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Uncertainty in the Weighted Gap Metric: A Geometric Approach*

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A geometric approach is introduced to study robust controllers in the weighted gap metric. Maximally stabilizing controllers are analysed and an inverse problem is studied.

Key Words—Control theory; frequency domain; multivariable systems; robust control.

Abstract—The stability of control systems is studied in the context of weighted input–output signal spaces. Necessary and sufficient conditions for a controller to stabilize a plant are given in terms of geometric notions. These geometric quantities can be calculated by solving $\mathcal{H}_\infty$ optimization problems. Maximally stabilizing controllers in a weighted signal space are introduced and characterized in terms of Nehari extensions. The robustness properties of maximally stabilizing controllers are analysed in terms of weighted coprime factor uncertainty. Necessary and sufficient conditions are established for a controller of a given plant to be the maximally stabilizing controller of the plant with respect to a weight. An upper bound for the mixed-sensitivity of a control system is given where the controller is the maximally stabilizing controller of the plant.

1. INTRODUCTION

THIS PAPER PRESENTS a detailed study of robust control from the point of view of robustness in the gap metric and coprime factor uncertainty. The following coprime factor uncertainty problem was first introduced and analysed by McFarlane and Glover (1989). It attracted a great deal of interest and stimulated research both in a theoretical and a practical direction. It can be summarized as follows. We assume that we are given the transfer function G of a plant with normalized right coprime factorization, i.e. $G = NM^{-1}$, where N, M are stable coprime rational functions with $N^*N + M^*M = I$ and M invertible. Then the problem is to find a controller that stabilizes a ball of maximal size given by

$$\left\{G_\Delta \text{ such that } G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1},\right.$$
$$\left.\Delta_N, \Delta_M \text{ stable, with } \left\|\begin{matrix}\Delta_N \\ \Delta_M\end{matrix}\right\|_\infty < \epsilon\right\}.$$

Another way of describing uncertainty is via the so-called gap metric, which has been introduced by Zames and El-Sakkary (1980) and El-Sakkary (1985). For more recent developments see for example Schumacher (1992) and the references therein. There have been a number of publications on studies of the connection between the coprime factor uncertainy problem and the description of uncertainty in the gap metric (see e.g. Zhu, 1989; Georgiou and Smith, 1990a; Sefton and Ober, 1993; Habets, 1991). It was shown (Ober and Sefton, 1991) that this problem is in fact equivalent to finding the maximally stabilizing controller K_0 of the plant G. The maximally stabilizing controller K_0 is defined as the controller that minimizes the distance in the gap metric between the graph of the plant and the orthogonal complement of the transposed graph of the controller. An equivalent way of defining the maximally stabilizing controller is by introducing the following geometric interpretation. We associated with the plant G the set of all possible input–output pairs that have bounded energy, called the graph space $\mathcal{G}(G)$ of G. In the frequency domain this amounts to considering all the possible input–output pairs corresponding to the plant G such that both the inputs and the ouputs have Laplace transforms that are in $\mathcal{H}_2$. The space $\mathcal{H}_2$ contains the space of rational functions which are square integrable on the imaginary axis and have analytic continuation to the right half-plane.

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Given a normalized right coprime factorization $G = NM^{-1}$ of the plant the graph space is given by (Vidyasagar, 1985)

$$\mathcal{G}(G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2.$$

Similarly, we can associate a graph space with a controller K. More precisely, we need to define the transposed graph $\mathcal{G}^T(K)$ associated with a controller which can be obtained from the graph of the controller by simply swapping the inputs with the outputs, i.e.

$$\mathcal{G}^T(K) = \begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2,$$

where $K = UV^{-1}$ is a normalized right coprime factorization of the controller K. In the space of all possible input–output pairs, i.e. the space $\mathcal{H}_2 \times \mathcal{H}_2$, we can introduce the standard geometric notions such as angles between subspaces. If we are given two (closed) subspaces A, B of the space of all possible input–output pairs the minimum angle $\theta_{\min}(A, B)$ between these two subspaces can be defined by

$$\cos \theta_{\min}(A, B) = \sup_{u \in A, v \in B} \frac{|\langle u, v \rangle|}{\|u\| \, \|v\|},$$

where $\langle u, v \rangle$ is the inner product between two elements in $\mathcal{H}_2 \times \mathcal{H}_2$ and $\|u\|$ is the norm of the element $u \in \mathcal{H}_2 \times \mathcal{H}_2$. It was shown (Ober and Sefton, 1991) that a controller K stabilizes the plant G if and only if the minimal angle between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of the controller is positive, i.e. if and only if

$$\theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp) > 0.$$

The maximally stabilizing controller can now also be characterized as the controller that maximizes the distance to instability, i.e. the distance to zero. It was shown (Ober and Sefton, 1991) that

$$\theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K_0)]^\perp) = \sup_K \theta_{\min}[(\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp).$$

The normalized coprime factor uncertainty problem imposes a very particular structure on the permissible perturbations in that perturbations of the numerator and perturbations of the denominator of the plant are equally weighted. The same comment applies to the definition of the maximally stabilizing controller in terms of the geometric framework that was just discussed. The topic of this paper is to extend the above results to include weighted coprime factor uncertainty. In the context of the geometric framework this amounts to weighting the input and output spaces with minimum phase weights. Introducing weights allows a substantial extension of the previously available results. Moreover, it permits the analysis and interpretation of a large class of control systems from the point of view of coprime factor perturbations or maximally stabilizing controllers with respect to a certain weight. Studying coprime factor perturbations in a weighted setting is not new. McFarlane and Glover (1989) introduced weights in the context of a loop-shaping design methodology (see also Georgiou and Smith, 1991). Georgiou and Smith (1990b) defined a weighted gap metric. Robust stabilization in the gap metric and an expression for the appropriate robustness margin has also been given in Georgiou and Smith (1990b). We are however not aware of a systematic study of this problem as it is undertaken here.

Our analysis will be done within the geometric framework. Our first objective will be to derive necessary and sufficient conditions for a controller to stabilize a plant. We will again have a theorem available that states that a controller will stabilize a plant if and only if the minimal angle between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of the controller is positive. Here the geometric notions are taken with respect to the weighted graphs. These results form the basis for our development. Of great importance, both for practical computations and for theoretical development, are the connections of these geometric notions to $\mathcal{H}_\infty$-optimization problems. One of these results permits the calculation of the minimum angle between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of a stabilizing controller by simply calculating the $\mathcal{L}_\infty$-norm of a transfer function that can be computed in a straightforward way from coprime factors of the plant and the controller. Similarly to the unweighted case it is possible to define and characterize a maximally stabilizing controller with respect to the given weights. The robustness of the maximally stabilizing controller can be analysed from the point of view of weighted coprime factor uncertainty.

One of the main contributions of the paper lies in the analysis of what is called the inverse weight problem. Here we study the question as to whether, given a control system with plant g and controller k, the controller can be considered to be the maximally stabilizing controller of g with respect to a particular weight. We give necessary and sufficient conditions for such a weight to exist. An interesting by-product of this investigation is a connection of the present problem with the mixed-sensitivity minimization problem. This observation may to an extent give a further explanation as to why the loop-shaping design methodology of McFarlane and Glover (1989) appears to produce very satisfactory designs (see e.g. Hyde *et al.*, 1990; Englehart and Smith, 1991).

In Section 2 we give a very brief summary of the geometric notions that will be used in this paper. The subsequent section is devoted to an introduction of weighted spaces, Toeplitz and Hankel operators on these weighted spaces and W-normalized coprime factorizations. These are normalized coprime factorizations with respect a weight W. In order to be able to derive explicit expressions for the various geometric notions that are of interest to us it is important to have explicit characterizations of orthogonal projections onto graph spaces of plants and onto the orthogonal complements of such graph spaces. This is done in Section 4. After these sections which are devoted to the development of the necessary background material, Section 5 contains one of the main results of this paper. It gives necessary and sufficient conditions for the closed-loop stability of a control system in terms of geometric notions in a weighted signal space. This is followed by a section which is concerned with the derivation of $\mathcal{H}_\infty$ formulae for the calculation of the geometric quantities that were shown to be of importance in the previous section in characterizing the closed-loop stability of a control system. Maximally stabilizing controllers with respect to weighted graph spaces are characterized in the following section. In Section 8 maximally stabilizing controllers are analysed from the point of view of weighted coprime factor uncertainty. In Section 9 and Section 10 the inverse weight problem is discussed. Necessary and sufficient conditions are derived for a controller to be a maximally stabilizing controller of a given plant g with respect to a weight W. In Section 10 a connection of the maximally stabilizing controller with the mixed-sensitivity problem is elaborated.

Since the submission of this paper Qiu and Davison (1992) have published an interesting contribution to the robust stabilization problem in the context of simultaneous unweighted gap metric uncertainties in the plant and the controller.

2. GEOMETRIC NOTIONS IN HILBERT SPACE

In this paper we use extensively the following geometric notions in a Hilbert space H. We will not give any proofs here but refer to for example Gohberg and Krein (1978), Nikolskii (1986) and Weidmann (1980). Let $A, B \subseteq H$ be two closed subspaces; then it is possible to define the minimal angle and the gap between these two spaces as follows:

$$\cos\theta_{\min}(A, B) = \sup_{u\in A, v\in B} \frac{|\langle u, v\rangle|}{\|u\|\,\|v\|}$$

and

$$\operatorname{gap}(A, B) = \|P_A - P_B\|,$$

where P_C denotes the orthogonal projection on the closed subspace C. Alternatively, the sine of the minimal angle can be defined by

$$\sin\theta_{\min}(A, B) = \|P_{A\|B}\|^{-1},$$

where the skew projection $P_{A\|B}$ is defined by $P_{A\|B}$: $A + B \to A$, $u + v \mapsto u$, $u \in A$, $v \in B$. The skew projection is well defined on the Hilbert space H if $H = A + B$ and $A \cap B = \emptyset$. The skew projection is bounded if and only if $\theta_{\min}(A, B) > 0$. The following relationships hold:

$$\cos\theta_{\min}(A, B) = \|P_A P_B\| = \|P_B P_A\| = \sup_{u\in B, \|u\|=1} \operatorname{dist}(u, A^\perp),$$

where $\operatorname{dist}(u, A^\perp) = \inf_{v\in A^\perp} \|u - v\|$. The gap between two spaces can be characterized as follows:

$$\begin{aligned}\operatorname{gap}(A, B) &= \max\{\|P_A P_{B^\perp}\|, \|P_{A^\perp} P_B\|\}\\ &= \max\{\cos\theta_{\min}(A, B^\perp), \cos\theta_{\min}(B, A^\perp)\}\\ &= \max\left\{\sup_{u\in A, \|u\|=1} \operatorname{dist}(u, B), \sup_{v\in B, \|v\|=1} \operatorname{dist}(u, A)\right\}.\end{aligned}$$

If $\operatorname{gap}(A, B) < 1$ then $\|P_A P_{B^\perp}\| = \|P_{A^\perp} P_B\|$.

3. WEIGHTS AND COPRIME FACTORIZATIONS

It is necessary to first define an admissible class of weighting functions. The function $W \in \mathcal{L}_\infty^{r\times r}$ will be considered to be in the class of weighting functions $\mathcal{W}^r$ if and only if it can be factored as $W_R^* W_R = W_L W_L^*$ where W_R, $W_L \in \mathcal{H}_\infty^{r \times r}$ are continuous on the imaginary axis, including at ∞, and invertible in $\mathcal{H}_\infty$. The Hardy space $\mathcal{H}_\infty^{p \times m}$ contains all $p \times m$ bounded rational functions on the imaginary axis with analytic continuation in the right-half plane. It is a subspace of $\mathcal{L}_\infty^{p \times m}$, the space of all $p \times m$ essentially bounded functions on the imaginary axis. These functions all have finite $\mathcal{L}_\infty$-norm defined by $\|G\|_\infty := \operatorname{ess\,sup}_{\omega \in \mathcal{R}} \sigma_{\max}[G(j\omega)]$. Given any weighting function $W \in \mathcal{W}^r$, the Hardy spaces $\mathcal{H}_2^{r, W_R}$, $\mathcal{H}_{2,-}^{r, W_L}$ contains all r-vector valued rational functions f that are defined on the imaginary axis, are square-integrable with respect to the weight, W, (i.e. $\int_{-\infty}^{\infty} f^* W f d\omega < \infty$) and have an analytic continuation into the right and left half planes, respectively. These are Hilbert spaces with inner product,

$$[f, g]_W := \int_{-\infty}^{\infty} f^* W g d\omega, \quad f, g \in \mathcal{H}_2^{r, W_R} \, (\mathcal{H}_{2,-}^{2, W_L}),$$

and induced norm $\|f\|_W^2 := [f, f]_W$. The subscript W will be dropped if there is no danger of ambiguity concerning the specific weight. The space $\mathcal{L}_2^{r, W}$ is defined similarly. It consists of all r-vector valued functions f on the imaginary axis that are bounded with respect to $[\cdot, \cdot]_W$, i.e.

$$[f, f]_W = \int_{-\infty}^{\infty} f^* W f d\omega < \infty.$$

Clearly, $\mathcal{L}_2^{r, W}$ admits the generally not orthogonal decomposition $\mathcal{L}_2^{r, W} = \mathcal{H}_2^{r, W_R} + \mathcal{H}_{2,-}^{r, W_L}$. The prefix $\mathcal{R}$ before any of these spaces denotes the subspace of real-rational function in the respective space.

By the definition of the elements in $\mathcal{W}^r$, there exist for each element $W \in \mathcal{W}^r$ right and left spectral factors W_R, W_L, i.e. W_R, $W_L \in \mathcal{H}_\infty^{r \times r}$ such that W_R^{-1}, $W_L^{-1} \in \mathcal{H}_\infty^{r \times r}$ and $W = W_R^* W_R = W_L W_L^*$. Since the spectral factors W_R, W_L are invertible in $\mathcal{H}_\infty^{r \times r}$ it is clear that the unweighted spaces $\mathcal{H}_2^{r, I}$, $\mathcal{H}_{2,-}^{r, I}$ and $\mathcal{L}_2^{r, I}$ coincide as sets with the spaces $\mathcal{H}_2^{r, W_R}$, $\mathcal{H}_{2,-}^{r, W_L}$ and $\mathcal{L}_2^{r, W}$. The usual inner product on the unweighted space $\mathcal{L}_{2,-}^{r, I}$ is denoted by $\langle \cdot, \cdot \rangle$.

The maps

$$R: \mathcal{L}_2^{r, W} \to \mathcal{L}_2^{r, I}; \quad x \mapsto W_R x$$

and

$$L: \mathcal{L}_2^{r, W} \to \mathcal{L}_2^{r, I}; \quad x \mapsto W_L^* x$$

are unitary maps. Similarly, $R_+ := R|_{\mathcal{H}_2^{r, W_R}}$ and $L_- := L|_{\mathcal{H}_{2,-}^{r, W_L}}$ are unitary maps. The adjoints, respectively inverses of R and L (R_+, L_-), are given by

$$R^{-1}: \mathcal{L}_2^{r, I} \to \mathcal{L}_2^{r, W}; \quad x \mapsto W_R^{-1} x,$$

and

$$L^{-1}: \mathcal{L}_2^{r, I} \to \mathcal{L}_2^{r, W}; \quad x \mapsto W_L^{-*} x,$$

$(R_+^{-1} = R^{-1}|_{\mathcal{H}_2^{r, I}}$ and $L_-^{-1} = L^{-1}|_{\mathcal{H}_{2,-}^{r, I}})$.

