr{H}_2^m$ if and only if

$$T_{[\begin{smallmatrix}N_1\\M_1\end{smallmatrix}]}T_{[N_1^*\ M_1^*]}\left(\begin{bmatrix}N_2\\M_2\end{bmatrix}\mathscr{H}_2^m\right) = \begin{bmatrix}N_1\\M_1\end{bmatrix}\mathscr{H}_2^m.$$

But this is the case if and only if $T_{(N_1^* N_2 + M_1^* M_2)}(\mathcal{H}_2^m) = \mathcal{H}_2^m$.

The following proposition is central to the further development. It establishes a criterion for the directed gap $\vec{\delta}(G_2, G_1)$ to be one when $\vec{\delta}(G_1, G_2)$ is less than one. Hence it gives a condition for the gap between two transfer functions to be one.

Proposition 4.6. Given two $p \times m$ systems G_1, G_2 with normalized right coprime factors (N_1, M_1) and (N_2, M_2), respectively, then if there exists a $Q_0 \in \mathcal{RH}_\infty$ such that

$$\vec{\delta}(G_1, G_2) = \inf_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\| \le \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q_0 \right\|_\infty < 1,$$

and $Q_0^{-1} \notin \mathcal{RH}_\infty$ then this implies that,

$$\vec{\delta}(G_2, G_1) = \inf_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\|_\infty = 1.$$

Proof. Let $Q_0 \in \mathcal{RH}_\infty$ be such that

$$\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q_0 \right\|_\infty < 1,$$

and $Q_0^{-1} \notin \mathcal{RH}_\infty$. It will first be shown that $Q_0^{-1} \in \mathcal{RL}_\infty$. Assume that this is not the case, i.e. $Q_0^{-1} \notin \mathcal{RL}_\infty$. Then $\inf_{\omega \in \Re} \sigma_{\min}(Q_0(i\omega)) = 0$ and therefore there exists a sequence $(v_i)_{i \ge 1}$, $v_i \in \mathcal{H}_2^m$, $\|v_i\|_2 = 1$, such that $\lim_{i \to \infty} \|Q_0 v_i\|_2 = 0$. But therefore we have the contradiction

$$1 > \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q_0 \right\|_\infty \ge \lim_{i \to \infty} \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} v_i - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q_0 v_i \right\|_2 = 1,$$

and hence $Q_0^{-1} \in \mathcal{RL}_\infty$.

Having established $Q_0^{-1} \in \mathcal{RL}_\infty$, an inner–outer factorization $Q_0 = \Theta_0 U_0$ can be obtained, where $U_0 \in \mathcal{RH}_\infty$ with $U_0^{-1} \in \mathcal{RH}_\infty$ and Θ_0 is a square inner function. The assumption $Q_0^{-1} \notin \mathcal{RH}_\infty$ implies that $\Theta_0 \ne I$. Now the result will be proved, i.e.

$$\vec{\delta}(G_2, G_1) = \inf_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\|_\infty = 1.$$

First note that by Proposition 4.3 and the assumption

$$\begin{aligned} &\|P_{\left(\begin{bmatrix} N_2 \\ M_2 \end{bmatrix}(\Theta_0 \mathcal{H}_2^m)\right)^\perp} P_{\left(\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}\mathcal{H}_2^m\right)}\| \\ &= \inf_{U \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \left[\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \Theta_0 \right] U \right\|_\infty \\ &\le \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \left[\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \Theta_0 \right] U_0 \right\|_\infty < 1. \end{aligned}$$

By Lemma 3.7 this inequality implies that

$$\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m = P_{\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}\mathcal{H}_2^m}\left(\left[\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \Theta_0\right] \mathcal{H}_2^m\right),$$

and hence

$$\begin{aligned} \mathcal{H}_2^m &= T_{[N_1^* \; M_1^*]} \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m \\ &= T_{[N_1^* \; M_1^*]} P_{\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}\mathcal{H}_2^m}\left(\left[\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \Theta_0\right] \mathcal{H}_2^m\right) \\ &= T_{[N_1^* \; M_1^*]} T_{\begin{bmatrix} N_1 \\ M_1 \end{bmatrix}} T_{[N_1^* \; M_1^*]}\left(\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \Theta_0 \mathcal{H}_2^m\right) \\ &= T_{(N_1^* N_2 + M_1^* M_2)}(\Theta_0 \mathcal{H}_2^m). \end{aligned}$$

It will now be shown that this implies that $\mathrm{Ker}\,(T_{(N_1^* N_2 + M_1^* M_2)}) \ne \{0\}$. Given $x \in (\Theta_0 \mathcal{H}_2^m)^\perp$, $x \ne 0$, let $y := T_{(N_1^* N_2 + M_1^* M_2)} x$. Now there exists $z \in \Theta_0 \mathcal{H}_2^m$ such that $T_{(N_1^* N_2 + M_1^* M_2)} z = y$ and therefore $T_{(N_1^* N_2 + M_1^* M_2)}(z - x) = 0$. But $z - x \ne 0$ and $z - x \in \mathrm{Ker}\,(T_{(N_1^* N_2 + M_1^* M_2)})$. This implies that $T_{(N_2^* N_1 + M_2^* M_1)}(\mathcal{H}_2^m) \ne \mathcal{H}_2^m$ as

$$\mathrm{Range}^\perp\,(T_{(N_2^* N_1 + M_2^* M_1)}) = \mathrm{Ker}\,(T_{(N_1^* N_2 + M_1^* M_2)}) \ne \{0\}$$

and hence by Lemma 4.5, $\vec{\delta}(G_2, G_1) = 1$.

Of particular interest in our context are characterizations when the gap between two shift invariant spaces is less than one. Nikolskii (1985) gives a number of characterizations for the gap between scalar shift invariant spaces to be less than one. The proofs of those characterizations can, however, not always be directly generalized to the situation discussed here. One of the reasons is that in contrast to the scalar case the orthogonal complements of the shift invariant spaces $\Theta \mathcal{H}_2^m$, $\Theta = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix}$, are infinite dimensional.