As an immediate consequence of these identities we obtain expressions for the projections on the weighted Hardy spaces.

Lemma 3.1. Given a weight $W \in \mathcal{W}^r$ and the spectral factorizations $W = W_L W_L^* = W_R^* W_R$ then the orthogonal projections $P_+^W: \mathcal{L}_2^{r, W} \to \mathcal{L}_2^{r, W}$ and $P_-^W: \mathcal{L}_2^{r, W} \to \mathcal{L}_2^{r, W}$ on the closed subspaces $\mathcal{H}_2^{r, W_R}$ and $\mathcal{H}_{2,-}^{r, W_L}$, respectively, are given by

$$P_+^W = R^{-1} P_+ R$$
$$P_-^W = L^{-1} P_- L,$$

where $P_+ := P_+^I$, $P_- := P_-^I$.

Inner functions with respect to weights will be of particular importance for our development. They are defined as follows.

Definition 3.2. Given a $p \times m$ function θ then,

(1) if $\theta \in \mathcal{H}_\infty^{p \times m}$ and $p \geq m$ then θ is called W-inner if and only if $\theta^* W \theta = I_m$, for $W \in \mathcal{W}^p$.
(2) If $\theta \in \mathcal{H}_\infty^{p \times m}$ and $m \geq p$ then θ is called W-co-inner if and only if $\theta W \theta^* = I_p$, for $W \in \mathcal{W}^m$.
(3) If $\theta \in \mathcal{L}_\infty^{p \times m}$ and $p = m$ then θ is called W-all-pass if and only if $\theta^* W \theta = I_m$, for $W \in \mathcal{W}^p$.

A major tool in our study will be coprime factorizations of transfer functions.

Definition 3.3. Let G be a not necessarily rational transfer function. The pair (N, M) where $N, M \in \mathcal{H}_\infty$ constitutes a right coprime factorization (RCF) of the transfer function G [similarly, the pair $(\tilde{N}, \tilde{M})$ where $\tilde{N}, \tilde{M} \in \mathcal{H}_\infty$, is a left coprime factorization (LCF) of G] if

(1) M, $(\tilde{M})$, is square and $\det(M(\infty)) \neq 0$ $(\det(\tilde{M}(\infty)) \neq 0)$.
(2) $G = NM^{-1} (G = \tilde{M}^{-1} \tilde{N})$.
(3) N and M are right coprime, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathcal{H}_\infty$ such that $-\tilde{X} N + \tilde{Y} M = I$ ($\tilde{N}$ and $\tilde{M}$ are left coprime, i.e. there exist $X, Y \in \mathcal{H}_\infty$ such that $-\tilde{N} X + \tilde{M} Y = I$).

Let $W \in \mathcal{W}^{p+m}$. Then, a right coprime factorization $G = NM^{-1}$ of the $p \times m$ transfer function G is called a W-normalized right coprime factorization (W-NRCF) of G if $\begin{bmatrix} N \\ M \end{bmatrix}$ is W-inner. A left coprime factorization $G = \tilde{M}^{-1} \tilde{N}$ of G is called a W-normalized left coprime factorization (W-NLCF) if $[\tilde{M} \quad \tilde{N}]$ is W^{-1}-co-inner. We also need a definition that involves a transposition of the coprime factors. A right coprime factorization $G = NM^{-1}$ of G is called a W-t-normalized right coprime factorization (W-t-NRCF) if $\begin{bmatrix} M \\ N \end{bmatrix}$ is W-inner. A left coprime factorization $G = \tilde{M}^{-1}\tilde{N}$ of G is called a W-T-normalized left coprime factorization (W-T-NLCF) if $[\tilde{N} \quad \tilde{M}]$ is W^{-1}-co-inner. For rational transfer functions and rational weights the existence of such factorizations can be shown in exactly the same way as for the case $W = I$ [see e.g. Vidyasagar, 1985].

We now define the graph that corresponds to a transfer function G. We consider the multiplication operator that corresponds to this transfer function and maps inputs to the respective outputs. We consider this multiplication operator as acting between weighted Hardy spaces. The multiplication operator associated with the not necessarily stable transfer function G is defined as follows:

$$M_G : \mathscr{H}_2^{m,W_i} \to \mathscr{H}_2^{p,W_o}; \quad f \mapsto Gf.$$

If G is not in $\mathscr{H}_\infty$ then M_G will be an unbounded operator with domain $D(M_G) = \{f \in \mathscr{H}_2^{m,W_i} \mid Gf \in \mathscr{H}_2^{p,W_o}\}$. The graph of the operator M_G is denoted by $\mathscr{G}(G)$. If $G = NM^{-1}$ is a right coprime factorization of G then the graph is given by

$$\mathscr{G}(G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^{p,I}.$$

We will also have occasion to use the so-called transposed graph of a transfer function. It is given by exchanging the two components of the graph, i.e.

$$\mathscr{G}^T(G) := \begin{bmatrix} M \\ N \end{bmatrix} \mathscr{H}_2^{p,I}.$$

The geometric analysis that is undertaken in this paper will be done on the space of all possible input–output pairs, i.e. the space $\mathscr{H} := \begin{bmatrix} \mathscr{H}_2^{p,W_o} \\ \mathscr{H}_2^{m,W_i} \end{bmatrix}$. We now define a number of special Toeplitz and Hankel operators. The Toeplitz operators map into the space $\mathscr{H}$ or are defined on the space $\mathscr{H}$. Let F be a $(p+m) \times m$ or $(p+m) \times p$ transfer function. Then the Toeplitz operator T_F is defined by

$$T_F : \mathscr{H}_2^{m,I}(\mathscr{H}_2^{p,I}) \to \mathscr{H}; \quad f \mapsto T_F f := P_+^W F f.$$

Similarly, if E is an $m \times (p+m)$ or $p \times (p+m)$ transfer function, then T_E is defined by

$$T_E : \mathscr{H} \to \mathscr{H}_2^{m,I}(\mathscr{H}_2^{p,I}); \quad f \mapsto T_E f := P_+^I E f.$$

The symbol H_F^W denotes the Hankel operator

$$H_F^W : \mathscr{H}_2^{p,I} \to \mathscr{H}_{2,-}^{p+m,W_L}; \quad f \mapsto H_F^W f := P_-^W F f$$

with symbol $F \in L_\infty^{(p+m)\times p}$.

Let $A : H_1 \to H_2$ be a bounded operator between the Hilbert spaces H_1 and H_2. Let $\sigma > 0$, $f \in H_1$ with $\|f\| = 1$ and $g \in H_2$ with $\|g\| = 1$. Then (f, g) is a Schmidt pair of A with singular value σ if $Af = \sigma g$ and $A^* g = \sigma f$.

4. ORTHOGONAL PROJECTIONS

It follows from the expressions on the geometric identities in a Hilbert space as introduced in Section 2 that it is of importance to have explicit representations of the projections onto the various graph spaces. The aim of this section is therefore to examine the projections onto the graph spaces of a plant and a controller in some detail. The results of this section are analogous to the corresponding results of Zhu (1989). Here, we prove the results for the specific application of projections onto weighted spaces.

Lemma 4.1. Given a weight

$$W = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^* W_i \end{bmatrix} \in \mathscr{W}^{p+m},$$

a $p \times m$ system G and an $m \times p$ controller K, let $(\tilde{N}, \tilde{M})$ be a W-NLCF of G and $(\tilde{U}, \tilde{V})$ a W-T-NLCF of K. Then the operators

$$Z_G : \mathscr{H}_2^{m,I} \to \mathscr{H}_2^{m,I}; \quad f \mapsto T_{[\tilde{M} \quad -\tilde{N}]} T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} f$$

$$Z_K : \mathscr{H}_2^{p,I} \to \mathscr{H}_2^{p,I}; \quad f \mapsto T_{[-\tilde{U} \quad \tilde{V}]} T_{W^{-1}\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]} f$$

are self-adjoint positive and have bounded inverses.

Proof. The proof is identical to the proof of Lemma 3.1.1 (Zhu, 1989). □

In the following definition a number of Toeplitz operators are defined with range space $\mathscr{H} = \begin{bmatrix} H_2^{p,W_o} \\ H_2^{m,W_i} \end{bmatrix}$.

Definition 4.2. Assume the notation of Lemma 4.1. Let $Z_G^{1/2}$ and $Z_K^{1/2}$ be square roots of the operator Z_G and Z_K, respectively, i.e. $(Z_G^{1/2})^* Z_G^{1/2} = Z_G$ and $(Z_K^{1/2})^* Z_K^{1/2} = Z_K$. Then define

$$E_G : \mathscr{H}_2^{m,I} \to \mathscr{H} \quad E_G : f \mapsto T_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} f = \begin{bmatrix} N \\ M \end{bmatrix} f$$

$$\tilde{E}_G : \mathscr{H}_2^{p,I} \to \mathscr{H} \quad \tilde{E}_G : f \mapsto T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} Z_G^{-1/2} f$$

$$F_K : \mathscr{H}_2^{m,I} \to \mathscr{H} \quad F_K : f \mapsto T_{W^{-1}\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]} Z_K^{-1/2} f$$

$$\tilde{F}_K : \mathscr{H}_2^{p,I} \to \mathscr{H} \quad \tilde{F}_K : f \mapsto T_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} f = \begin{bmatrix} V \\ U \end{bmatrix} f.$$

In the next Lemma it is shown that E_G is an isometric operator that maps the space $\mathcal{H}_2^{m,l}$ onto the weighted graph space of the system. Similarly $\tilde{E}_G$ is an isometric operator mapping the space $\mathcal{H}_2^{p,l}$ onto the orthogonal complement of the weighted graph space of the system, i.e. onto $[\mathcal{G}(G)]^\perp$. For the expression of the graph $\mathcal{G}(G)$ in terms of the coprime factors of G see Vidyasagar (1985).

Lemma 4.3. Using the notation of Definition 4.2 we have

$$\text{Range}\,(E_G) = \mathcal{G}(G) \quad \text{Range}\,(\tilde{E}_G) = [\mathcal{G}(G)]^\perp.$$

For any right and left coprime factorization $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ we have

$$\mathcal{G}(G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^{m,l}$$

$$[\mathcal{G}(G)]^\perp = P_+^W W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} \mathcal{H}_2^{p,l}.$$

Proof. The first expression is obvious from the definition of E_G and $\mathcal{G}(G)$. To prove the second expression first note that $\text{Range}\,(\tilde{E}_G) = P_+^W W^{-1}\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}\mathcal{H}_2^{p,l}$. Let now $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{H}$ be such that $f \perp \mathcal{R}(\tilde{E}_G)$. Then, for all $g \in \mathcal{H}_2^{p,l}$

$$\begin{aligned} 0 &= \left[P_+^W W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} g, f \right] \\ &= \left\langle W_R^{-1} P_+ W_R^{-*} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} g, Wf \right\rangle \\ &= \langle g, [\tilde{M} \quad -\tilde{N}] f \rangle, \end{aligned}$$

which implies that $[\tilde{M} \quad -\tilde{N}]f = 0$. Hence, $f_1 = \tilde{M}^{-1}\tilde{N}f_2 = Gf_2$, but this shows that $f \in \mathcal{G}(G)$. The result now follows by observing that $\tilde{E}_G$ has a closed range. The first expression of the second set of identities follows immediately from the definitions and the previous results by recalling that different coprime factorizations of the same transfer function are related by a pre- or post-multiplication by a function that is invertible in $\mathcal{H}_\infty$. The second expression is proved in the same way as the analogous expression above. □

We now give explicit characterizations of the projections onto the various graph spaces. These characterizations involve Toeplitz operators whose symbols are the coprime factors of the transfer functions. These results are generalizations of the results on the unweighted case which are due to Zhu (1989).

Theorem 4.4. Given the definitions in Definition 4.2, E_G, $\tilde{E}_G$, F_G and $\tilde{F}_G$ are isometric operators satisfying

$$P_{\mathcal{G}(G)} = E_G E_G^* = T_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} T_{[N^* \quad M^*]W}$$

$$P_{[\mathcal{G}(G)]^\perp} = \tilde{E}_G \tilde{E}_G^* = T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \times \left[T_{[\tilde{M} \quad -\tilde{N}]} T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right]^{-1} T_{[\tilde{M} \quad -\tilde{N}]}$$

$$P_{\mathcal{G}^T(K)} = \tilde{F}_K \tilde{F}_K^* = T_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} T_{[V^* \quad U^*]W}$$

$$P_{[\mathcal{G}^T(K)]^\perp} = F_K F_K^* = T_{W^{-1}\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]} \times \left[T_{[-\tilde{U} \quad \tilde{V}]} T_{W^{-1}\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]} \right]^{-1} T_{[-\tilde{U} \quad \tilde{V}]}.$$

Proof. First, the adjoints of the operators E_G, $\tilde{E}_G$ are calculated. For $f \in \mathcal{H}_2^{m,l}$, $g \in \mathcal{H}$

$$\begin{aligned} [E_G f, g] &= \left\langle \begin{bmatrix} N \\ M \end{bmatrix} f, Wg \right\rangle = \langle f, [N^* \quad M^*] Wg \rangle \\ &= \langle f, T_{[N^* \quad M^*]W} g \rangle = \langle f, E_G^* g \rangle \end{aligned}$$

and therefore $E_G^* = T_{[N^* \quad M^*]W}$. To calculate the adjoint of the operator $\tilde{E}_G$ let $f \in \mathcal{H}_2^{p,l}$, $g \in \mathcal{H}$. Then

$$\begin{aligned} [\tilde{E}_G f, g] &= \left\langle R^{-1} P_+ R W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} Z_G^{-1/2} f, Wg \right\rangle \\ &= \langle Z_G^{-1/2} f, [\tilde{M} \quad -\tilde{N}] g \rangle \\ &= \langle f, Z_G^{-1/2} T_{[\tilde{M} \quad -\tilde{N}]} g \rangle \\ &= \langle f, \tilde{E}_G g \rangle \end{aligned}$$

and therefore $\tilde{E}_G^* = (Z_G^*)^{-1/2} T_{[\tilde{M} \quad -\tilde{N}]}$. It can be shown that the operators are isometries as for $f \in \mathcal{H}_2^{m,l}$

$$\begin{aligned} E_G^* E_G f &= T_{[N^* \quad M^*]W} T_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} f \\ &= P_+ [N^* \quad M^*] W \begin{bmatrix} N \\ M \end{bmatrix} f = f \end{aligned}$$

as $\begin{bmatrix} N \\ M \end{bmatrix}$ is W-inner. Since Z_G is invertible and therefore bijective, $\tilde{E}_G$ is clearly an isometry by construction.

By Lemma 4.3 $\text{Range}\,(E_G) = \mathcal{G}(G)$ and $\text{Range}\,(\tilde{E}_G) = [\mathcal{G}(G)]^\perp$. Therefore, the operator $E := [E_G \quad \tilde{E}_G]$ is a unitary operator from $\begin{bmatrix} \mathcal{H}_2^{m,l} \\ \mathcal{H}_2^{p,l} \end{bmatrix}$ to $\mathcal{H}$.

Consider the self-adjoint operator $E_G E_G^*$. It follows from Lemma 4.3 that $\text{Range}\,(E_G E_G^*) = \mathcal{G}(G)$ and $\mathcal{K}(E_G E_G^*) = [\mathcal{G}(G)]^\perp$. Therefore $E_G E_G^* = P_{\mathcal{G}(G)}$ (see e.g. Weidmann, 1980, p. 82). Similarly, we obtain $\tilde{E}_G \tilde{E}_G^* = P_{[\mathcal{G}(G)]^\perp}$.

The results for F_K, $\tilde{F}_K$ can be proved analogously. □

5. GEOMETRIC CHARACTERIZATION OF CLOSED-LOOP STABILITY

We are now going to discuss how the stability of control systems can be phrased in terms of the geometric notions that were introduced in Section 2.

We will need the following lemma.

Lemma 5.1 (*Nikolskii*, 1986, *p*. 201). Let H be a Hilbert space and let A, B be closed subspaces of H. Denote the orthogonal projection operator onto the space A by $P_A: H \to A$. Use analogous notation for the similar operations onto the subspace B. Then the following statements are equivalent: (i) $P_{B^\perp}A = B^\perp$; (ii) $H = A + B$; (iii) $\|P_{A^\perp}P_{B^\perp}\| < 1$. Also the following statements are equivalent:
(i) $P_{B^\perp}A = B^\perp$, $A \cap B = \{0\}$; (ii) $H = A + B$, $H = A^\perp + B^\perp$; (iii) $\|P_{A^\perp}P_{B^\perp}\| < 1$, $\|P_A P_B\| < 1$.

Given a plant G with right and left coprime factorizations $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and a controller K with right and left coprime factorizations $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$, there are well-known criteria for a controller K to internally stabilize a plant G (see e.g. Vidyasagar, 1985). By an internally stable control system (G, K) we mean the pair (G, K) of a plant G and an internally stabilizing controller K.

Necessary and sufficient conditions for the control system (G, K) to be internally stable, i.e.

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-GK)^{-1} & -(I-GK)^{-1}G \\ -K(I-GK)^{-1} & (I-KG)^{-1} \end{bmatrix} \in \mathcal{H}_\infty^{(p+m)\times(p+m)},$$

are the invertibility of $\begin{bmatrix} V & N \\ U & M \end{bmatrix}$, the invertibility of $\tilde{V}M - \tilde{U}N$ or the invertibility of $\tilde{M}V - \tilde{N}U$ in $\mathcal{H}_\infty$ (see e.g. Vidyasagar, 1985).

We are now going to show how stability criteria for control systems can be stated in terms of geometric notions in the Hilbert space $\mathcal{H} := \begin{bmatrix} \mathcal{H}_2^{p,W_o} \\ \mathcal{H}_2^{m,W_i} \end{bmatrix}$. These are generalizations of the results of the unweighted case (Ober and Sefton, 1990, 1991) to the case of weighted spaces. A further equivalent condition in the unweighted case was given by Foias *et al.* (1990).

Theorem 5.2. Let $\mathcal{H}_2^{p,W_o}$ and $\mathcal{H}_2^{m,W_i}$ be the output and input space, respectively, of the $p \times m$ transfer function G. Let K be a $m \times p$ transfer function and denote by $\mathcal{H}$ the space

$$\mathcal{H} := \begin{bmatrix} \mathcal{H}_2^{p,W_o} \\ \mathcal{H}_2^{m,W_i} \end{bmatrix}.$$

For a closed subspace $\mathcal{A} \subseteq \mathcal{H}$ let $P_\mathcal{A}$ denote the orthogonal projection onto $\mathcal{A}$. The following statements are equivalent:

(S0) The pair (G, K) is internally stable.
(S1) $\mathcal{G}(G) + \mathcal{G}^T(K) = \mathcal{H}$.
(S2) $P_{[\mathcal{G}^T(K)]^\perp}\mathcal{G}(G) = [\mathcal{G}^T(K)]^\perp$.
(S3) $\|P_{[\mathcal{G}(G)]^\perp}P_{[\mathcal{G}^T(K)]^\perp}\| < 1$.
(S4) $\theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp) > 0$.
(S5) $[\mathcal{G}(G)]^\perp \cap [\mathcal{G}^T(K)]^\perp = \emptyset$ and $P_{[\mathcal{G}(G)]^\perp \| [\mathcal{G}^T(K)]^\perp}$ is bounded.
(S6) $\text{gap}\,(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) < 1$.

Proof. (S0) and (S1) are equivalent: let $G = NM^{-1}$ be a right coprime factorization of G and $K = UV^{-1}$ a right coprime factorization of K. Then by a standard result (see e.g. Vidyasagar, 1985) (G, K) is internally stable if and only if $\begin{pmatrix} N & V \\ M & U \end{pmatrix}$ is invertible in $\mathcal{H}_\infty^{(p+m)\times(p+m)}$. Note that

$$\mathcal{G}(G) + \mathcal{G}^T(K) = \begin{pmatrix} N & V \\ M & U \end{pmatrix} \begin{pmatrix} \mathcal{H}_2^{m,I} \\ \mathcal{H}_2^{p,I} \end{pmatrix}.$$

If (S0) holds then the resulting invertibility of $\begin{pmatrix} N & V \\ M & U \end{pmatrix}$ implies (S1). If conversely (S1) holds then

$$\begin{pmatrix} \mathcal{H}_2^{p,I} \\ \mathcal{H}_2^{m,I} \end{pmatrix} = \mathcal{H} = \mathcal{G}(G) + \mathcal{G}^T(K) = \begin{pmatrix} N & V \\ M & U \end{pmatrix} \begin{pmatrix} \mathcal{H}_2^{m,I} \\ \mathcal{H}_2^{p,I} \end{pmatrix}.$$

However, this shows that $\begin{pmatrix} N & V \\ M & U \end{pmatrix}$ is invertible in $\mathcal{H}_\infty^{(p+m)\times(p+m)}$ (see e.g. Francis, 1987), which implies (S0).

The equivalence of (S1), (S2) and (S3) follows from Lemma 5.1. The equivalence of (S3) and (S4) follows from the fact that for two closed subspaces A, B of a Hilbert space H we have $\cos\theta_{\min}(A, B) = \|P_A P_B\|$. (S4) is equivalent to (S5): if (S4) holds then $[\mathcal{G}(G)]^\perp \cap [\mathcal{G}^T(K)]^\perp = \emptyset$. Therefore, the skew projection $P_{[\mathcal{G}(G)]^\perp \| [\mathcal{G}^T(K)]^\perp}$ is well defined. (S5) is a now a consequence of the fact that for two closed subspaces A, B of a Hilbert space H we have $\sin\theta_{\min}(A, B) = \|P_{A\|B}\|^{-1}$. That (S5) implies (S4) follows from the same identity and the fact that $P_{A\|B}$ is bounded if and only if $\theta_{\min}(A, B) > 0$. (S3) is equivalent to (S6): in order to show that (S3) implies (S6) we have to show that $\|P_{\mathcal{G}(G)}P_{\mathcal{G}^T(K)}\| < 1$ since $\text{gap}\,(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) = \max\{\|P_{\mathcal{G}(G)}P_{\mathcal{G}^T(K)}\|, \|P_{[\mathcal{G}(G)]^\perp}P_{[\mathcal{G}^T(K)]^\perp}\|\}$. Showing this amounts by Lemma 5.1 to showing that $P_{\mathcal{G}^T(K)}[\mathcal{G}(G)]^\perp = \mathcal{G}^T(K)$. Let $K = UV^{-1}$ be a W-t-normalized coprime factorization of K. Since the assumption (S3) implies internal stability of (G, K) there exists a left coprime factorization $G = \tilde{M}^{-1}\tilde{N}$ such that $\tilde{M}V - \tilde{N}U = I$

(see e.g. Vidyasagar, 1985). Note that

$$P_{\mathcal{G}^T(K)}[\mathcal{G}(G)]^\perp = T_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} T_{[V^* \quad U^*]W} T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N} \end{smallmatrix}\right]} \mathcal{H}_2^{p,l}.$$

For $f \in \mathcal{H}_2^{p,l}$ we have that

$$\begin{aligned} T_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} T_{[V^* \quad U^*]W} T_{W^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N} \end{smallmatrix}\right]} f &= \begin{bmatrix} V \\ U \end{bmatrix} P_+ [V^* \quad U^*] W_R^* P_+ W_R^{-*} \begin{bmatrix} \tilde{M}^* \\ -N^* \end{bmatrix} f \\ &= \begin{bmatrix} V \\ U \end{bmatrix} P_+ [V^* \quad U^*] \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} f \\ &= \begin{bmatrix} V \\ U \end{bmatrix} f. \end{aligned}$$

This shows the claim that $P_{\mathcal{G}^T(K)}[\mathcal{G}(G)]^\perp = \mathcal{G}^T(K)$. By Lemma 5.1 we therefore have that $\|P_{\mathcal{G}(G)} P_{\mathcal{G}^T(K)}\| < 1$. Hence, $\operatorname{gap}(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) < 1$. That (S6) implies (S3) follows by definition. □

In general we have for closed subspaces A, B of a Hilbert space that $\operatorname{gap}(A, B^\perp) = \max\{\|P_A P_B\|, \|P_{A^\perp} P_{B^\perp}\|\}$. The proof of the previous theorem shows however that if the subspaces are graph spaces then the gap can be expressed in terms of just one of these expressions. This is summarized in the following corollary.

Corollary 5.3. We have that

$$\begin{aligned} &\cos \theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp) \\ &\quad = \|P_{[\mathcal{G}(G)]^\perp} P_{[\mathcal{G}^T(K)]^\perp}\| = \operatorname{gap}(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp). \end{aligned}$$

6. $\mathcal{H}_\infty$-OPTIMIZATION

In this section, it is shown how the various quantities that are of importance in this paper can be calculated in terms of $\mathcal{H}_\infty$ problems. The approach taken is to relate the weighted problems to unweighted problems. Then, the results can be applied that are already available for the unweighted problem.

The basic connection between weighted and unweighted problems follows from the following lemma. It is explained here how weighted normalized copri ne factorizations are related to unweighted normalized coprime factorizations.

Lemma 6.1. Given a weight

$$W = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^* W_i \end{bmatrix} \in \mathcal{W}^{p+m},$$

with W_o, W_o^{-1}, W_i, $W_i^{-1} \in \mathcal{H}_\infty$, a $p \times m$ plant G and an $m \times p$ controller K, then let (N, M) be a W-NRCF and $(\tilde{N}, \tilde{M})$ a W-NLCF of G. Let (U, V) be a W-T-NRCF and $(\tilde{U}, \tilde{V})$ be W-T-NLCF of K. Then

(1) $(W_o N, W_i M)$ and $(\tilde{M} W_o^{-1}, \tilde{N} W_i^{-1})$ are an I-NRCF and an I-NLCF, respectively, of $W_o G W_i^{-1}$.
(2) $(W_i U, W_o V)$ and $(\tilde{U} W_o^{-1}, \tilde{V} W_i^{-1})$ are an I-NRCF and an I-NLCF, respectively, of $W_i K W_o^{-1}$.

Proof. As $\begin{bmatrix} N \\ M \end{bmatrix}$ is W-inner we have

$$\begin{aligned} I &= [N^* \quad M^*] W \begin{bmatrix} W \\ M \end{bmatrix} = [N^* \quad M^*] W_R^* W_R \begin{bmatrix} N \\ M \end{bmatrix} \\ &= \begin{bmatrix} W_o N \\ W_i M \end{bmatrix}^* \begin{bmatrix} W_o N \\ W_i M \end{bmatrix}. \end{aligned}$$

Hence, $(W_o N, W_i M)$ is I-normalized; it is coprime as (N, M) is coprime and W_i, W_o are invertible in H_∞. We also obtain $(W_o N)(W_i M)^{-1} = W_o G W_i^{-1}$. The other identities are proved analogously. □

The previous lemma suggests that the analysis of the weighted problem for the plant and controller (G, K) is closely related to the analysis of the unweighted problem for the weighted plant and weighted controller $(W_o G W_i^{-1}, W_i K W_o^{-1})$. This is further confirmed by the following proposition where the projection operators onto the various graph spaces are related.

Proposition 6.2. Given a weight $W = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^* W_i \end{bmatrix} \in \mathcal{W}^{p+m}$, with W_o, W_o^{-1}, W_i, $W_i^{-1} \in H_\infty$, an $p \times m$ system G and a $m \times p$ controller K, then

(1) $P^W_{\mathcal{G}(G)} = R_+^{-1} P^I_{\mathcal{G}(W_o G W_i^{-1})} R_+$
(2) $P^W_{[\mathcal{G}(G)]^\perp} = R_+^{-1} P^I_{[\mathcal{G}(W_o G W_i^{-1})]^\perp} R_+$
(3) $P^W_{[\mathcal{G}^T(K)]^\perp} = R_+^{-1} P^I_{[\mathcal{G}^T(W_i K W_o^{-1})]^\perp} R_+$
(4) $P^W_{\mathcal{G}^T(K)} = R_+^{-1} P^I_{\mathcal{G}^T(W_i K W_o^{-1})} R_+$.

Proof. To prove (1) let $f \in \mathcal{H}$; then, with $G = NM^{-1}$ a W-NRCF of G,

$$\begin{aligned} P^W_{\mathcal{G}(G)} f &= \begin{bmatrix} N \\ M \end{bmatrix} P_+ [N^* \quad M^*] W f \\ &= W_R^{-1} W_R \begin{bmatrix} N \\ M \end{bmatrix} P_+ [N^* \quad M^*] W_R^* W_R f \\ &= R_+^{-1} P^I_{\mathcal{G}(W_o G W_i^{-1})} R_+ f. \end{aligned}$$

Here we used $W_R \begin{bmatrix} N \\ M \end{bmatrix}$ is an I-NRCF of $W_o G W_i^{-1}$. The identity (2) follows since for

$f \in \mathcal{H}$

$$P^W_{[\mathcal{G}(G)]^\perp} f = W_R^{-1} P_+ W_R^{-*} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N} \end{bmatrix} \Big[P_+ [\tilde{M} \quad -\tilde{N}] \times W_R^{-1} P_+ W_R^{-*} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N} \end{bmatrix} \Big]^{-1} \times P_+ [\tilde{M} \quad -\tilde{N}] W_R^{-1} W_R f = R_+^{-1} P^I_{[\mathcal{G}(W_o G W_i^{-1})]^\perp} R_+ f.$$

The remaining two identities are proved analogously. □

The gap between the graphs of two plants G and G_Δ is a well-known measure of the distance between the plants and has been used to study robustness properties of control systems (Zames and El-Sakkary, 1980; El-Sakkary, 1985; Zhu, 1989; Georgiou and Smith, 1990a; Sefton and Ober, 1993). We are now going to study the gap between two plants also in a weighted setting.

The weighted gap metric between two $p \times m$ systems, G and G_Δ, defined on a weighted Hilbert space is defined by

$$\delta^W(G, G_\Delta) := \text{gap}\,(\mathcal{G}(G), \mathcal{G}(G_\Delta)) := \|P^W_{\mathcal{G}(G)} - P^W_{\mathcal{G}(G_\Delta)}\|$$

where $W = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^* W_i \end{bmatrix} \in \mathcal{W}^{p \times m}$. As previously the orthogonal projection $P^W_{\mathcal{G}(G)}$ is the orthogonal projection onto the graph space of G in the weighted space $\mathcal{H} := \begin{bmatrix} \mathcal{H}_2^{p, W_o} \\ \mathcal{H}_2^{m, W_i} \end{bmatrix}$. It will be shown in fact that

$$\delta^W(G, G_\Delta) = \delta^I(W_o G W_i^{-1}, W_o G_\Delta W_i^{-1}).$$

Therefore, the weighted gap between two systems can be calculated from the unweighted gap between two shaped systems.