Theorem 4.7. Given two $p \times m$ systems G_1, G_2 with normalized right coprime factors (N_1, M_1) and (N_2, M_2), respectively, then the following statements are equivalent,

(1) $\delta(G_1, G_2) < 1$;

(2) the Toeplitz operator $T_{(N_1^* N_2 + M_1^* M_2)}$ is invertible;

(3) there exists a $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$ and $Q^{-1} \in \mathcal{RH}_\infty$;

(4) there exists a $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$ and $Q^{-1} \in \mathcal{RH}_\infty$. For all $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$, $Q^{-1} \in \mathcal{RH}_\infty$;
(5) there exists a $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\| < 1$ and $Q^{-1} \in \mathcal{RH}_\infty$;
(6) there exists a $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\| < 1$ and $Q^{-1} \in \mathcal{RH}_\infty$. For all $Q \in \mathcal{RH}_\infty$ such that $\left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\| < 1$, $Q^{-1} \in \mathcal{RH}_\infty$;
(7) $\begin{pmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{pmatrix} = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m + \left(\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \mathcal{H}_2^m \right)^\perp$
and

$$\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m \cap \left(\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \mathcal{H}_2^m \right)^\perp = \{0\};$$

(8) $\begin{pmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{pmatrix} = \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \mathcal{H}_2^m + \left(\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m \right)^\perp$
and

$$\begin{bmatrix} N_2 \\ M_2 \end{bmatrix} \mathcal{H}_2^m \cap \left(\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} \mathcal{H}_2^m \right)^\perp = \{0\}.$$

Proof. (1)$\Leftrightarrow$(2) This is proved by Zhu (1989).
(1)$\Rightarrow$(3), (4) Assume $\delta(G_1, G_2) < 1$, then this implies by Corollary 4.4 that there exists $Q \in \mathcal{RH}_\infty$, such that

$$\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1.$$

Now by Proposition 4.6 if $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$ and $Q^{-1} \notin \mathcal{RH}_\infty$ we have that $\vec{\delta}(G_2, G_1) = 1$. But this contradicts the assumption that $\delta(G_1, G_2) = \max \{\vec{\delta}(G_1, G_2), \vec{\delta}(G_2, G_1)\} < 1$. Hence we have that $Q^{-1} \in \mathcal{RH}_\infty$.
(4)$\Rightarrow$(3) is obvious.
(3)$\Rightarrow$(2) Assume $Q \in \mathcal{RH}_\infty$ is such that $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$ and $Q^{-1} \in \mathcal{RH}_\infty$. This implies that

$$1 > \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty \geq \| T_{[N_1^* \ M_1^*]} T_{([\substack{N_1 \\ M_1}] - [\substack{N_2 \\ M_2}] Q)} \| = \| T_{I - (N_1^* N_2 + M_1^* M_2) Q} \|.$$

Therefore the operator $I - T_{I-(N_1^* N_2 + M_1^* M_2)Q} = T_{(N_1^* N_2 + M_1^* M_2)Q}$ is invertible. However

$$T_{(N_1^* N_2 + M_1^* M_2)Q} = T_{(N_1^* N_2 + M_1^* M_2)} T_Q,$$

and as $Q^{-1} \in \mathcal{RH}_\infty$, T_Q is invertible and hence $T_{(N_1^* N_2 + M_1^* M_2)}$ is invertible.
(1)$\Leftrightarrow$(5)$\Leftrightarrow$(6) is proved in an analogous manner.
(1)$\Leftrightarrow$(7) and (1)$\Leftrightarrow$(8) follow immediately from Lemma 3.7.

The previous theorem also simplifies the computation of the gap metric between two plants. The expression in Corollary 4.4 due to Georgiou (1988) implies that two H_∞ optimizations have to be performed to compute the gap between two plants.

As a consequence of the previous theorem only one such computation has to be done. Let G_1, G_2 be two $p \times m$ plants with normalized right coprime factorizations $G_1 = N_1 M_1^{-1}$ and $G_2 = N_2 M_2^{-1}$. In order to compute the gap metric $\delta(G_1, G_2)$ between G_1 and G_2 one has to compute one of the directed gaps, e.g.

$$\vec{\delta}(G_1, G_2) := \inf_{Q \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty.$$

If $\vec{\delta}(G_1, G_2) = 1$ then $\delta(G_1, G_2) = 1$, since the gap is the maximum of the directed gaps. If $\vec{\delta}(G_1, G_2) < 1$, let Q_0 be any function $Q_0 \in \mathcal{RH}_\infty$ such that

$$\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q_0 \right\|_\infty < 1.$$

If $Q_0^{-1} \in \mathcal{RH}_\infty$ then by the Theorem $\delta(G_1, G_2) < 1$ and therefore by Proposition 3.6, $\delta(G_1, G_2) = \vec{\delta}(G_1, G_2)$. If $Q_0^{-1} \notin \mathcal{RH}_\infty$, then $\delta(G_1, G_2) = 1$ by the theorem.

5. GAP METRIC AND COPRIME FACTOR PERTURBATIONS

In this section the connection between coprime factor perturbations and the gap metric is explored. There has been considerable interest recently in phrasing robust stabilization problems in terms of these classes of perturbations as this allows a greater amount of phenomena to be modelled than in other uncertainty descriptions, such as those using additive or multiplicative perturbations (see e.g. McFarlane and Glover (1989) and Georgiou and Smith (1990)).

Given a system $G = NM^{-1}$ then any other system of the same input/output dimensions can be written in the form $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}$ for $\Delta_N, \Delta_M \in \mathcal{H}_\infty$. It is shown here that if the restriction is placed on the coprime factor perturbations to lead to a coprime factorization of the perturbed plant, then a coprime factor ball of a certain radius coincides with a ball in the gap metric of the same radius.

The definition of the gap ball and the directed gap ball are now recalled.