In the following proposition it is shown how the weighted gap can be calculated by solving two weighted $\mathcal{H}_\infty$-optimization problems. Similar results are given for the gap between the graph of the plant and the orthogonal complement of the transposed graph of the controller. In order to indicate whether the gap is calculated with respect to the weight W or the weight I, the superscript W or I, respectively, is used.

Proposition 6.3. Given a weight

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \in \mathcal{W}^{p+m},$$

a $p \times m$ system G and an $m \times p$ controller K, let (N, M) be a W-NRCF and $(\tilde{N}, \tilde{M})$ a W-NLCF of G.

(1) Let (U, V) be a W-T-NRCF and $(\tilde{U}, \tilde{V})$ be W-T-NLCF of K. Then

$$\|P^W_{[\mathcal{G}(G)]^\perp} P^W_{[\mathcal{G}^T(K)]^\perp}\| = \inf_{Q \in \mathcal{H}_\infty^{g \times p}} \left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right] \right\|_\infty$$

$$\|P^W_{\mathcal{G}(G)} P^W_{\mathcal{G}^T(K)}\| = \inf_{Q \in \mathcal{H}_\infty^{g \times p}} \left\| W_R \left[\begin{bmatrix} N \\ M \end{bmatrix} - W^{-1} \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} Q \right] \right\|_\infty$$

and hence

$$\text{gap}^W(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) = \text{gap}^I(\mathcal{G}(W_o G W_i^{-1}), [\mathcal{G}^T(W_i K W_o^{-1})]^\perp) = \inf_{Q \in \mathcal{H}_\infty^{g \times p}} \left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right] \right\|_\infty.$$

(2) Let $(\tilde{N}_\Delta, \tilde{M}_\Delta)$ be a W-NLCF and $\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix}$ a W-NRCF of G_Δ. Then

$$\|P^W_{[\mathcal{G}(G_\Delta)]^\perp} P^W_{\mathcal{G}(G)}\| = \inf_{Q \in \mathcal{H}_\infty} \left\| W_R \left(\begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix} Q \right) \right\|_\infty$$

$$\left\|P^W_{\mathcal{G}(G_\Delta)} P^W_{[\mathcal{G}(G)]^\perp}\right\| = \inf_{Q \in \mathcal{H}_\infty} \left\| W_R \left(\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right) \right\|_\infty$$

and hence

$$\delta^W(G, G_\Delta) = \delta^I(W_o G W_i^{-1}, W_o G_\Delta W_i^{-1}) = \max \left\{ \inf_{Q \in \mathcal{H}_\infty} \left\| W_R \left(\begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix} Q \right) \right\|_\infty, \inf_{Q \in \mathcal{H}_\infty} \left\| W_R \left(\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right) \right\|_\infty \right\}.$$

Proof. Part (1). It follows from Proposition 6.2 that

$$\|P^W_{[\mathcal{G}(G)]^\perp} P^W_{[\mathcal{G}^T(K)]^\perp}\| = \|P^I_{[\mathcal{G}(W_o G W_i^{-1})]^\perp} P^I_{[\mathcal{G}^T(W_i K W_o^{-1})]^\perp}\|.$$

Now by Proposition 6.3 (Ober and Sefton, 1991)

$$\|P^I_{[\mathcal{G}(W_o G W_i^{-1})]^\perp} P^I_{[\mathcal{G}^T(W_i K W_o^{-1})]^\perp}\| = \inf_{Q \in \mathcal{H}_\infty^{g \times p}} \left\| W_R^{-*} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - W_R \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \inf_{Q \in \mathcal{H}_\infty^{g \times p}} \left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right] \right\|_\infty$$

where $(\tilde{M} W_i^{-1}, \tilde{N} W_o^{-1})$ and $(W_i U, W_o V)$ are I-NLCF and I-NRCF of the plant $W_o G W_i^{-1}$ and the controller $W_i K W_o^{-1}$, respectively, with $W = \text{diag}\,(W_o^* W_o, W_i^* W_i)$, W_o, W_o^{-1}, W_i, $W_i^{-1} \in \mathcal{H}_\infty$. The second expression follows identically.

To prove part (2) of the proposition note, by Proposition 6.2

$$\|P^W_{[\mathcal{G}(G_\Delta)]^\perp} P^W_{\mathcal{G}(G)}\| = \|P^I_{[\mathcal{G}(W_o G_\Delta W_i^{-1})]^\perp} P^I_{\mathcal{G}(W_o G W_i^{-1})}\|.$$

Since $(W_o N, W_i M)$ and $(W_o N_\Delta, W_i M_\Delta)$ are

I-NRCFs of G and G_Δ, respectively, we have (see Georgiou, 1988)

$$\|P^I_{[\mathscr{G}(W_oG_\Delta W_i^{-1})]^\perp}P^I_{\mathscr{G}(W_oGW_i^{-1})}\| = \inf_{Q\in\mathscr{H}_\infty}\left\|W_R\begin{bmatrix}N\\M\end{bmatrix} - W_R\begin{bmatrix}N_\Delta\\M_\Delta\end{bmatrix}Q\right\|_\infty,$$

which proves the first identity. The second identity follows similarly. To complete the proof note that

$$\begin{aligned}\delta^W(G, G_\Delta) &= \max\{\|P^W_{[\mathscr{G}(G_\Delta)]^\perp}P^W_{\mathscr{G}(G)}\|, \|P^W_{[\mathscr{G}(G)]^\perp}P^W_{\mathscr{G}(G_\Delta)}\|\}\\ &= \max\{\|P^I_{[\mathscr{G}(W_oG_\Delta W_i^{-1})]^\perp}P^I_{\mathscr{G}(W_oGW_i^{-1})}\|, \|P^I_{[\mathscr{G}(W_oGW_i^{-1})]^\perp}P^I_{\mathscr{G}(W_oG_\Delta W_i^{-1})}\|\}.\end{aligned}$$

□

In case the controller stabilizes the plant, the above expressions reduce to an evaluation of the $\mathscr{L}_\infty$-norm of a transfer function. We will need the following Lemma.

Lemma 6.4. Given a weight $W = \begin{bmatrix}W_1 & 0\\ 0 & W_2\end{bmatrix} \in \mathscr{W}^{p+m}$, a $p\times m$ system G and an $m\times p$ controller K, let (N, M) be a W-NRCF and $(\tilde{N}, \tilde{M})$ a W-NLCF of G. Let (U, V) be a W-T-NRCF and $(\tilde{U}, \tilde{V})$ a W-T-NLCF of K. Then

$$(NW_1V^* + MW_2U^*)^*(NW_1V^* + MW_2U^*) + (\tilde{V}M - \tilde{U}N)^*(\tilde{V}M - \tilde{U}N) = I$$

and

$$(\tilde{M}V - \tilde{N}U)(\tilde{M}V - \tilde{N}U)^* + (\tilde{M}W_1^{-1}\tilde{U}^* + \tilde{N}W_2^{-1}\tilde{V}^*) \times (\tilde{M}W_1^{-1}\tilde{U}^* + \tilde{N}W_2^{-1}\tilde{V}^*)^* = I.$$

Proof. Note that the following product of two all-pass matrices is all-pass:

$$Y = \begin{bmatrix}[N^* \quad M^*]W_R^*\\ [\tilde{M} \quad -N]W_R^{-1}\end{bmatrix}\begin{bmatrix}W_R\begin{bmatrix}V\\U\end{bmatrix} & (W_R^{-1})^*\begin{bmatrix}-\tilde{U}^*\\ \tilde{V}^*\end{bmatrix}\end{bmatrix} = \begin{bmatrix}[N^* \quad M^*]W\begin{bmatrix}V\\U\end{bmatrix} & (\tilde{V}M - \tilde{U}N)^*\\ (\tilde{M}V - \tilde{N}U) & -[\tilde{M} \quad \tilde{N}]W^{-1}\begin{bmatrix}\tilde{U}^*\\ \tilde{V}^*\end{bmatrix}\end{bmatrix}.$$

The result follows by considering the diagonal components of the identity $YY^* = I$. □

Proposition 6.5. Given a weight $W = \begin{bmatrix}W_1 & 0\\ 0 & W_2\end{bmatrix} \in \mathscr{W}^{p+m}$, a $p\times m$ system G and an $m\times p$ controller K, let (N, M) be a W-NRCF and $(\tilde{N}, \tilde{M})$ a W-NLCF of G. Let (U, V) be a W-T-NRCF and $(\tilde{U}, \tilde{V})$ a W-T-NLCF of K. Assume that (G, K) is internally stable, then

(1) $$\cos\theta^W_{\min}([\mathscr{G}(G)]^\perp, [\mathscr{G}(K)]^\perp) = \text{gap}^W(\mathscr{G}(G), [\mathscr{G}(K)]^\perp) = \left\|[N^* \quad M^*]W\begin{bmatrix}V\\U\end{bmatrix}\right\|_\infty = \left\|[\tilde{V} \quad \tilde{U}]W\begin{bmatrix}\tilde{N}^*\\ \tilde{M}^*\end{bmatrix}\right\|_\infty.$$

(2) $$\sin\theta^W_{\min}([\mathscr{G}(G)]^\perp, [\mathscr{G}(K)]^\perp) = \tau(\tilde{M}V - \tilde{N}U) = \tau(\tilde{V}M - \tilde{U}N),$$

where $\tau(F) = \text{ess inf}\{\sigma_{\min}(F(s)) \mid \text{Re}(s) = 0\}$.

Proof. That $\cos\theta^W_{\min}([\mathscr{G}(G)]^\perp, [\mathscr{G}(K)]^\perp) = \text{gap}^W(\mathscr{G}(G), [\mathscr{G}(K)]^\perp)$ was proved in Corollary 5.3. We know that

$$\text{gap}^W(\mathscr{G}(G), [\mathscr{G}(K)]^\perp) = \text{gap}^I(\mathscr{G}(W_oGW_i^{-1}), [\mathscr{G}(W_iKW_o^{-1})]^\perp),$$

where $W = \text{diag}(W_o^*W_o, W_i^*W_i)$, with W_o, W_o^{-1}, W_i, $W_i^{-1} \in H_\infty$. Using the fact that $[W_oN \quad W_iM]$ ($[\tilde{M}W_o^{-1} \quad \tilde{N}W_i^{-1}]$) is an I-NRCF (I-NLCF) of $W_oGW_i^{-1}$ and that $[W_iU \quad W_oV]$ ($[\tilde{U}W_o^{-1} \quad \tilde{V}W_i^{-1}]$) is an I-T-NRCF (I-T-NLCF) of $W_iKW_o^{-1}$, we can apply the results by Ober and Sefton (1991) to obtain

$$\begin{aligned}\text{gap}^I(\mathscr{G}(W_oGW_i^{-1}), [\mathscr{G}(W_iKW_o^{-1})]^\perp) &= \|(W_oN)^*(W_oV) + (W_iM)^*(W_iU)\|_\infty\\ &= \left\|[N^* \quad M^*]W\begin{bmatrix}V\\U\end{bmatrix}\right\|_\infty\\ &= \|(\tilde{U}W_i)(\tilde{M}W_i)^* + (\tilde{V}W_o)(\tilde{N}W_o)^*\|_\infty\\ &= \left\|[\tilde{V} \quad \tilde{U}]W\begin{bmatrix}\tilde{N}^*\\ \tilde{M}^*\end{bmatrix}\right\|_\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\sin\theta^W_{\min}([\mathscr{G}(G)]^\perp, [\mathscr{G}(K)]^\perp) &= 1 - [\cos\theta^W_{\min}([\mathscr{G}(G)]^\perp, [\mathscr{G}(K)]^\perp)]^2\\ &= \tau(\tilde{M}V - \tilde{N}U) = \tau(\tilde{V}M - \tilde{U}N),\end{aligned}$$

by Lemma 6.4. □

7. ROBUST CONTROL

We are now in a position to consider the question of robust stabilization in our framework. It has turned out that the minimum angle between the orthogonal complement of the graph space of the plant and the orthogonal complement of the transposed graph space of the controller is an important quantity in the analysis of a control system. The control system is stable if and only if this minimum angle is positive. It therefore appears natural to ask the question whether or not is it possible to find a controller that is maximally stabilizing in the sense that it maximizes this minimum angle. The first objective of this section is to prove that such a maximally stabilizing controller exists. Analysing this controller in this and the subsequent sections the name maximally stabilizing can indeed be justified. It will also become clear that the concept of the maximally stabilizing controller is very closely related to the optimally robust controller for normalized coprime factor uncertainty as studied by McFarlane and Glover (1989).

The maximally stabilizing controller is defined as follows.

Definition 7.1. Given a $p \times m$ system G and a weight $W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \in \mathcal{W}$, the optimal minimal angle $\theta_{\min}^{\text{opt}}$ with respect to the weight W is defined by

$$\cos \theta_{\min}^{\text{opt}} := \inf_K \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp).$$

Further, a controller, K, achieving this infimum is called a maximally stabilizing controller with respect to the weight W.

It is now possible to give an analytical expression for the optimal minimal angle and to calculate a maximally stabilizing controller with respect to a weight W.

Theorem 7.2. Given a weight

$$W = \begin{bmatrix} W_o^* W_o & 0 \\ 0 & W_i^* W_i \end{bmatrix} \in \mathcal{W}^{p+m},$$

a $p \times m$ system G and let $G = \tilde{M}^{-1}\tilde{N}$ be a W-NLCF, then the optimal minimal angle with respect to the weight W is

$$\cos \theta_{\min}^{\text{opt}} := \inf_K \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) = \sigma_1^W,$$

where $\sigma_1^W := \left\| H^{W_R^{-1}}_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right\| = \left\| H^I_{W_R^{-1}\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right\|$. Any maximally stabilizing controller satisfying the infimum above has a right coprime factorization (U, V) satisfying the extension

$$\left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right] \right\|_\infty = \sigma_1^W.$$

Conversely, if $U, V \in \mathcal{H}_\infty$ are such that

$$\left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right] \right\|_\infty = \sigma_1^W,$$

then $K = UV^{-1}$ is a W-maximally stabilizing controller of G.