Definition 5.1. Given a $p \times m$ system G_1 with normalized r.c.f. (N_1, M_1) then the following classes of transfer functions are defined for $\epsilon > 0$,

$$\mathcal{B}_{G_1}^\epsilon := \{G_2 : \delta(G_1, G_2) < \epsilon\},$$
$$\vec{\mathcal{B}}_{G_1}^\epsilon := \{G_2 : \vec{\delta}(G_1, G_2) < \epsilon\},$$

are called the gap ball and directed gap ball of G_1, respectively. Also define the following classes

$$\mathscr{G}^{\epsilon}_{G_1} := \left\{(N_1+\Delta_N)(M_1+\Delta_M)^{-1}: \begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix} \in \mathscr{H}_\infty^{(p+m)\times m}; ((N_1+\Delta_N),(M_1+\Delta_M)) \text{ right coprime}; \left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\|_\infty < \epsilon\right\},$$

$$\vec{\mathscr{G}}^{\epsilon}_{G_1} := \left\{(N_1+\Delta_N)(M_1+\Delta_M)^{-1}: \begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix} \in \mathscr{H}_\infty^{(p+m)\times m}; \left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\|_\infty < \epsilon\right\}.$$

One of the main contributions of this paper is a complete characterization of the gap ball in terms of coprime factor perturbations. Georgiou and Smith (1990) showed that for given ϵ the directed gap ball $\vec{\mathscr{G}}^{\epsilon}_{G_1}$ and the coprime factor ball $\vec{\mathscr{G}}^{\epsilon}_{G_1}$ are identical. In the same paper it was also shown that for small enough ϵ the gap ball $\mathscr{B}^{\epsilon}_{G_1}$ and the coprime factor ball $\mathscr{G}^{\epsilon}_{G_1}$ are identical. In the following theorem we are going to show that gap balls $\mathscr{B}^{\epsilon}_{G_1}$ can be fully characterized in terms of the balls $\mathscr{G}^{\epsilon}_{G_1}$.

Theorem 5.2. Given a $p\times m$ system G_1 with normalized r.c.f. (N_1, M_1) then for $\epsilon>0$,
(1) $\vec{\mathscr{B}}^{\epsilon}_{G_1} = \vec{\mathscr{G}}^{\epsilon}_{G_1}$,
(2) $\mathscr{B}^{\epsilon}_{G_1} = \mathscr{G}^{\epsilon}_{G_1}$.

Proof. (1) This statement was proved by Georgiou and Smith (1990).

(2) Note that the result follows immediately for $\epsilon>1$. It is therefore assumed that $0<\epsilon\le 1$. It is first shown that $G_2\in\mathscr{B}^{\epsilon}_{G_1}$ implies that $G_2\in\mathscr{G}^{\epsilon}_{G_1}$. If $G_2\in\mathscr{B}^{\epsilon}_{G_1}$ then by Corollary 4.4 and Theorem 4.6 there exists a $Q\in\mathscr{RH}_\infty$, with $Q^{-1}\in\mathscr{RH}_\infty$, such that

$$\left\|\begin{bmatrix}N_1\\ M_1\end{bmatrix} - \begin{bmatrix}N_2\\ M_2\end{bmatrix}Q\right\| < \epsilon,$$

where (N_2, M_2) is a normalized coprime factorization of G_2. Let

$$\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix} = \begin{bmatrix}N_1\\ M_1\end{bmatrix} - \begin{bmatrix}N_2\\ M_2\end{bmatrix}Q,$$

and therefore $\begin{bmatrix}N_2\\ M_2\end{bmatrix}Q = \begin{bmatrix}N_1\\ M_1\end{bmatrix} - \begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}$ is also a coprime factorization of G_2 as $Q^{-1}\in\mathscr{RH}_\infty$. Now as $\left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\| < \epsilon$ this implies that $G_2\in\mathscr{G}^{\epsilon}_{G_1}$.

It is now shown that $G_2\in\mathscr{G}^{\epsilon}_{G_1}$ implies that $G_2\in\mathscr{B}^{\epsilon}_{G_1}$. If $G_2\in\mathscr{G}^{\epsilon}_{G_1}$ there exist $\Delta_N, \Delta_M\in\mathscr{H}_\infty$ such that $\left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\|_\infty < \epsilon$ and $G_2 = N_2M_2^{-1}$ is a coprime factorization of G_2 where $\begin{bmatrix}N_2\\ M_2\end{bmatrix} := \begin{bmatrix}N_1\\ M_1\end{bmatrix} - \begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}$. This implies that there exists a unit $\tilde{Q}\in\mathscr{RH}_\infty$ such that $(N_2\tilde{Q}^{-1}, M_2\tilde{Q}^{-1})$ is a normalized coprime factorization of G_2. Hence

$$\begin{aligned}\vec{\delta}(G_1,G_2) &= \inf_{Q\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}N_1\\ M_1\end{bmatrix} - \left[\begin{bmatrix}N_2\\ M_2\end{bmatrix}\tilde{Q}^{-1}\right]Q\right\|_\infty \\ &\le \left\|\begin{bmatrix}N_1\\ M_1\end{bmatrix} - \left[\begin{bmatrix}N_2\\ M_2\end{bmatrix}\tilde{Q}^{-1}\right]\tilde{Q}\right\|_\infty \\ &= \left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\|_\infty < \epsilon.\end{aligned}$$

As $\tilde{Q}^{-1}\in\mathscr{RH}_\infty$ we have that $\delta(G_1,G_2)<1$ by Theorem 4.7. Therefore $\vec{\delta}(G_1,G_2) = \vec{\delta}(G_2,G_1) < \epsilon$ implying $\delta(G_1,G_2)<\epsilon$ and hence $G_2\in\mathscr{B}^{\epsilon}_{G_1}$.

The following example illustrates the concepts introduced in this paper and demonstrates that the gap metric is not continuous with respect to a linear addition of a perturbation to the coprime factors.

Example 5.1. Consider the system

$$G(s) = \frac{\sqrt{3}}{s^2+1},$$

which has a normalized coprime factorization

$$\begin{bmatrix}n\\ m\end{bmatrix} = \frac{1}{s^2+\sqrt{2}s+2}\begin{bmatrix}\sqrt{3}\\ s^2+1\end{bmatrix}.$$

Also consider the perturbation,

$$\begin{bmatrix}\Delta_n\\ \Delta_m\end{bmatrix} = \begin{bmatrix}\frac{-1}{2\sqrt{3}}\\ \frac{-1}{2}\end{bmatrix} \quad \text{where} \quad \left\|\begin{bmatrix}\Delta_n\\ \Delta_m\end{bmatrix}\right\|_\infty = \frac{1}{\sqrt{3}},$$

and define a perturbed system as,

$$G_{\Delta_\epsilon} := (n+\epsilon\Delta_n)(m+\epsilon\Delta_m)^{-1}.$$

The function $f(\epsilon)$ is defined as $f(\epsilon) = \delta(G, G_{\Delta_\epsilon})$. It is now shown that the function $f(\epsilon)$ has a discontinuity at $\epsilon=1$ and further that,

$$\begin{cases} f(\epsilon) \le \dfrac{\epsilon}{\sqrt{3}} & \text{for } 0\le\epsilon<1,\ 1<\epsilon\le\sqrt{3} \\ f(\epsilon)=1 & \text{for } \epsilon=1.\end{cases}$$

A factorization of G_{Δ_ϵ} is given by,

$$\begin{aligned}\begin{bmatrix}n_{\Delta_\epsilon}\\ m_{\Delta_\epsilon}\end{bmatrix} &:= \begin{bmatrix}n\\ m\end{bmatrix} + \epsilon\begin{bmatrix}\Delta_n\\ \Delta_m\end{bmatrix} \\ &= \frac{1}{2\sqrt{3}(s^2+\sqrt{2}s+2)} \times \begin{bmatrix}-(\epsilon s^2+\epsilon\sqrt{2}s-2(3-\epsilon))\\ \sqrt{3}((2-\epsilon)s^2-\epsilon\sqrt{2}s+2(1-\epsilon))\end{bmatrix}.\end{aligned}$$

First it is necessary to show that this is a coprime factorization of G_{Δ_ϵ} for all values of $\epsilon \neq 1$. If $p_n(\epsilon, s)$ and $p_m(\epsilon, s)$ are the numerator polynomials of n_{Δ_ϵ} and m_{Δ_ϵ}, respectively, then $(n_{\Delta_\epsilon}, m_{\Delta_\epsilon})$ is coprime if and only if the polynomials $p_n(\epsilon, s)$ and $p_m(\epsilon, s)$ do not have a common zero in the closed right half plane. Any common zero is also a zero of the linear combination

$$-p_n(\epsilon, s) + \frac{1}{\sqrt{3}} p_m(\epsilon, s) = 2s^2 - 4.$$

This implies that any common zero must be at the point $s = \sqrt{2}$. As equation $p_n(\epsilon, \sqrt{2}) = 0$ has a unique solution, $\epsilon = 1$, this implies that $(n_{\Delta_\epsilon}, m_{\Delta_\epsilon})$ is a coprime factorization for all $\epsilon \neq 1$. For $\epsilon = 1$

$$\begin{bmatrix} n_{\Delta_\epsilon} \\ m_{\Delta_\epsilon} \end{bmatrix}\Bigg|_{\epsilon=1} = \frac{1}{2\sqrt{3}(s^2 + \sqrt{2}s + 2)} \times \begin{bmatrix} -(s + 2\sqrt{2})(s - \sqrt{2}) \\ \sqrt{3}s(s - \sqrt{2}) \end{bmatrix},$$

and therefore $G_{\Delta_\epsilon} = \dfrac{-(s + 2\sqrt{2})}{\sqrt{3}s}$ and $n_{\Delta_\epsilon}, m_{\Delta_\epsilon}$ are not coprime for $\epsilon = 1$ as they share a common zero at $s = \sqrt{2}$.

It is now possible to prove the claim. For $\epsilon < \sqrt{3}$ and $\epsilon \neq 1$ $(n_{\Delta_\epsilon}, m_{\Delta_\epsilon})$ are coprime and $\left\| \epsilon \begin{bmatrix} \Delta_n \\ \Delta_m \end{bmatrix} \right\|_\infty \leq \epsilon/\sqrt{3}$ and therefore $G_{\Delta_\epsilon} \in \mathscr{G}_G^\delta$ for any $\delta > \epsilon/\sqrt{3}$. This implies by Theorem 5.2 that $G_{\Delta_\epsilon} \in \mathscr{B}_G^\delta$ for any $\delta > \epsilon/\sqrt{3}$. In turn this implies that $\delta(G, G_{\Delta_\epsilon}) < \delta$ for any $\delta > \epsilon/\sqrt{3}$ implying $\delta(G, G_{\Delta_\epsilon}) < \epsilon/\sqrt{3}$. Hence $f(\epsilon) = \delta(G, G_{\Delta_\epsilon}) \leq \epsilon/\sqrt{3}$.

For $\epsilon = 1$, $G_{\Delta_\epsilon} = \dfrac{-(s + 2\sqrt{2})}{\sqrt{3}s}$ has normalized coprime factors

$$\begin{bmatrix} n_1 \\ m_1 \end{bmatrix} = \frac{1}{2(s + \sqrt{2})} \begin{bmatrix} -(s + 2\sqrt{2}) \\ \sqrt{3}s \end{bmatrix}.$$

Now note that

$$\left\| \begin{bmatrix} n \\ m \end{bmatrix} - \frac{(s + \sqrt{2})(s - \sqrt{2})}{\sqrt{3}(s^2 + \sqrt{2}s + 2)} \begin{bmatrix} n_1 \\ m_1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \frac{1}{2\sqrt{3}} \\ \frac{1}{2} \end{bmatrix} \right\|_\infty = \frac{1}{\sqrt{3}} < 1,$$

and hence there exists a $q \in \mathscr{R}\mathscr{H}_\infty$ and $q^{-1} \notin \mathscr{R}\mathscr{H}_\infty$ such that $\left\| \begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n_1 \\ m_1 \end{bmatrix} q \right\| < 1$. This implies by Theorem 4.6 that $\vec{\delta}(G_{\Delta_\epsilon}, G) = 1$ implying that $f(\epsilon) = \delta(G, G_{\Delta_\epsilon}) = 1$.