Proof. It was shown in Proposition 6.3 that

$$\operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) = \operatorname{gap}^I (\mathcal{G}(W_o G W_i^{-1}), [\mathcal{G}^T(W_i K W_o^{-1})]^\perp)$$

and therefore

$$\inf_K \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) = \inf_K \operatorname{gap}^I (\mathcal{G}(W_o G W_i^{-1}), [\mathcal{G}^T(W_i K W_o^{-1})]^\perp).$$

Ober and Sefton (1991) solved the unweighted problem. In particular it was shown that

$$\inf_K \operatorname{gap}^I (\mathcal{G}(G_I), [\mathcal{G}^T(K)]^\perp) = \|H^I_{\left[\begin{smallmatrix} \tilde{M}_I^* \\ -\tilde{N}_I^* \end{smallmatrix}\right]}\|$$

with $G_I = \tilde{M}_I^{-1}\tilde{N}_I$ an I-NLCF of G_I. Also an I-maximally stable controller exists for G_I and each I-maximally stable controller K_I has a right coprime factorization $[U_I \quad V_I]$ that satisfies

$$\inf_{U,V \in \mathcal{H}_\infty} \left\| \begin{bmatrix} \tilde{M}_I^* \\ -\tilde{N}_I^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{M}_I^* \\ -\tilde{N}_I^* \end{bmatrix} - \begin{bmatrix} V_I \\ U_I \end{bmatrix} \right\|_\infty = \|H^I_{\left[\begin{smallmatrix} \tilde{M}_I^* \\ -\tilde{N}_I^* \end{smallmatrix}\right]}\|.$$

Set $G_I := W_o G W_i^{-1}$, where $W = \operatorname{diag}(W_o^* W_o, W_i^* W_i)$, with $W_o, W_o^{-1}, W_i, W_i^{-1} \in H_\infty$, and note that $[\tilde{N} W_i^{-1} \quad \tilde{M} W_o^{-1}]$ is an I-NLCF of G_I, then

$$\begin{aligned}
\inf_K \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) &= \inf_K \operatorname{gap}^I (\mathcal{G}(W_o G W_i^{-1}), [\mathcal{G}^T(W_i K W_o^{-1})]^\perp) \\
&= \inf_K \operatorname{gap}^I (\mathcal{G}(G_I), [\mathcal{G}^T(K)]^\perp) \\
&= \operatorname{gap}^I (\mathcal{G}(G_I), [\mathcal{G}^T(K_I)]^\perp) \\
&= \inf_{U,V \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_o^{-*}\tilde{M}^* \\ -W_i^{-*}\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty \\
&= \left\| \begin{bmatrix} W_o^{-*}\tilde{M}^* \\ -W_i^{-*}\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V_I \\ U_I \end{bmatrix} \right\|_\infty \\
&= \|H^I_{\left[\begin{smallmatrix} W_o^{-*}\tilde{M}^* \\ -W_i^{-*}\tilde{N}^* \end{smallmatrix}\right]}\| = \|H^{W_R^{-1}}_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\|.
\end{aligned}$$

Therefore, with $K_o = W_i^{-1} K_I W_o$ where $K_I = U_I V_I^{-1}$ we have that

$$\begin{aligned}
\inf_K \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) &= \operatorname{gap}^I (\mathcal{G}(G_I), [\mathcal{G}^T(K_I)]^\perp) \\
&= \operatorname{gap}^W (\mathcal{G}(G), [\mathcal{G}^T(K_o)]^\perp).
\end{aligned}$$

The remaining statements of the theorem are proved analogously by relating the weighted problem to the unweighted problem and by then applying the solution to the unweighted problem (Ober and Sefton, 1991). □

Specializing the above result to the unweighted case, i.e. $W = I$, it is seen that the maximally stabilizing controller is identical to the optimally robust controller with respect to coprime factor uncertainty as studied by McFarlane and Glover, 1989. The above result also gives an interpretation of the loop-shaping procedure of McFarlane and Glover, 1989. It shows that designing an optimal controller by their loop-shaping procedure is equivalent to designing a maximally stable controller on a weighted space. Their procedure includes a design methodology for choosing these weights to achieve certain closed-loop performance objectives.

8. COPRIME FACTOR PERTURBATIONS

The purpose of this section is to give the maximally stabilizing controller an interpretation from the point of view of allowable coprime factor perturbations. While deriving this interpretation we will also obtain further properties of the maximally stabilizing controller. These properties are important points of motivation for the questions that are being considered in the subsequent sections. From now on all the analysis will be restricted to the case of scalar systems.

We first need to analyse further the maximally stabilizing controller. The following theorem states an interesting property of the Schmidt vectors of a Hankel operator with symbol $[\tilde{m} \quad -\tilde{n}]^*$.

Theorem 8.1. Let $g_I = \tilde{m}_I^{-1}\tilde{n}_I$ be an I-NLCF of the scalar transfer function g_I. Let σ_i be the ith singular value of the Hankel operator $H^I_{\left[\begin{smallmatrix}\tilde{m}_I^*\\-\tilde{n}_I^*\end{smallmatrix}\right]}$ with Schmidt pairs (f_i, h_i), $i = 1, \ldots, n$. Let $\begin{bmatrix} v_I \\ u_I \end{bmatrix}$ be the Nehari extension of $\begin{bmatrix} \tilde{m}_I^* \\ -\tilde{n}_I^* \end{bmatrix}$, i.e.

$$\inf_{u,v \in \mathscr{H}_\infty} \left\| \begin{bmatrix} \tilde{m}_I^* \\ -\tilde{n}_I^* \end{bmatrix} - \begin{bmatrix} v \\ u \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{m}_I^* \\ -\tilde{n}_I^* \end{bmatrix} - \begin{bmatrix} v_I \\ u_I \end{bmatrix} \right\|_\infty = \sigma_1.$$

Then

(1) the Schmidt vectors satisfy $[\tilde{m}_I - \tilde{n}_I]f_i = \sigma_i h_i$.
(2) $\tilde{m}_I v_I - \tilde{n}_I u_I = (1 - \sigma_1^2)$.
(3) $u_I^* u_I + v_I^* v_I = (1 - \sigma_1^2)$.
(4) If $r_I^* := m_I^* u_I + n_I^* v_I$ then $r_I^* r_I = \sigma_1^2(1 - \sigma_1^2)$.

Proof. The sub- and superscripts I are dropped for simplicity of presentation. To prove (1) first note that

$$H^*_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} H_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} + T^*_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} T_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} = I$$

and hence for the input Schmidt vector h_i

$$[\tilde{m} \quad -\tilde{n}] T_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} h_i = T^*_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} T_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} h_i = (1 - \sigma_i^2) h_i.$$

Since

$$\begin{bmatrix} \tilde{m}^* \\ -\tilde{n}^* \end{bmatrix} h_i - T_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} h_i = H_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} h_i = \sigma_i f_i.$$

Premultiplying this expression by $[\tilde{m} \quad -\tilde{n}]$ implies that

$$\sigma_i [\tilde{m} \quad -\tilde{n}] f_i = h_i - (1 - \sigma_i^2) h_i = \sigma_i^2 h_i.$$

(2) follows in a straightforward way from the expression $\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \tilde{m}^* \\ -\tilde{n}^* \end{bmatrix} - \sigma_1 \dfrac{f_1}{h_1}$ for the Nehari extension (see Foias and Frazho, 1990) and (1):

$$[\tilde{m} \quad -\tilde{n}] \begin{bmatrix} v \\ u \end{bmatrix} = 1 - \sigma_1 [\tilde{m} \quad -\tilde{n}] \frac{f_1}{h_1} = 1 - \sigma_1^2.$$

(3) is shown by the following calculations:

$$\begin{aligned} [v^* \quad u^*] \begin{bmatrix} v \\ u \end{bmatrix} &= \left([\tilde{m} \quad -\tilde{n}] - \sigma_1 \frac{f_1^*}{h_1^*} \right) \left(\begin{bmatrix} \tilde{m}^* \\ -\tilde{n}^* \end{bmatrix} - \sigma_1 \frac{f_1}{h_1} \right) \\ &= 1 - \sigma_1^2 - \sigma_1^2 + \sigma_1^2 \frac{f_1^* f_2}{h_1^* h_1} = 1 - \sigma_1^2, \end{aligned}$$

as $f_1^* f_1 = h_1^* h_1$ (see Foias and Frazho, 1990).

(4) follows since $Y = \begin{bmatrix} \tilde{m} & -\tilde{n} \\ n^* & m^* \end{bmatrix}$ is square and unitary and therefore

$$\sigma_1^2 = [m^* - v^* \quad -\tilde{n} - u^*] \begin{bmatrix} \tilde{m}^* & n \\ -\tilde{n}^* & m \end{bmatrix} \times \begin{bmatrix} \tilde{m} & -\tilde{n} \\ n^* & m^* \end{bmatrix} \begin{bmatrix} \tilde{m}^* - v \\ -\tilde{n}^* - u \end{bmatrix} = \sigma_1^4 + rr^*,$$

which implies the required result. □

In the following corollary the analogous results are obtained for the case of a W-maximally stabilizing controller.

Corollary 8.2. Let $g = nm^{-1}$ and $g = \tilde{m}^{-1}\tilde{n}$ be a W-NRCF and a W-NLCF, respectively, of the scalar transfer function g. Then the W-maximally stabilizing controller k_0 has a right coprime factorization $k_0 = uv^{-1}$ that has the following properties:

(1) $\left\| W_R \left[W^{-1} \begin{bmatrix} \tilde{m}^* \\ -\tilde{n}^* \end{bmatrix} - \sqrt{1 - (\sigma_1^W)^2} \begin{bmatrix} v \\ u \end{bmatrix} \right] \right\|_\infty = \sigma_1^W$,

(2) $k_0 = uv^{-1}$ is a W-T-NRCF.

(3) $c^* c = (\sigma_1^W)^2$ where $c := [n^* \quad m^*] W \begin{bmatrix} v \\ u \end{bmatrix}$.

(4) $\tilde{m}v - \tilde{n}u = \sqrt{1 - (\sigma_1^W)^2}$

where $\sigma_1^W = \| H^{W^{-1}}_{\left[\begin{smallmatrix}\tilde{m}^*\\-\tilde{n}^*\end{smallmatrix}\right]} \|$.

Proof. The proof follows in the usual way from the results for the I-normalized case in the previous theorem and by suitably normalizing the coprime factorization of the controller. □

We are now going to give an interpretation of the maximally stabilizing controller from the point of view of coprime factor perturbations. In Vidyasagar and Kimura (1986) the coprime factor uncertainty model has been introduced. Here, we define a weighted coprime factor uncertainty ball as follows. Let $g = \tilde{m}^{-1}\tilde{n}$ be a W-NLCF of g. Let

$$\mathcal{P}_g^\epsilon = \{(\tilde{m} + \Delta\tilde{m})^{-1}(\tilde{n} + \Delta\tilde{n}) \mid \Delta\tilde{m}, \Delta\tilde{n} \in \mathcal{H}_\infty; \|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}\|_\infty < \epsilon\}.$$

With this definition we can prove the following theorem that characterizes the robustness of a maximally stabilizing controller in terms of coprime factor uncertainty.

Theorem 8.3. Let $g = \tilde{m}^{-1}\tilde{n}$ be a W-NLCF of the rational plant g, with $W = \operatorname{diag}(w_1, w_2) \in \mathcal{W}^2$ a rational weight. If the rational controller k_0 is the W-maximally stabilizing controller and $\cos\theta^W_{\min}([\mathcal{G}(g)]^\perp, [\mathcal{G}(k_0)]^\perp) = \operatorname{gap}^W(\mathcal{G}(g), [\mathcal{G}(k_0)]^\perp) := \sigma_1^W$, then the size $\epsilon_{\max}$ of the largest uncertainty ball $\mathcal{P}_g^{\epsilon_{\max}}$ so that k_0 stabilizes all $g_\Delta \in \mathcal{P}_g^{\epsilon_{\max}}$ is given by $\epsilon_{\max} = \sqrt{1 - (\sigma_1^W)^2}$.

Proof. Let $[\Delta\tilde{m} \quad \Delta\tilde{n}] \in \mathcal{H}_\infty \times \mathcal{H}_\infty$ such that $\|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}\|_\infty < \sqrt{1 - (\sigma_1^W)^2}$, and let $k_0 = uv^{-1}$ be the W-T-NRCF of k_0 characterized in Corollary 8.2; then

$$\begin{aligned}
&\|\sqrt{1 - (\sigma_1^W)^2} - [(\tilde{m} + \Delta\tilde{m})v - (\tilde{n} + \Delta\tilde{n})u]\|_\infty \\
&\quad = \left\|\sqrt{1 - (\sigma_1^W)^2} - (\tilde{m}v - \tilde{n}u) - [\Delta\tilde{m} \quad \Delta\tilde{n}]\begin{bmatrix} v \\ -u \end{bmatrix}\right\|_\infty \\
&\quad = \left\|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}W_R\begin{bmatrix} v \\ -u \end{bmatrix}\right\|_\infty \\
&\quad \le \|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}\|_\infty \left\|W_R\begin{bmatrix} v \\ -u \end{bmatrix}\right\|_\infty \\
&\quad \le \|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}\|_\infty \\
&\quad < \sqrt{1 - (\sigma_1^W)^2}.
\end{aligned}$$

However, this implies that

$$\begin{aligned}
\sqrt{1 - (\sigma_1^W)^2} &- [\sqrt{1 - (\sigma_1^W)^2} \\
&- [(\tilde{m} + \Delta\tilde{m})v - (\tilde{n} + \Delta\tilde{n})u]] \\
&= (\tilde{m} + \Delta\tilde{m})v - (\tilde{n} + \Delta\tilde{n})u
\end{aligned}$$

is invertible in $\mathcal{H}_\infty$. Hence k_0 stabilizes $g_\Delta = (\tilde{m} + \Delta\tilde{m})^{-1}(\tilde{n} + \Delta\tilde{n})$. We now show that $\epsilon_{\max} = \sqrt{1 - (\sigma_1^W)^2}$ is the largest possible radius by constructing a perturbation of size $\sqrt{1 - (\sigma_1^W)^2}$ that destabilizes the control system. Let

$$[\Delta\tilde{m} \quad \Delta\tilde{n}] := \sqrt{1 - (\sigma_1^W)^2}\,\Theta[-v^* \quad u^*]W$$

where Θ is a Blaschke product, i.e. a stable rational allpass function, chosen in such a way that $[\Delta\tilde{m} \quad \Delta\tilde{n}] \in \mathcal{H}_\infty \times \mathcal{H}_\infty$. Note that $\|[\Delta\tilde{m} \quad \Delta\tilde{n}]W_R^{-1}\|_\infty = \sqrt{1 - (\sigma_1^W)^2}$. Then the controller k_0 does not stabilize $g_\Delta = (\tilde{m} + \Delta\tilde{m})^{-1}(\tilde{n} + \Delta\tilde{n})$ since

$$\begin{aligned}
&(\tilde{m} + \Delta\tilde{m})v - (\tilde{n} + \Delta\tilde{n})u \\
&\quad = \sqrt{1 - (\sigma_1^W)^2} - \sqrt{1 - (\sigma_1^W)^2}\,\Theta[-v^* \quad u^*]W\begin{bmatrix} v \\ u \end{bmatrix} \\
&\quad = \sqrt{1 - (\sigma_1^W)^2}(1 - \Theta),
\end{aligned}$$

which is not invertible in $\mathcal{H}_\infty$, as $(1 - \Theta)$ has zeros on the imaginary axis. □

Note that the notion of a destabilizing perturbation used in the above proof also includes perturbations that perturb a plant to a plant with pole-zero cancellation in the closed right half plane.

It was shown (Ober and Sefton, 1991) that the I-maximally stabilizing controller of a plant g is the optimally robust controller with respect to unweighted normalized coprime factor perturbations. It was shown by McFarlane and Glover (1989) that the size of the maximal uncertainty ball that can be tolerated by this controller is given by $\sqrt{1 - \sigma_1^2}$, where σ_1 is the Hankel operator with symbol $[\tilde{m} \quad -\tilde{n}]^*$, $g = \tilde{m}^{-1}\tilde{n}$, an I-normalized coprime factorization of g. The previous theorem shows that for W-maximally stabilizing controllers the analogous formula holds, where in this case σ_1^W is the first singular value of the corresponding weighted Hankel operator (see Theorem 7.2).

9. THE INVERSE WEIGHT PROBLEM

In the previous sections we studied maximally stabilizing controllers of a plant g with respect to a given weight. These controllers were analysed regarding their robust stability properties. In this section we are concerned with the inverse problem: given a plant and a stabilizing controller, is it possible to interpret this controller as the maximally stabilizing controller with respect to a certain weight. An unrelated inverse weight problem in the $\mathcal{H}_\infty$ framework was considered by Lenz *et al.* (1988).