6. ROBUST STABILIZATION

In this section robust stabilization in the context of uncertainty in the gap metric is considered. Before analysing this situation a number of results have to be recalled (Ober and Sefton (1990, 1991)) concerning the formulation of internal stability of a control system in terms of projections and gaps involving the graphs of the plant and the controller.

Theorem 6.1. Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$. Then:

(1) the following statements are equivalent,

(S1) the pair (G, K) is internally stable,

(S2) $\begin{bmatrix} V & N \\ U & M \end{bmatrix}^{-1} \in \mathscr{R}\mathscr{H}_\infty$,

(S3) $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathscr{R}\mathscr{H}_\infty$,

(S4) $\operatorname{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) < 1$.

Moreover

$$\cos\theta_{\min}\left(\left[\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m\right]^\perp, \left[\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right]^\perp\right) = \operatorname{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) = \|P_{([{N \atop M}]\mathscr{H}_2^m)^\perp} P_{([{V \atop U}]\mathscr{H}_2^p)^\perp}\|.$$

(2) If (G, K) is internally stable then

$$\operatorname{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) = \|N^*V + M^*U\|_\infty = \sqrt{1 - (\tau(\tilde{V}M - \tilde{U}N))^2},$$

where for $F \in \mathscr{L}_\infty$ we set $\tau(F) = ess \inf \{\sigma_{\min}(F(s)) \mid \operatorname{Re}(s) = 0\}$.

Typically transfer function descriptions are used in the context of dealing with observable and controllable systems. In order to be able to deal with robustness issues in a realistic setting it is however also very important to allow the description of systems with unstable unobservable and uncontrollable unstable modes in the uncertainty description. From a control point of view these systems are the worst systems to be encountered since they are internally unstable and they cannot be stabilized by feedback. Such systems can be described in the coprime factor framework by allowing for non-coprime factorizations of a transfer function. Let a system G have a right factorization $G = NM^{-1}$ where $N, M \in \mathscr{R}\mathscr{H}_\infty$ are not coprime. Therefore N, M share a right-half plane zero, i.e. for s_0 with $\operatorname{Re}(s_0) \geq 0$ there exists a non-zero vector x_0 such that $\begin{bmatrix} N \\ M \end{bmatrix}(s_0)x_0 = 0$. Hence in the product NM^{-1} there is a right-half plane pole–zero cancellation. This corresponds to an unstable uncontrollable/unobservable mode.

In order to be able to analyse the stability/instability of control systems with perturbed plant we summarize some instability results. Before we state this proposition we need the following lemma.

Lemma 6.2. Let $(U, V) \in \mathcal{RH}_\infty$ be a normalized right coprime factorization of a $m \times p$ transfer function. Let $N, M \in \mathcal{RH}_\infty$ be a not necessarily coprime right factorization of a $p \times m$ transfer function, such that $\begin{bmatrix} N \\ M \end{bmatrix}$ is inner. If

$$\|P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp} P_{([\begin{smallmatrix} N \\ M \end{smallmatrix}]\mathcal{H}_2^m)^\perp}\| < 1,$$

then N, M are right coprime.

Proof. From Lemma 3.7 we have that $\|P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp} P_{([\begin{smallmatrix} N \\ M \end{smallmatrix}]\mathcal{H}_2^m)^\perp}\| < 1$ if and only if

$$\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp = P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right).$$

But

$$\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp = T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}\mathcal{H}_2^m,$$

and

$$P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)$$

$$= T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}[T^*_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]} T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}]^{-1} T^*_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)$$

$$= T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}[T^*_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]} T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}]^{-1} T_{(\tilde{V}M - \tilde{U}N)}\mathcal{H}_2^m.$$

Equating these expressions implies that

$$\mathcal{H}_2^m = [T^*_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]} T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}]^{-1} T_{(\tilde{V}M - \tilde{U}N)}\mathcal{H}_2^m,$$

and hence

$$\mathcal{H}_2^m = T_{(\tilde{V}M - \tilde{U}N)}\mathcal{H}_2^m,$$

as

$$[T^*_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]} T_{[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}]}]^{-1},$$

is bijective (see e.g. Ober and Sefton (1990)). This is the case if and only if $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathcal{RH}_\infty$. This implies that N, M are right coprime.

We are now going to give characterizations for when a control system (G, K) is not stable, i.e. for when it is not internally stable or G has an unobservable and uncontrollable mode in the closed right half plane.

Proposition 6.3. Suppose the $p \times m$ transfer function G does not necessarily have a coprime right factorization (N, M) and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$, then the following statements are equivalent;

(1) the pair (G, K) is not stable,

(2) $\begin{bmatrix} V & N \\ U & M \end{bmatrix}^{-1} \notin \mathcal{RH}_\infty$;

(3) $(\tilde{V}M - \tilde{U}N)^{-1} \notin \mathcal{RH}_\infty$;

(4) $\text{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp\right)$

$$= \|P_{([\begin{smallmatrix} N \\ M \end{smallmatrix}]\mathcal{H}_2^m)^\perp} P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp}\| = 1.$$

Proof. The only part of the proof that is not trivially covered by Theorem 6.1 is to show that if G has an uncontrollable and unobservable mode in the closed right half plane then (2), (3) and (4) hold. But if this is the case, i.e. if the factorization of G is not coprime then this immediately implies (2) and (3). Lemma 6.2 implies (4).

Before we can prove the main theorem of this section we have to prove the following proposition which is an important part in the construction of the proof for the theorem.

Proposition 6.4. Given a $p \times m$ system G with r.c.f. (N, M) and normalized l.c.f. $(\tilde{N}, \tilde{M})$, and an $m \times p$ stabilizing controller with normalized r.c.f. (U, V) and normalized l.c.f. $(\tilde{U}, \tilde{V})$ then,

$$\inf_{\substack{Q_1, Q_2 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \left\|\begin{bmatrix} N & V \\ M & U \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right\|_\infty$$

$$= \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\|\begin{bmatrix} N & V \\ M & U \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right\|_\infty$$

$$= \tau(\tilde{M}V - \tilde{N}U).$$

Further there exist $Q_1, Q_2 \in \mathcal{RH}_\infty$ that achieve the infimum in the above expression.

Proof. The first equality is standard. First note that for any $Q_1 \in \mathcal{RH}_\infty$,

$$\inf_{Q_2 \in \mathcal{RH}_\infty} \left\|\begin{bmatrix} N & V \\ M & U \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right\|_\infty$$

$$= \inf_{Q_2 \in \mathcal{RH}_\infty} \left\|\begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix}\begin{bmatrix} N & V \\ M & U \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}\right\|_\infty$$

$$= \inf_{Q_2 \in \mathcal{RH}_\infty} \left\|\begin{bmatrix} (V^*N + U^*M)Q_1 - Q_2 \\ (\tilde{V}M - \tilde{U}N)Q_1 \end{bmatrix}\right\|_\infty$$

$$= \left\|\begin{bmatrix} H_{(V^*N + U^*M)Q_1} \\ L_{(\tilde{V}M - \tilde{U}N)Q_1 | \mathcal{H}_2^m} \end{bmatrix}\right\|,$$

where the last equation follows from Proposition

3.8. This implies that,

$$A := \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right\|_\infty$$

$$= \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \left\| \begin{bmatrix} H_{(V^*N+U^*M)Q_1} \\ L_{(\tilde{V}M-\tilde{U}N)Q_1 | \mathcal{H}_2^m} \end{bmatrix} \right\|.$$

Note that if $(I+Q_1)^{-1} \notin \mathcal{RH}_\infty$ then $\|Q_1\|_\infty \geq 1$. This implies that,

$$A \geq \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \|(\tilde{V}M - \tilde{U}N)Q_1\|_\infty$$

$$\geq \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \{\tau(\tilde{V}M - \tilde{U}N)\, \|Q_1\|_\infty\},$$

$$\geq \tau(\tilde{V}M - \tilde{U}N) \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \|Q_1\|_\infty,$$

$$\geq \tau(\tilde{V}M - \tilde{U}N).$$

The reverse inequality is now proved by construction. As $V^*N + U^*M \in \mathcal{RL}_\infty$ there exists a factorization $V^*N + U^*M = (V^*N + U^*M)_N \Theta_M^*$, with $(V^*N + U^*M)_N \in \mathcal{RH}_\infty$, $\Theta_M \in \mathcal{RH}_\infty$, right coprime and Θ_M square inner. Then,

$$A = \inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \left\| \begin{bmatrix} H_{(V^*N+U^*M)_N \Theta_M^* Q_1} \\ L_{(\tilde{V}M-\tilde{U}N)Q_1} \end{bmatrix} \Bigg| \mathcal{H}_2^m \right\|$$

$$\leq \inf_{\substack{\hat{Q}_1 \in \mathcal{RH}_\infty \\ (I+\Theta_M\hat{Q}_1)^{-1} \notin \mathcal{RH}_\infty}} \left\| \begin{bmatrix} H_{(V^*N+U^*M)_N \hat{Q}_1} \\ L_{(\tilde{V}M-\tilde{U}N)\Theta_M \hat{Q}_1 | \mathcal{H}_2^m} \end{bmatrix} \right\|$$

$$= \inf_{\substack{\hat{Q}_1 \in \mathcal{RH}_\infty \\ (I+\Theta_M\hat{Q}_1)^{-1} \notin \mathcal{RH}_\infty}} \|(\tilde{V}M - \tilde{U}N)\Theta_M \hat{Q}_1\|_\infty,$$

as $(V^*N + U^*M)_N \hat{Q}_1 \in \mathcal{RH}_\infty$. Let $B_R = \tilde{V}M - \tilde{U}N$, then as $(B_R\Theta_M)^{-1} \in \mathcal{RL}_\infty$ there exists an inner–outer factorization $B_R\Theta_M = \tilde{\Theta}B_L$ where $B_L \in \mathcal{RH}_\infty$ with $B_L^{-1} \in \mathcal{RH}_\infty$ and $\tilde{\Theta}$ is a square inner function. It is clear that, $\sigma_i(B_R)(j\omega) = \sigma_i(B_R\Theta_M)(j\omega) = \sigma_i(\tilde{\Theta}B_L)(j\omega) = \sigma_i(B_L)(j\omega)$ for all $i = 1, 2, \ldots, m$ and for all $\omega \in \Re$. As $B_R(j\omega)$ is a continuous function in ω, there exists $\omega_0 \in \Re$ such that, $\sigma_{\min}(B_R(j\omega_0)) = \sigma_{\min}(B_L(j\omega_0)) = \tau(B_R)$. At $j\omega_0$, let the singular value decomposition of B_L be,

$$B_L(j\omega_0) = U \begin{pmatrix} \sigma_{\max}^{\omega_0} & 0 & \cdots & 0 \\ 0 & \sigma_2^{\omega_0} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \sigma_{\min}^{\omega_0} \end{pmatrix} V^*.$$

It is always possible to construct a function $U_2 \in \mathcal{RH}_\infty$ such that

$$U_2(j\omega_0) = -U \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \sigma_{\min}^{\omega_0} \end{pmatrix} V^*,$$

and $\|U_2\|_\infty = \sigma_{\min}^{\omega_0}$. It is also always possible to construct an inner function $\Theta_2 \in \mathcal{RH}_\infty$ such that $\Theta_2(j\omega_0) = \tilde{\Theta}(j\omega_0)$. It will be shown that the function $\hat{Q}_1^{\text{opt}} = B_L^{-1}U_2\Theta_2 \in \mathcal{RH}_\infty$ achieves the infimum in equation (1). To prove this it is necessary to show that $\hat{Q}_1^{\text{opt}}$ satisfies $\|(\tilde{V}M - \tilde{U}N)\Theta_M\hat{Q}_1^{\text{opt}}\|_\infty = \tau(\tilde{V}M - \tilde{U}N)$ and $(I + \Theta_M\hat{Q}_1^{\text{opt}})^{-1} \notin \mathcal{RH}_\infty$. Now note that,