The first step to the solution of this problem is to notice that if k is a W-maximally stabilizing controller, $W = \operatorname{diag}(w_1, w_2)$, of the plant g then $c^*c = \alpha^2$, for some $\alpha \in \mathcal{R}$, where $c := n^*w_1v + m^*w_2u$ and where $[n \quad m]$ is a W-NRCF of g and $[u \quad v]$ is a W-T-NRCF of k. We will therefore first investigate under which conditions there exists a weight $W = \mathrm{diag}(w_1, w_2)$, such that $c := n^* w_1 v + m^* w_2 u$ is the scalar multiple of an allpass function where $[n \;\; m]$ is a W-NRCF of the given transfer function g and $[u \;\; v]$ is a W-T-NRCF of a stabilizing controller k. The next lemma connects this problem with the solution to a quadratic equation.

Lemma 9.1. Given a rational SISO system g, a rational stabilizing controller, k, and an α, $1 > \alpha > 0$, there exists a $W = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \in \mathscr{W}^2$ such that $c^* c = \alpha^2$, with $c := n^* w_1 v + m^* w_2 u$ if and only if W is of the form $W = w_1 \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} \in \mathscr{W}^2$ where w_1 is any weighting function in $\mathscr{W}^1$ and $w \in \mathscr{W}^1$ satisfies the quadratic equation

$$kk^* w^2 + [(1-\alpha^2)^{-1} gk + (1-\alpha^2)^{-1} g^* k^* - \alpha^2 (1-\alpha^2)^{-1} gg^* kk^* - \alpha^2 (1-\alpha^2)^{-1}] w + gg^* = 0. \quad (1)$$

Proof. Assume that given $1 > \alpha > 0$ there exists a $W = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \in \mathscr{W}^2$ such that $c^* c = \alpha^2$ where $c = n^* w_1 v + m^* w_2 u$, with $g = nm^{-1}$ a W-NRCF and $k = uv^{-1}$ a W-T-NRCF. Note then that $m^* m = (g w_1 g^* + w_2)^{-1}$ and $v^* v = (w_1 + k w_2 k^*)^{-1}$. Therefore

$$\begin{aligned} \alpha^2 = c^* c &= (n^* w_1 v + m^* w_2 u)^* (n^* w_1 v + m^* w_2 u) \\ &= v^* (w_1 g + k^* w_2) m m^* (g^* w_1 + w_2 k) v \\ &= (w_1 + k w_2 k^*)^{-1} (g w_1 g^* + w_2)^{-1} (w_1 g + k^* w_2) \\ &\quad \times (g^* w_1 + w_2 k). \end{aligned}$$

Hence

$$\alpha^2 (w_1 + k w_2 k^*)(g w_1 g^* + w_2) = (w_1 g + k^* w_2)(g^* w_1 + w_2 k)$$

and therefore,

$$0 = (1-\alpha^2) w_1^2 gg^* + (1-\alpha^2) w_2^2 kk^* + w_1 w_2 gk + w_1 w_2 g^* k^* - \alpha^2 w_1 w_2 gg^* kk^* - \alpha^2 w_1 w_2.$$

Dividing both sides by $1-\alpha^2$ and by w_1^2 and setting $w := w_2 w_1^{-1}$ we obtain

$$kk^* w^2 + [(1-\alpha^2)^{-1} gk + (1-\alpha^2)^{-1} g^* k^* - \alpha^2 (1-\alpha^2)^{-1} gg^* kk^* - \alpha^2 (1-\alpha^2)^{-1}] w + gg^* = 0.$$

Conversely, assume that there is a $w \in \mathscr{W}^1$ that solves the quadratic equation. Then clearly the above steps can be reversed to show the reverse implication. □

The problem of finding a weight for which a given controller k is the W-maximally stabilizing controller of the given plant g can by the above Lemma only have a solution if there exists a solution, $w \in \mathscr{W}^1$ to (1). The following lemma summarizes some basic properties of the solutions to this equation.

Lemma 9.2. Given the assumptions of Lemma 1 and let $\omega \in [-\infty, \infty]$. Then the equation

$$\begin{aligned} & k(j\omega) k(j\omega)^* w(j\omega)^2 + [(1-\alpha^2)^{-1} g(j\omega) k(j\omega) \\ & \quad + (1-\alpha^2)^{-1} g(j\omega)^* k(j\omega)^* - \alpha^2 (1-\alpha^2)^{-1} \\ & \quad \times g(j\omega) g(j\omega)^* k(j\omega) k(j\omega)^* - \alpha^2 (1-\alpha^2)^{-1}] \\ & \quad \times w(j\omega) + g(j\omega) g(j\omega)^* = 0 \end{aligned}$$

has

(1) one (and therefore two) non-negative solutions if

$$\sqrt{1-\alpha^2} \le \frac{|1 - g(j\omega) k(j\omega)|}{1 + |g(j\omega)|\,|k(j\omega)|}.$$

(2) One repeated non-negative solution if and only if

$$\sqrt{1-\alpha^2} = \frac{|1 - g(j\omega) k(j\omega)|}{1 + |g(j\omega)|\,|k(j\omega)|}.$$

(3) A solution $w(j\omega) = 0$ if and only if either $g(j\omega) = 0$ or $k(j\omega) = \infty$.

(4) A solution $w(j\omega) = \infty$ if and only if either $g(j\omega) = \infty$ or $k(j\omega) = 0$.

Proof. Notice that equation (1) evaluated on the imaginary axis takes the form of a quadratic equation $ax^2 + bx + c = 0$ with real coefficients such that $a \ge 0$ and $c \ge 0$. Note that the real solutions to the equation are either both non-negative or both non-positive. The equation has positive solutions if and only if $-b \ge 2(ac)^{1/2}$. Therefore a necessary and sufficient condition for the equation to have two positive solutions is given by the following inequality (the variables are dropped from the expressions for simplicity of notation):

$$\frac{\alpha^2}{1-\alpha^2} + \frac{\alpha^2}{1-\alpha^2} g^* g k^* k - \frac{1}{1-\alpha^2} gk - \frac{1}{1-\alpha^2} g^* k^* \ge 2\,|g|\,|k|,$$

which is equivalent to

$$\alpha^2 + \alpha^2 g^* g k^* k - gk - g^* k^* \ge 2\,|g|\,|k| - \alpha^2 2\,|g|\,|k|.$$

However, this is equivalent to

$$\alpha^2 [1 + |g|\,|k|]^2 \ge 2\,|g|\,|k| + gk + g^* k^* = [1 + |g|\,|k|]^2 - |1 - gk|^2,$$

which is the case if and only if

$$\sqrt{1-\alpha^2} \le \frac{|1-gk|}{1+|g|\,|k|}.$$

This shows statement (1). Statement (2) follows by replacing the inequality signs in the previous derivation by equality signs. Statement (3) can be seen by dividing the equation by k^*k. Because of the internal stability of the control system it is not possible to have simultaneously $g(j\omega)=0$ and $k(j\omega)=\infty$. Statement (4) follows analogously by dividing the equation by g^*g and by w^2. □

The next theorem studies under which conditions there are solutions to equation (1).

Theorem 9.3. Given a rational plant g, and a rational stabilizing controller k, such that $\inf_{\omega\in\mathfrak{R}} \frac{|1-g(j\omega)k(j\omega)|}{1+|g(j\omega)k(j\omega)|}<1$. Define α_C, $0<\alpha_C<1$ such that

$$\sqrt{1-\alpha_C^2} = \inf_{\omega\in\mathfrak{R}} \frac{|1-g(j\omega)k(j\omega)|}{1+|g(j\omega)k(j\omega)|}.$$

Let ω_i, $1\le i\le r$, be the points at which this infimum is attained. Order the elements $x_i\in X$, $1\le i\le n$ and $y_i\in Y$, $1\le i\le m$ of the sets

$$X := \{x\in\mathfrak{R} \mid jx \text{ is a pole of } g \text{ or a zero of } k\}$$
$$Y := \{y\in\mathfrak{R} \mid jy \text{ is a zero of } g \text{ or a pole of } k\}.$$

so that $x_1<\cdots<x_i<\cdots<x_n$ and $y_1<\cdots<y_i<\cdots<y_m$.

(1) If there exists a solution $w\in\mathcal{W}^1$ to (1) for a $0<\alpha<1$, then $0<\alpha_C\le\alpha<1$.
(2) If α is such that $0<\alpha_C<\alpha<1$ then there exists
  (a) No solution $w\in\mathcal{W}^1$ to (1) if $X\neq\emptyset$ and $Y\neq\emptyset$.
  (b) Exactly one solution $w\in\mathcal{W}^1$ to (1) if either $X=\emptyset$ or $Y=\emptyset$.
  (c) Two solutions $w\in\mathcal{W}^1$ to (1) if $X=\emptyset$ and $Y=\emptyset$.
(3) Let $\alpha=\alpha_C$.
  (a) If $X\neq\emptyset$ and $Y\neq\emptyset$ then there exists one solution $w\in\mathcal{W}^1$ to (1) if and only if the following conditions are satisfied: (i) if $x_i<y_j$ then there exists a ω_k such that $x_i<\omega_k<y_j$, (ii) if $y_j<x_i$ then there exists a ω_k such that $y_j<\omega_k<x_i$. Otherwise there is no solution in $\mathcal{W}^1$ to (1).
  (b) If either $X=\emptyset$ or $Y=\emptyset$ then there exists exactly one solution $w\in\mathcal{W}^1$ to (1).
  (c) If both $X=\emptyset$ and $Y=\emptyset$ then there exist two solutions $w\in\mathcal{W}^1$ to (1).

Proof. First note that $\inf_{\omega\in\mathfrak{R}} \frac{|1-g(j\omega)k(j\omega)|}{1+|g(j\omega)k(j\omega)|}>0$ by the stability of the control system (g,k). (1) If there is a solution $w\in\mathcal{W}^1$ to (1) for $0<\alpha<1$ then Lemma 9.2 implies that $\alpha_C\le\alpha<1$. (2) Lemma 9.2 implies that if $\alpha\ge\alpha_C$ then for each $\omega\in[-\infty,\infty]$ there exist two solutions to equation (1) with values in $[0,\infty]$. Since g and k are rational this implies that for $\alpha_C\le\alpha<1$ there exist two continuous functions with values in $[0,\infty]$ that solve (1). If a continuous function on the imaginary axis has values in $]0,\infty[$ including at ∞ such a function admits a spectral factorization with a spectral factor in $\mathcal{W}^1$ (see e.g. Theorem 5, Helson, 1964). Therefore, for a solution of (1) to be in $\mathcal{W}^1$ it has to be shown that the continuous solution has values in $]0,\infty[$ including at ∞.

If $\alpha_C<\alpha<1$ then there are two continuous solutions to (1) with values in $[0,\infty]$. By Lemma 9.2 these solutions do not intersect. If $X\neq\emptyset$ and $Y\neq\emptyset$ then by Lemma 9.2 one of the solutions has poles on the imaginary axis and the other solution has zeros on the imaginary axis. In this case there is therefore no solution in $\mathcal{W}^1$ to (1). If $X=\emptyset$ or $Y\neq\emptyset$ then one of the two solutions has zeros on the imaginary axis. However, none of the solutions has poles on the imaginary axis. Since the two solutions have no intersection the larger of the two solutions will be in $\mathcal{W}^1$. In a similar way it is shown that there is one solution in $\mathcal{W}^1$ to (1) if $X\neq\emptyset$ and $Y=\emptyset$. If $X=\emptyset$ and $Y=\emptyset$ then by the same argument neither of the two solutions has poles or zeros on the imaginary axis. In this case there are therefore two solutions in $\mathcal{W}^1$ to (1). (3) First note that for $\omega\in X\cup Y$ we have that $\frac{|1-g(j\omega)k(j\omega)|}{1+|g(j\omega)k(j\omega)|}=1$. Hence, by Lemma 9.2 we have that the two solutions of (1) have no intersection point at the frequencies in $X\cup Y$.

If $X\neq\emptyset$ and $Y\neq\emptyset$ then a continuous solution to (1) can only be found if the poles and zeros of the solutions of (1) interlace with the intersection points of the two solutions such that a solution can be constructed from the two solutions that are in $\mathcal{W}^1$. This is done by partitioning the imaginary axis into pieces bounded by the intersection points of the two solutions. The interlacing condition in the statement guarantees that in between each two intersection points there is a solution that has neither poles nor zeros. Putting these pieces together (we think of the point $+j\infty$ to be identified with the point $-j\infty$), we obtain a solution to (1). Clearly, if the interlacing condition is not satisfied then there is no solution in $\mathcal{W}^1$.

The remaining statements follow analogously to the respective statements in (2). □

The previous theorem gave amongst others a criterion for the existence of two solutions in $\mathcal{W}^2$ to equation (1). Given one solution the second is easily calculated as follows.

Corollary 9.4. Given the assumptions of the previous theorem, assume that $X=\emptyset$ and $Y=\emptyset$. Let $0<\alpha<1$ be such that there exists a solution $w\in\mathcal{W}^1$ to equation (1) then $\frac{1}{w}\frac{g^*g}{k^*k}$ is another solution to equation (1) for the same value of α.

Proof. The assumptions imply that $\frac{1}{w}\frac{g^*g}{k^*k}\in\mathcal{W}^1$. Since the product of two solutions to equation (1) is $\frac{g^*g}{k^*k}$ the second solution is given by $w\frac{g^*g}{k^*k}$. □

Given a plant g and a controller k, the previous theorem gives necessary and sufficient conditions for the existence of a weight W such that $c=m^*w_1v+n^*w_2u$ is an allpass function scaled by the scalar α. It will now be investigated under which conditions the controller k is the maximally stabilizing controller of g with respect to W.

The solution of this problem is based on the work on so-called badly approximable functions (Poreda, 1972; Garnett, 1981; Helton and Marshall, 1990). The following theorem gives the continuous-time equivalent of a result usually stated in the discrete-time version. The winding number is here defined to be the change in argument of a function, which is continuous on the imaginary axis, as the frequency ω is changed from $-\infty$ to $+\infty$.

Theorem 9.5. Given an all-pass SISO transfer function $c\in\mathcal{L}_\infty^{1\times 1}$ then

$$\inf_{q\in\mathcal{H}_\infty}\|c+q\|_\infty=\|c\|_\infty$$

if and only if the winding number of c is strictly positive.

We can now give a necessary and sufficient condition for a controller to be a W-maximally stabilizing controller given that $c=n^*w_1v+n^*w_2u$ is a scaled all-pass function. We first need the following Lemma.