$$\|(\tilde{V}M - \tilde{U}N)\Theta_M\hat{Q}_1^{\text{opt}}\|_\infty$$
$$= \|B_R\Theta_M\hat{Q}_1^{\text{opt}}\|_\infty = \|\tilde{\Theta}B_L\hat{Q}_1^{\text{opt}}\|_\infty$$
$$= \|\tilde{\Theta}B_L^{-1}U_2\Theta_2\|_\infty = \|U_2\|_\infty$$
$$= \tau(\tilde{V}M - \tilde{U}N),$$

by construction of the function U_2. Further,

$$(I + \Theta_M\hat{Q}_1^{\text{opt}})(j\omega_0) = (I + \Theta_M B_L^{-1}U_2\Theta_2)(j\omega_0)$$
$$= [\Theta_M(I + B_L^{-1}U_2)\Theta_2](j\omega_0)$$
$$= \Theta_M(j\omega_0)V$$
$$\times \left(I - \begin{pmatrix} (\sigma_{\max}^{\omega_0})^{-1} & 0 & \cdots & 0 \\ 0 & (\sigma_2^{\omega_0})^{-1} & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & (\sigma_{\min}^{\omega_0})^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \sigma_{\min}^{\omega_0} \end{pmatrix} \right) V^*\Theta_2(j\omega_0)$$

$$= \Theta_M(j\omega_0)V \left(\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right)$$
$$\times V^*\Theta_2(j\omega_0),$$

and hence $(I + \Theta_M\hat{Q}_1^{\text{opt}})^{-1} \notin \mathcal{RH}_\infty$. This completes the proof of the reverse equality as

$$A \leq \inf_{\substack{\hat{Q}_1 \in \mathcal{RH}_\infty \\ (I+\Theta_M\hat{Q}_1)^{-1} \notin \mathcal{RH}_\infty}} \|(\tilde{V}M - \tilde{U}N)\Theta_M\hat{Q}_1\|_\infty$$
$$\leq \|(\tilde{V}M - \tilde{U}N)\Theta_M\hat{Q}_1^{\text{opt}}\|_\infty$$
$$= \tau(\tilde{V}M - \tilde{U}N).$$

Summarizing note that $Q_1 = \Theta_M B_L^{-1}U_2\Theta_2$ and $Q_2 = (V^*M + U^*N)_N B_L^{-1}U_2\Theta_2$ achieve this infimum.

We also need the following lemma.

Lemma 6.5. Given a $p \times m$ system G with r.c.f. (N, M), a $m \times p$ controller K with r.c.f (U, V) and a perturbed system $G_\Delta = N_\Delta M_\Delta^{-1}$ where (N_Δ, M_Δ) are not necessarily coprime, then

$$\|P_{([\begin{smallmatrix} N_\Delta \\ M_\Delta \end{smallmatrix}]\mathcal{H}_2^m)^\perp} P_{([\begin{smallmatrix} V \\ U \end{smallmatrix}]\mathcal{H}_2^p)^\perp}\| = 1,$$

implies that

$$\|P_{([{N \atop M}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}\|^2 + \|P_{([{N \atop M}]\mathcal{H}_2^m)}P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}\|^2 \geq 1.$$

Proof. We assume without loss of generality that there exists a function $v \in \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$ with $\|v\|_2 = 1$ such that the norm $\|P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}\| = 1$ is attained, i.e.

$$\|P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}v\|_2 = 1.$$

If the norm is not attained the result can be obtained by a perturbation argument. We therefore have that $v \in \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp \cap \left(\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix}\mathcal{H}_2^m\right)^\perp$. Now note that this implies

$$\begin{aligned}
&\|P_{([{N \atop M}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}\|^2 + \|P_{([{N \atop M}]\mathcal{H}_2^m)}P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}\|^2 \\
&\quad \geq \|P_{([{N \atop M}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}v\|_2^2 \\
&\qquad + \|P_{([{N \atop M}]\mathcal{H}_2^m)}P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}v\|_2^2 \\
&\quad = \|P_{([{N \atop M}]\mathcal{H}_2^m)^\perp}v\|_2^2 + \|P_{([{N \atop M}]\mathcal{H}_2^m)}v\|_2^2 \\
&\quad = \|v\|_2^2 = 1.
\end{aligned}$$

We are now in a position to prove the main theorem of this section. For a given control system the theorem gives a characterization of the uncertainty in terms of balls in the gap metric that can be stabilized by the controller. Notice that the theorem can be rephrased in the following way. All perturbed plants such that their graphs and the graph of the nominal plant form an angle less than α are stabilized by the controller if and only if $\sin\alpha \leq \sin\alpha_0$ where α_0 is the angle between the graph of the plant and the orthogonal complement of the graph of the controller. A robustness result phrased in terms of coprime factor uncertainty was given by Vidyasagar and Kimura (1986). The following result can be seen to be a geometric interpretation of the robustness results by McFarlane and Glover (1989). Another geometric approach was given by Foias *et al.* (1990).

Theorem 6.6. Given a $p \times m$ system G, a $m \times p$ stabilizing controller K then for all perturbed systems $G_\Delta \in \vec{\mathcal{B}}_G^\epsilon$,

$$(G_\Delta, K) \text{ is internally stable,}$$

if and only if

$$\epsilon \leq (1 - \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2)^{\frac{1}{2}}.$$

Proof. Let (N, M) and (U, V) be normalized r.c.f. of the transfer function G and K, respectively. Let $G_\Delta \in \vec{\mathcal{B}}_G^\epsilon$ and assume that $\epsilon \leq (1 - \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2)^{\frac{1}{2}}$. By assumption there exists $\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{RH}_\infty$ with $\left\|\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}\right\|_\infty < \epsilon$ such that $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}$. Then by assumption,

$$\begin{aligned}
&\|P_{([{N \atop M}]\mathcal{H}_2^m)^\perp}P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}\|^2 + \|P_{([{N \atop M}]\mathcal{H}_2^m)}P_{([{N_\Delta \atop M_\Delta}]\mathcal{H}_2^m)^\perp}\|^2 \\
&\quad = \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2 + \vec{\delta}(G, G_\Delta)^2 \\
&\quad < \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2 + \epsilon^2 \\
&\quad < 1.
\end{aligned}$$

By Lemma 6.5 this inequality implies that $\|P_{([{V \atop U}]\mathcal{H}_2^p)^\perp}P_{([{N+\Delta_N \atop M+\Delta_M}]\mathcal{H}_2^m)^\perp}\| < 1$. By Theorem 6.1 this shows that (G_Δ, K) is internally stable if $(N + \Delta_N)$, $(M + \Delta_M)$ is a coprime factorization. But the coprimeness follows immediately from Lemma 6.2 and therefore also $G_\Delta \in \mathcal{B}_G^\epsilon$.

To prove the converse direction it is necessary to construct a perturbation $\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{RH}_\infty$ with

$$\begin{aligned}
\left\|\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}\right\|_\infty &= (1 - \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2)^{\frac{1}{2}} \\
&= \tau(\tilde{V}M - \tilde{U}N),
\end{aligned}$$

such that

$$\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix} := \begin{bmatrix} N \\ M \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix},$$

is a not necessarily coprime right factorization corresponding to a system G_Δ, such that (G_Δ, K) is not stable. Let

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = \begin{bmatrix} N & V \\ M & U \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

with Q_1, Q_2 as constructed in Proposition 6.4 such that $\left\|\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}\right\|_\infty = \tau(\tilde{V}M - \tilde{U}N)$. By construction $(I + Q_1)^{-1} \notin \mathcal{RH}_\infty$ and therefore $(\tilde{V}M_\Delta - \tilde{U}N_\Delta)^{-1} = [(\tilde{V}M - \tilde{U}N)(I + Q_1)]^{-1} \notin \mathcal{RH}_\infty$. Hence by Proposition 6.3 the factorization (N_Δ, M_Δ) represents a system G_Δ, such that (G_Δ, K) is not stable.

As a corollary we can now recover a result first obtained by Georgiou and Smith (1990).

Corollary 6.7. Let $\sigma_1 = \|H_{[{N \atop M}]}\|$. Then for $0 < \epsilon \leq \sqrt{1 - \sigma_1^2}$,

$$\vec{\mathcal{B}}_G^\epsilon = \mathcal{B}_G^\epsilon.$$

Proof. In the proof of the theorem it was shown that $\vec{\mathcal{B}}_G^\epsilon = \mathcal{B}_G^\epsilon$ for $\epsilon \leq (1 - \text{gap}(\mathcal{G}(M_G), (\mathcal{G}^T(M_K))^\perp)^2)^{\frac{1}{2}}$, for any stabilizing controller K. Ober and Sefton (1991) showed that there exists a controller K_0 such that gap $(\mathscr{G}(M_G), (\mathscr{G}^T(M_{K_0}))^\perp) = \sigma_1$. This implies the result.

7. CONCLUSIONS

Normalized coprime factor uncertainty models and the uncertainty model based on the gap metric are studied. A new condition is given that allows the calculation of the gap between two plants to be reduced to the computation of the directed gap. The connection between the coprime factor uncertainty models is fully clarified. The robustness of a control system is analysed from a geometric point of view.

REFERENCES

Cordes, H. and J. Labrousse (1963). The invariance of the index in the metric space of closed operators. *J. of Mathematics and Mechanics*, **12**, 693–719.

Doyle, J. C., K. Glover, P. Khargonekar and B. Francis (1989). State-space solutions to standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems. *IEEE Trans. Aut. Control*, **34**, 831–848.

El-Sakkary, A. (1985). The gap metric: robustness of stabilization of feedback systems. *IEEE Trans. Aut. Control*, 240–247.

Foias, C., T. T. Georgiou and M. C. Smith (1990). Geometric techniques for robust stabilization of linear time-varying systems. *Proc. CDC*, Honolulu, December 1990.

Francis, B. (1987). A course in $\mathscr{H}_\infty$ control theory. *Lecture Notes in Control and Information Sciences*, Vol. 88. Springer Verlag, Berlin.

Georgiou, T. (1988). On the computation of the gap metric. *Systems and Control Letters*, **11**, 253–257.

Georgiou, T. and M. Smith (1990). Optimal robustness in the gap metric. *IEEE Trans. Aut. Control*, 673–686.

Gohberg, I. and M. Krein (1978). Introduction to the theory of linear non-self adjoint operators. *Translations of Mathematical Monographs*, American Mathematical Society.

McFarlane, D. and K. Glover (1989). Robust controller design using normalized coprime factor plant descriptions. *Lecture Notes in Control and Information Sciences*, Vol. 110. Springer Verlag, Berlin.

Nikolskii, N. (1986). Treatise on the shift operator. *Grundlehren der Mathematischen Wissenschaft*. Springer Verlag, Berlin.

Ober, R. and J. Sefton (1990). Stability of control systems and graphs of linear systems. Technical Report 202, Programs in Mathematical Sciences, University of Texas at Dallas, August 1990.

Ober, R. and J. Sefton (1991). Stability of control systems and graphs of linear systems. *Systems and Control Letters*, 265–280.

Qui, L. and E. J. Davison (1992). Feedback stability under simultaneous gap metric uncertainties in plant and controller. *Systems and Control Letters*, **18**, 9–22.

Sefton, J. and R. Ober (1991). On the gap metric and coprime factor perturbations. Technical Report No. 208, Programs in Mathematical Sciences, University of Texas at Dallas.

Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach*. MIT Press, MA.

Vidyasagar, M. and H. Kimura (1986). Robust controllers for uncertain linear multivariable systems. *Automatica*, **22**, 85–94.

Weidmann, J. (1980). *Linear Operators in Hilbert Space*. Springer Verlag, Berlin.

Zhu, S. (1989). Robustness of feedback stabilization: a topological approach. Ph.D. thesis, Eindhoven, The Netherlands.