Lemma 9.6. Given a SISO plant g, rational stabilizing controller k_0 and weight $W=\begin{bmatrix} w_1 & 0\\ 0 & w_2\end{bmatrix}\in\mathcal{W}^2$, let $g=nm^{-1}$ and $g=\tilde{m}^{-1}\tilde{n}$ be a W-NRCF and a W-NLCF, respectively, and let $k_0=u_0v_0^{-1}$ be a W-T-NRCF. If $c^*c=\alpha$, for $0<\alpha<1$ and $c:=n^*w_1v_0+m^*w_2u_0$ then

$$\tilde{m}v_0-\tilde{n}u_0=\sqrt{1-\alpha^2}.$$

Proof. We only give the proof in the I-normalized case. The general case follows by the usual reduction method. Since $\begin{bmatrix}\tilde{m} & -\tilde{n}\\ n^* & m^*\end{bmatrix}$ is unitary and since $\begin{bmatrix} v_0\\ u_0\end{bmatrix}$ is normalized we have that

$$\begin{aligned}1&=[v_0^*\ \ u_0^*]\begin{bmatrix} v_0\\ u_0\end{bmatrix}\\ &=[v_0^*\ \ u_0^*]\begin{bmatrix}\tilde{m} & -\tilde{n}\\ n^* & m^*\end{bmatrix}^*\begin{bmatrix}\tilde{m} & -\tilde{n}\\ n^* & m^*\end{bmatrix}\begin{bmatrix} v_0\\ u_0\end{bmatrix}\\ &=(\tilde{m}v_0-\tilde{n}u_0)^*(\tilde{m}v_0-\tilde{n}u_0)=1-c^*c=1-\alpha^2.\end{aligned}$$

As k_0 stabilizes g we have that $\tilde{m}v_0-\tilde{n}u_0$ is a unit. But the only units that have constant modulus are constants. Hence $\tilde{m}v_0-\tilde{n}u_0=\sqrt{1-\alpha^2}$. □

Theorem 9.7. Given a rational SISO plant g, rational stabilizing controller k_0, and weight $W=\begin{bmatrix} w_1 & 0\\ 0 & w_2\end{bmatrix}\in\mathcal{W}^2$ such that $c^*c=\alpha^2$ for some $0<\alpha<1$, where $c:=n^*w_1v_0+m^*w_2u_0$ with $g=nm^{-1}$ a W-NRCF and $k_0=u_0v_0^{-1}$ a W-T-NRCF; then, k_0 is the W-maximally stabilizing controller of g if and only if the winding number of c is strictly positive.

Proof. In order to prove the theorem we have to show that

$$\inf_k \operatorname{gap}^W(\mathcal{G}(g),[\mathcal{G}(k)]^\perp)=\operatorname{gap}^W(\mathcal{G}(g),[\mathcal{G}(k_0)]^\perp)$$

if and only if the winding number of c is strictly positive. However

$$\begin{aligned}\inf_k \operatorname{gap}^W(\mathcal{G}(g),[\mathcal{G}(k)]^\perp)^2 &=\inf_{k\text{ stabilizing}}\operatorname{gap}^W(\mathcal{G}(g),[\mathcal{G}(k)]^\perp)^2\\ &=\inf_{\substack{k\text{ stabilizing}\\ k=uv^{-1}\ W\text{-T-NRCF}}}\|n^*w_1v+m^*w_2u\|_\infty^2\\ &=1-\sup_{\substack{k\text{ stabilizing}\\ k=uv^{-1}\ W\text{-T-NRCF}}}\tau^2(\tilde{m}v-\tilde{n}u),\end{aligned}$$

where $\tau(\tilde{m}v-\tilde{n}u)=\inf_\omega|(\tilde{m}v-\tilde{n}u)(j\omega)|$ and $g=$

$\tilde{m}^{-1}\tilde{n}$ a W-NLCF of g. However

$$\sup_{\substack{k \text{ stabilizing} \\ k=uv^{-1}\ W\text{-T-NRCF}}} \tau^2(\tilde{m}v - \tilde{n}u)$$

is attained for the same k_0 that achieves

$$\inf_{\substack{k \text{ stabilizing} \\ k=uv^{-1}\ W\text{-T-NRCF}}} \|(\tilde{m}v - \tilde{n}u)^{-1}\|_\infty^2$$

$$= \inf_{k \text{ stabilizing}} \left\|(1-gk)^{-1}(1-gk)^{-*} \times \frac{(gw_1g^* + w_2)}{w_1w_2}(w_1 + kw_2k^*)\right\|_\infty.$$

By the Youla-parametrization each stabilizing controller has a representation $k = \dfrac{u_0 + mq}{v_0 + nq}$, $q \in \mathcal{H}_\infty$. Note that by Lemma 9.6, $\tilde{m}v_0 - \tilde{n}u_0 = \sqrt{1-\alpha^2}$. Then

$$(1-gk)^{-1} = \left[1 - \frac{\tilde{n}}{\tilde{m}}\frac{u_0 + mq}{v_0 + nq}\right]^{-1}$$

$$= \left[\frac{\tilde{m}v_0 + \tilde{m}nq - \tilde{n}u_0 - \tilde{n}mq}{\tilde{m}(v_0 + nq)}\right]^{-1} = \frac{\tilde{m}(v_0 + nq)}{\sqrt{1-\alpha^2}}.$$

Further

$$gw_1g^* + w_2 = \frac{\tilde{n}}{\tilde{m}} w_1 \frac{\tilde{n}^*}{\tilde{m}^*} + w_2$$

$$= \frac{1}{\tilde{m}\tilde{m}^*}(\tilde{n}w_1\tilde{n}^* + \tilde{m}w_2\tilde{m}^*) = \frac{w_1w_2}{\tilde{m}\tilde{m}^*}$$

and

$$w_1 + kw_2k^* = w_1 + \left[\frac{u_0 + mq}{v_0 + nq}\right]w_2\left[\frac{u_0 + mq}{v_0 + nq}\right]^*$$

$$= \frac{1}{(v_0 + nq)(v_0 + nq)^*}[(v_0 + nq)w_1(v_0 + nq)^* + (u_0 + mq)w_2(u_0 + mq)^*]$$

$$= \frac{1}{(v_0 + nq)(v_0 + nq)^*}[v_0w_1v_0^* + u_0w_2u_0^* + nqw_1v_0^* + mqw_2u_0^* + v_0w_1n^*q^* + u_0w_2m^*q^* + nw_1n^*qq^* + mw_2m^*qq^*]$$

$$= \frac{1}{(v_0 + nq)(v_0 + nq)^*}[1 + c^*q + cq^* + qq^*]$$

$$= \frac{1}{(v_0 + nq)(v_0 + nq)^*}[1 - \alpha^2 + (c+q)(c+q)^*].$$

Summarizing, we therefore have that $\inf_k \text{gap}^W(\mathcal{G}(g), [\mathcal{G}(k)]^\perp)$ is attained for k that attains

$$\inf_{k \text{ stabilizing}} \left\|(1-gk)^{-1}(1-gk)^{-*} \times \frac{(gw_1g^* + w_2)}{w_1w_2}(w_1 + kw_2k^*)\right\|_\infty$$

$$= \inf_{q\in\mathcal{H}_\infty} \frac{1}{1-\alpha^2}\|1 - \alpha^2 + (c+q)(c+q)^*\|_\infty^2$$

$$= 1 + \inf_{q\in\mathcal{H}_\infty} \frac{1}{1-\alpha^2}\|c+q\|_\infty^2.$$

Recall that the parameter $q = 0$ corresponds to $k = k_0$. The result is therefore established since $c^*c = \alpha^2$ and by Proposition 9.5 $\inf_{q\in\mathcal{H}_\infty}\|c+q\|_\infty^2 = \|c\|_\infty^2 = \alpha^2$ if and only if the winding number of c is strictly positive. □

The above theorem gave a test for the optimality of the controller k. This test can be made more concrete if the transfer functions and weights are rational.

Corollary 9.8. Given the assumptions of the previous theorem and assume moreover that g, k, and $W = \text{diag}(w_o^*w_o, w_i^*w_i)$ are rational, with w_o, w_o^{-1}, w_i, $w_i^{-1} \in H_\infty$; then k_0 is the W-maximally stabilizing controller of g if and only if the McMillan degree of $\dfrac{w_o}{w_i}g$ is strictly greater than the McMillan degree of $\dfrac{w_i}{w_o}k_0$, having performed possible pole-zero cancellations in $\dfrac{w_o}{w_i}g$ and $\dfrac{w_i}{w_o}k_0$.

Proof. Recall that $\dfrac{w_o}{w_i}g = w_on(w_im)^{-1}$ is an I-NRCF of $\dfrac{w_o}{w_i}g$ and $\dfrac{w_i}{w_o}k = w_iu(w_ov)^{-1}$ is an I-NRCF of $\dfrac{w_i}{w_o}k$. It is shown (Sefton, 1991; Sefton and Ober, 1991) that

$$c = (w_on)^*(w_ov) + (w_im)^*(w_in)$$
$$= n^*w_1v + m^*w_2u$$

has as McMillan degree the sum of the McMillan degrees of $\dfrac{w_o}{w_i}g$ and $\dfrac{w_i}{w_o}k$. Therefore, the stable poles of c are given by the poles of $\begin{bmatrix} w_iu \\ w_ov \end{bmatrix}$ and the unstable poles are given by the poles of $[(w_on)^* \quad (w_im)^*]$. Since c is allpass the winding number is given by the difference between the number of unstable poles and the number of stable poles. The number of poles of $\begin{bmatrix} w_iu \\ w_ov \end{bmatrix}$ is equal to the McMillan degree of $\dfrac{w_i}{w_o}k$ and the number of poles of

$[(w_o n)^* \quad (w_i m)^*]$ is equal to the McMillan degree of $\frac{w_o}{w_i} g$. Therefore the winding number of $n^* w_1 v + m^* w_2 u$ is strictly positive if and only if the McMillan degree of $\frac{w_o}{w_i} g$ is strictly greater than the McMillan degree of $\frac{w_i}{w_o} k$. □

The following corollary shows that it is not necessary to have coprime factorizations with respect to W in order to be able to check for the maximality of a controller.

Corollary 9.9. Given the assumptions of the previous theorem, (1) let $g = nm^{-1}$ be any RCF of g and $k = uv^{-1}$ any left coprime factorization of k. Then k is the W-maximally stabilizing controller of g if and only if the winding number of $n^* w_1 v + m^* w_2 u$ is strictly positive. (2) If $g \in \mathcal{H}_\infty$ and $k \in \mathcal{H}_\infty$ then k is the W-maximally stabilizing controller of g if and only if the winding number of $g^* w_1 + w_2 k$ is strictly positive.

Proof. (1) Different coprime factorizations are related by the multiplication by a unit in $\mathcal{H}_\infty$. However, multiplication of c by a unit in $\mathcal{H}_\infty$ and a unit in $\mathcal{H}_{\infty,-}$ does not change the winding number of c. (2) If $g \in \mathcal{H}_\infty$ $(k \in \mathcal{H}_\infty)$ then $\frac{g}{1}\left(\frac{k}{1}\right)$ is a coprime factorization of g (k). Hence the result follows from (1). □

10. FREQUENCY DOMAIN INTERPRETATION

In this section we are going to give further interpretations of some of the quantities that were introduced previously. The first relationship that we are going to study is the connection between the cosine of the minimal angle between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of the controller and the quantity α_C as introduced in the previous section.

Theorem 10.1. Let k be the W-maximally stabilizing controller of the plant g, $w = \mathrm{diag}\,(w_1, w_2)$, with $\alpha_0 := \mathrm{gap}^W\,(\mathcal{G}(g), [\mathcal{G}(k)]^\perp)$. Define $0 < \alpha_C < 1$ by

$$\sqrt{1-\alpha_C^2} = \inf_{\omega \in \Re} \frac{|1 - g(j\omega)k(j\omega)|}{1 + |g(j\omega)k(j\omega)|}.$$

Then

(1) $\alpha_0 \geq \alpha_C$.
(2) $\alpha_0 = \alpha_C$ if and only if $|w_2(j\omega)k(j\omega)| = |w_1(j\omega)g(j\omega)|$ for some $\omega \in [-\infty, \infty]$.

Proof. We have that

$$\begin{aligned}
&\frac{(w_1 + kw_2k^*)(gw_1g^* + w_2)}{w_1w_2\,|1-gk|^2} - \frac{(1+|gk|)^2}{|1-gk|^2} \\
&\quad= \frac{1}{w_1w_2\,|1-gk|^2}[(w_1 + w_2\,|k|^2)(w_2 + w_1\,|g|^2) \\
&\qquad - w_1w_2(1+|gk|)^2] \\
&\quad= \frac{1}{w_1w_2\,|1-gk|^2}[w_1w_2 + w_2^2\,|k|^2 + w_1^2\,|g|^2 \\
&\qquad + w_1w_2\,|g|^2\,|k|^2 - w_1w_2 - 2w_1w_2\,|g|\,|k| \\
&\qquad - w_1w_2\,|g|^2\,|k|^2] \\
&\quad= \frac{1}{w_1w_2\,|1-gk|^2} \\
&\qquad \times [w_1\,|g|^2 - 2w_1w_2\,|g|\,|k| + w_2\,|k|^2] \\
&\quad= \frac{1}{w_1w_2\,|1-gk|^2}[w_1\,|g| - w_2\,|k|]^2.
\end{aligned}$$

Note that with the notation of the proof of Lemma 9.1, $\alpha_0 = \mathrm{gap}^W\,(\mathcal{G}(g), [\mathcal{G}(k)]^\perp) = \|n^* w_1 v + m^* w_2 u\|_\infty$ and

$$\alpha_0^2 = \frac{(w_1 g + k^* w_2)(g^* w_1 + w_2 k)}{(w_1 + kw_2k^*)(w_2 + gw_1g^*)}.$$

Hence

$$\frac{1}{1-\alpha_0^2} = \frac{(w_1 + kw_2k^*)(gw_1g^* + w_2)}{w_1w_2\,|1-gk|^2}$$

and therefore

$$\begin{aligned}
&\frac{1}{1-\alpha_0^2} - \frac{1}{1-\alpha_C^2} \\
&\quad= \frac{(w_1 + w_2\,|k|^2)(w_2 + w_1\,|g|^2)}{w_1w_2\,|1-gk|^2} - \left\|\frac{(1+|gk|)^2}{|1-gk|^2}\right\|_\infty \\
&\quad= \tau\left(\frac{(w_1 + w_2\,|k|^2)(w_2 + w_1\,|g|^2)}{w_1w_2\,|1-gk|^2} - \frac{(1+|gk|)^2}{|1-gk|^2}\right) \\
&\quad= \tau\left(\frac{1}{w_1w_2\,|1-gk|^2}[w_1\,|g| - w_2\,|k|]^2\right) \\
&\quad\geq 0.
\end{aligned}$$

This shows that $\frac{1}{1-\alpha_0^2} \geq \frac{1}{1-\alpha_C^2}$ and hence (1). This identity also shows that $\frac{1}{1-\alpha_0^2} = \frac{1}{1-\alpha_C^2}$ if and only if $w_1(j\omega)\,|g(j\omega)| = w_2(j\omega)\,|k(j\omega)|$ for some $\omega \in [-\infty, \infty]$, but this implies (2). □

As a corollary to the previous theorem we can give an interesting interpretation of α_0 in terms of the mixed-sensitivity of the control system with the maximally stabilizing controller. Recall that the mixed-sensitivity of a control system is defined by

$$\|\,|S| + |T|\,\|_\infty = \left\|\,\left|\frac{1}{1-gk}\right| + \left|\frac{gk}{1-gk}\right|\,\right\|_\infty$$

with sensitivity function $S = \dfrac{1}{1-gk}$ and complementary sensitivity function $T = \dfrac{gk}{1-gk}$.

Corollary 10.2. Assume the notation of the previous theorem. Then

$$\| \, |S| + |T| \, \|_\infty \leq \frac{1}{\sqrt{1-\alpha_0^2}},$$

with equality if and only if $w_1(j\omega)\,|g(j\omega)| = w_2(j\omega)\,|k(j\omega)|$ for some $\omega \in [-\infty, \infty]$.

Proof. Note that

$$\| \, |S| + |T| \, \|_\infty = \frac{1}{\sqrt{1-\alpha_C^2}} \leq \frac{1}{\sqrt{1-\alpha_0^2}}.$$

The statements then follow from the previous theorem. □

We have established the importance of the existence of an intersection between $w_1\,|g|$ and $w_2\,|k|$. The following theorem will give a sufficient condition for the existence of such an intersection in the unweighted case.

Theorem 10.3. A scalar plant g with I-NRCF $g = nm^{-1}$ and the I-maximally stabilizing controller k are given. Assume that both g and k are continuous on the imaginary axis. If

$$|g(j\omega_1| \geq \frac{1 + (1-\sigma_1^2)^{1/2}}{\sigma_1}$$

for some frequency $\omega_1 \in [-\infty, \infty]$, and

$$|g(j\omega_2)| \leq \frac{1 - (1-\sigma_1^2)^{1/2}}{\sigma_1}$$

for a frequency $\omega_2 \in [-\infty, \infty]$, where $\sigma_1 = \|H_{\left[\begin{smallmatrix} m^* \\ -n^* \end{smallmatrix}\right]}\|$, then there exists a frequency $\omega_0 \in [-\infty, \infty]$ such that

$$|g(j\omega_o)| = |k(j\omega_o)|.$$

Proof. The first step of the proof is to find an expression for the controller, k, in terms of the modulus and the phase of the plant g. At each point $j\omega$ on the imaginary axis we have $g = re^{j\theta_0}$ for $r \geq 0$ and $\theta_0 \in [0, 2\pi[$. For simplicity of presentation we suppress the frequency dependence of the expressions. The normalized coprime factors of g are given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{(1+r^2)^{1/2}} \begin{bmatrix} re^{j\theta_1} \\ e^{-j\theta_2} \end{bmatrix}$$

where $\theta_1 + \theta_2 = \theta_0$. If the controller, k, has normalized coprime factors (u, v) they must satisfy by Corollary 8.2

$$mv - nu = (1-\sigma_1^2)^{1/2},$$
$$m^*u + n^*v = \sigma_1 e^{j\theta_r}$$

where $e^{j\theta_r}$ is the phase of the scaled all pass function $m^*u + n^*v$. Solving the simultaneous equations for u, v gives

$$v = \frac{1}{(1+r^2)^{1/2}} (\sigma_1 re^{j(\theta_1+\theta_r)} + (1-\sigma_1^2)^{1/2} e^{j\theta_2})$$

$$u = \frac{1}{(1+r^2)^{1/2}} (\sigma_1 e^{j(\theta_r-\theta_2)} - (1-\sigma_1^2)^{1/2} re^{-j\theta_1})$$

and hence

$$k = uv^{-1} = \frac{(\sigma_1 e^{j(\theta_r-\theta_2)} - (1-\sigma_1^2)^{1/2} re^{-j\theta_1})}{(\sigma_1 re^{j(\theta_1+\theta_r)} + (1-\sigma_1^2)^{1/2} e^{j\theta_2})}$$
$$= e^{-j\theta_0} \frac{(\sigma_1 e^{j\tilde{\theta}} - (1-\sigma_1^2)^{1/2} r)}{(\sigma_1 re^{j\tilde{\theta}} + (1-\sigma_1^2)^{1/2})}$$

where $\tilde{\theta} = \theta_1 - \theta_2 + \theta_r$.

The theorem is now proved by finding a condition for $r = |g| \geq |k|$ at a frequency ω_1 and a condition for $r = |g| \leq |k|$ at a frequency ω_2. If both conditions are satisfied then by the continuity of g and k the existence of a frequency ω_0 is guaranteed such that $|g| = |k|$. To find a sufficient condition for $r = |g| \geq |k|$ at frequency ω_1 note that by the above characterization of k

$$\frac{\sigma_1 + (1-\sigma_1^2)^{1/2} r}{|\sigma_1 r - (1-\sigma_1^2)^{1/2}|} \geq |k|.$$

Hence, $r = |g| \geq \dfrac{\sigma_1 + (1-\sigma_1^2)^{1/2} r}{|\sigma_1 r - (1-\sigma_1^2)^{1/2}|} \geq |k|$ if

$$\sigma_1 r^2 - 2r(1-\sigma_1^2)^{1/2} - \sigma_1 \geq 0.$$

This is the case if and only if

$$(\sigma_1 r + (1 - (1-\sigma_1^2)^{1/2}))(\sigma_1 r - (1 + (1-\sigma_1^2)^{1/2})) \geq 0.$$

Therefore, a sufficient condition for the existence of a frequency ω_1 such that $|g| \geq |k|$ is the existence of a frequency such that

$$|g| \geq \frac{1 + (1-\sigma_1^2)^{1/2}}{\sigma_1}.$$

Similarly to find a sufficient condition for the existence of a frequency ω_2 such that $|g| \leq |k|$ note that for all frequencies

$$\frac{\sigma_1 - (1-\sigma_1^2)^{1/2} r}{\sigma_1 r + (1-\sigma_1^2)^{1/2}} \leq |k|.$$

Hence, there exists a frequency ω_2 such that $r = |g| \leq |k|$ if

$$\sigma_1 r^2 + 2r(1-\sigma_1^2)^{1/2} - \sigma_1 \leq 0.$$

This is the case if and only if

$$(\sigma_1 r + (1 + (1 - \sigma_1^2)^{1/2}))(\sigma_1 r - (1 - (1 - \sigma_1^2)^{1/2})) \leq 0.$$

Therefore, a sufficient condition for the existence of a frequency ω_2 such that $r = |g| \leq |k|$ is the existence of a frequency such that

$$r = |g| \leq \frac{1 - (1 - \sigma_1^2)^{1/2}}{\sigma_1}. \qquad \square$$

The previous theorem gives a sufficient condition to ensure a cross-over frequency ω_o such that $|g(j\omega_o)| = |k(j\omega_o)|$. This condition will be satisfied for most systems which have a high gain at low frequencies and a small gain at high frequencies. Given there exists a cross-over frequency then the theorem gives an interesting interpretation of the stability margin, ϵ_{max}, as a worst case weighted distance of the Nyquist locus $-gk(j\omega)$ from the instability point $(-1, 0)$; moreover, this worst case occurs at the cross-over frequency.

The following example is an illustration of a simple case when there does not exist a cross-over frequency between the system, g, and its maximally stabilizing controller, k. Included within the example is a parameter λ, the gain of the system at $\omega = 0$. As this is decreased to a value less than $\sqrt{3}$ the cross-over frequency no longer exists.

Example 10.4. Consider the SISO system $g = \frac{\lambda}{s-1}$, $\lambda \in \mathscr{R}$. Its normalized coprime factors are

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{s + (1 + \lambda^2)^{1/2}} \begin{bmatrix} \lambda \\ s - 1 \end{bmatrix}.$$

A state space realization of $\begin{bmatrix} n \\ m \end{bmatrix}$ is given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \left[\begin{array}{c|c} -\sqrt{1+\lambda^2} & 1 \\ \hline \lambda & 0 \\ -(\sqrt{1+\lambda^2} + 1) & 1 \end{array}\right].$$

Balancing this realization (see e.g. Francis, 1987) the first Hankel singular value σ_1 can be calculated to be

$$\sigma_1 = \|H_{[\begin{smallmatrix} m^* \\ -n^* \end{smallmatrix}]}\| = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{\sqrt{1+\lambda^2}} + 1}$$

and therefore

$$\epsilon_{max} = \frac{|\lambda|}{\sqrt{2(1 + \lambda^2 + \sqrt{1+\lambda^2})}}.$$

A coprime factorization of the maximally stabilizing controller is characterized as the Nehari extension of $\begin{bmatrix} m^* \\ -n^* \end{bmatrix}$. But the Nehari extension of the McMillan degree one function $\begin{bmatrix} m^* \\ -n^* \end{bmatrix}$ has McMillan degree 0 (see e.g. Glover, 1984; Fuhrmann and Ober, 1991). By Theorem 8.1 we therefore have to find scalars u, v such that $mv - nu = 1 - \sigma_1^2$ and $u^2 + v^2 = 1 - \sigma_1^2$. These unique quantities are given by

$$u = \sqrt{1 - \sigma_1^2}\, \frac{1 + \sqrt{1+\lambda^2}}{-\lambda} \qquad v = \sqrt{1 - \sigma_1^2}$$

and therefore

$$k = \frac{-1}{\lambda}[1 + \sqrt{1+\lambda^2}].$$

This gives the expression

$$\frac{|1 - g(j\omega)k(j\omega)|}{1 + |g(j\omega)k(j\omega)|} = \frac{(\omega^2 + (1+\lambda^2))^{\frac{1}{2}}}{(\omega^2+1)^{1/2} + (1 + (1+\lambda^2)^{1/2})},$$

$\omega \in [-\infty, \infty]$. Elementary calculations show that this expression attains its infimum at $\omega = 0$ if $\lambda \leq \sqrt{3}$ and at ω such that $\sqrt{1+\omega^2} = \lambda^2(1 + \sqrt{1+\lambda^2})^{-1}$ if $\lambda \geq \sqrt{3}$. We have that

$$\inf_{\omega \in \mathscr{R}} \frac{|1 - g(j\omega)k(j\omega)|}{1 + |g(j\omega)k(j\omega)|} = \begin{cases} \dfrac{\sqrt{1+\lambda^2}}{2 + \sqrt{1+\lambda^2}}, & |\lambda| \leq \sqrt{3}; \\ \dfrac{|\lambda|}{\sqrt{2(1 + \lambda^2 + \sqrt{1+\lambda^2})}}, & |\lambda| \geq \sqrt{3}. \end{cases}$$

Hence, α_C defined in Theorem 10.1 is given by

$$\alpha_C^2 = \begin{cases} 2\dfrac{\sqrt{1 + \sqrt{1+\lambda^2}}}{2 + \sqrt{1+\lambda^2}} & |\lambda| \leq \sqrt{3}; \\ \dfrac{1}{2}\left[1 + \dfrac{1}{\sqrt{1+\lambda^2}}\right], & |\lambda| \geq \sqrt{3}. \end{cases}$$

It is easy to check that there is a frequency ω such that $|g(j\omega)| = |k(j\omega)|$ if and only if $\lambda \geq \sqrt{3}$. This intersection point occurs at the same point as the infimum of $\frac{|1 - g(j\omega)k(j\omega))|}{1 + |g(j\omega)k(j\omega)|}$. We have therefore verified the statement of Theorem 10.1 for this example. We clearly have that for arbitrary λ, $\sigma_1 = \alpha_0 \geq \alpha_C$. Also, $\alpha_0 = \alpha_C$ if and only if there is a frequency $\omega_0 \in [-\infty, \infty]$ such that $|g(j\omega_0)| = |k(j\omega_0)|$, i.e. if and only $\lambda \geq \sqrt{3}$.

For $\lambda \leq \sqrt{3}$ it is possible to find the modulus of the weight on the imaginary axis by solving the quadratic equation (1) and choosing the solution that does not tend to zero as $\omega \to \infty$. This gives for all $\omega \in \mathscr{R}$

$$w_1(j\omega) = \frac{(1 + (1+\lambda^2)^{1/2})(2\omega^2 + (1+\lambda^2) - \sqrt{4\omega^4 + \omega^2(3-\lambda^2)(1+\lambda^2)})}{\lambda^2(1+\lambda^2)}$$

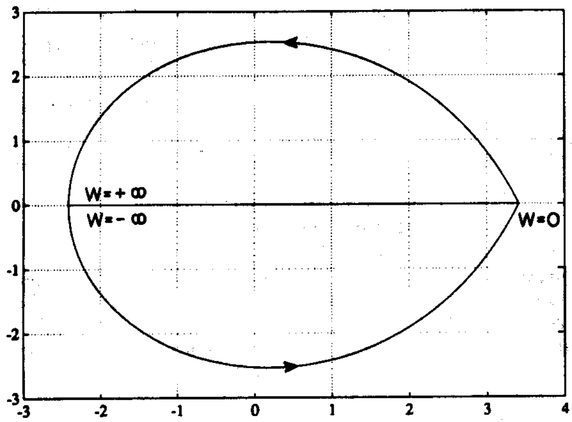

FIG. 1. Nyquist plot of $c = n^*w_1v + m^*w_2u$.

and

$$w_2(j\omega) = 1.$$

The weight w_1 is a non-rational function, varying between $w_1 = (1 + (1 + \lambda^2)^{1/2})/\lambda^2$ at $\omega = 0$, and decreasing to $w_1 = \dfrac{(1+\lambda^2)(1+(1+\lambda^2)^{1/2})}{4\lambda^2}$ at $\omega = \infty$.

Figure 1 shows the Nyquist plot of $c = n^*w_1v + m^*w_2u$, where the parameter λ was chosen to be $\lambda = 1$. Here n, m are the normalized coprime factors of g, but for computational simplicity we used the factorization $v = 1$ and $u = k$ for the controller k. The Nyquist plot shows that the winding number of c is positive. By Corollary 9.9 this implies that k is the maximally stabilizing controller of g with respect to the weight $W = \text{diag}(w_1, w_2)$. The coprime factor perturbations that can be tolerated by this control system can therefore be analysed by using Theorem 8.3.

In Fig. 2 the Nyquist plot is shown of the loop gain $-gk$, where again the parameter λ is chosen to be 1. (We have changed our convention of the earlier parts of the paper and are considering positive feedback here, in order for the Nyquist plot to have the usual interpretations.) The dotted lines show the loci of $z = \dfrac{1+|-s|}{|1-(-s)|} = \dfrac{1+|s|}{|1+s|}$ for different values of z. Using plots of such loci it is possible to assess the mixed-sensitivity of a design by finding the locus with highest value of z that intersects with the loop gain $-gk$. The above calculations show that

$$\| \, |S| + |T| \, \|_\infty = \sup_{\omega \in \Re} \frac{|1-gk|}{1+|gk|} = \sqrt{2} + 1 \le \frac{1}{\sqrt{1-\alpha_0^2}}$$
$$= \frac{1}{\sqrt{1-\sigma_1^2}} = \epsilon_{\max}^{-1} = \sqrt{2(2+\sqrt{2})}.$$

This is confirmed by the plot which shows that the locus of $-gk$ is just outside the locus of $z = \dfrac{1+|s|}{|1+s|}$ for $z = \sqrt{2} + 1$.

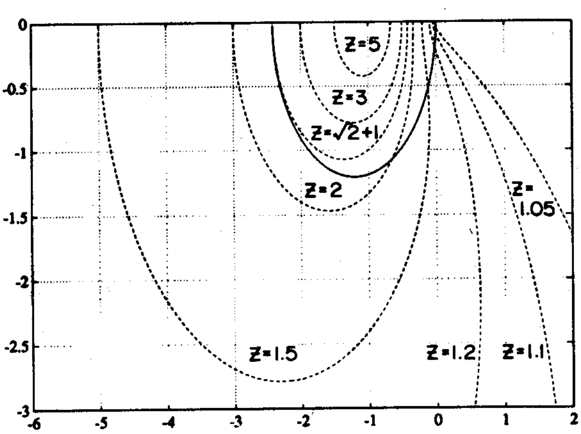

FIG. 2. Nyquist plot of $-gk$ (solid line); loci of $z = \dfrac{1+|s|}{|1+s|}$ (dotted lines).

11. CONCLUSIONS

A geometric approach has been developed to study robust control in the weighted gap metric. Normalized coprime factorizations with respect to weights have been introduced. Maximally stabilizing controllers in the weighted gap metric have been defined and characterized through the solution of a weighted Nehari extension problem.

A coprime factor perturbation model has been introduced. An analytic expression for the largest uncertainty ball that can be tolerated by a W-maximally stabilizing controller was given in terms of the first singular value of a Hankel operator on a suitably defined weighted Hardy space. Given a plant and an arbitrary stabilizing controller the problem was considered under which conditions there exists a weight W such that the controller is the W-maximally stabilizing controller of the plant.

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