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ASYMPTOTIC STABILITY OF INFINITE-DIMENSIONAL DISCRETE-TIME BALANCED REALIZATIONS*

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Abstract. The question of power and asymptotic stability of infinite-dimensional discrete-time state space systems is investigated. It is shown that every balanced realization is asymptotically stable. Conditions are given for balanced, input normal, or output normal realizations to be asymptotically and/or power stable.

Key words. linear infinite-dimensional systems, balanced realizations, stability, Hankel operator

AMS subject classifications. 93B15, 93B20, 93B28, 93D20

1. Introduction. Balanced realizations for finite-dimensional systems have received a great deal of attention. They were introduced as a means of performing model reduction in an easy fashion [7] and have subsequently been used in H^∞ control theory, for example, to evaluate the Hankel norm of a linear system [4], [5]. Recently, they have been used to study parametrization problems of certain sets of linear systems [9].

The elegant results obtained for finite-dimensional balanced systems aroused interest in the problem of the extension of the notion of a balanced realization to infinite-dimensional systems. Glover, Curtain, and Partington [5] derived continuous-time balanced realizations for a class of systems with nuclear Hankel operators. Young [13] developed a very general realization theory for infinite discrete-time systems. Similar results were obtained in the continuous-time case by Ober and Montgomery-Smith [10].

One of the fundamental problems in systems theory is the question of stability of the system. In this paper, we will address this problem in the case of infinite-dimensional balanced realizations and the closely related input and output normal realizations. We show that every balanced realization is asymptotically stable. In general, input normal and output normal realizations do not have the same stability properties as balanced realizations, but we can also give necessary and sufficient conditions for them to be asymptotically and/or power stable. The result is that an input normal or output normal realization is power stable if and only if its transfer function is rational and proper with poles inside the open unit disk, whereas the power stability of a parbalanced realization is more complicated to characterize in terms of the properties of the transfer function.

The approach we take in the proofs of the results is to relate balanced realizations and, in particular, the input and output normal realizations to restricted shift realizations. We start in §2 with the restricted and *-restricted shift realizations and study their connections with Hankel operators, shift operators, and the Douglas–Shapiro–Shields factorizations of analytical functions. In §3, using these connections and the spectral theory of shift operators, we are able to give the above-mentioned necessary and sufficient conditions for the asymptotic and power stability of the output normal

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and input normal realizations. Young [13] established the existence of parbalanced realization for any function $G \in TLD^{U,Y}$. In §4 we prove that parbalanced realizations are always asymptotically stable. We also give examples that show the difficulty to analyze the power stability of a parbalanced realization in connection with its transfer function. A concluding remark is given on how to restrict ourselves to a slightly smaller class of discrete-time transfer functions and linear systems so that we can transpose all our results to the continuous-time case using a bilinear mapping (see [10]).

The following symbols are used:

$\mathbb{D}$	the open unit disk,
$\partial\mathbb{D}$	the unit circle,
$\mathbb{D}_e$	the exterior of $(\partial\mathbb{D}) \cup \mathbb{D}$,
$D_X^{U,Y}$	defined in §2,
$G^{\perp}(z)$	$(1/z)[G(1/z) - G(\infty)]$, $z \in \mathbb{D}$ for $G \in TLD^{U,Y}$,
H_K	the Hankel operator with symbol K,
$H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$	$\{F \mid F : \mathbb{D} \to \mathcal{L}(U,Y)$ analytic and bounded on $\mathbb{D}\}$,
$H^2_Y(\mathbb{D})$	$\{f \mid f : \mathbb{D} \to Y$ analytic on $\mathbb{D}$ and $\sup_{0<r<1} \int_0^{2\pi} \|f(re^{it})\|^2 dt < \infty\}$,
J	$L^2_U(\partial\mathbb{D}) \to L^2_Y(\partial\mathbb{D})$, $(Jf)(z) = f(z^{-1})$,
$\tilde{K}(z)$	$(K(\bar{z}))^*$,
$\mathcal{L}(U,Y)$	$\{A \mid A : U \to Y$ a bounded operator$\}$,
$L^2_Y(\partial\mathbb{D})$	$\{f \mid f : \partial\mathbb{D} \to Y$ square integrable on $\partial\mathbb{D}\}$,
$L^{\infty}_{\mathcal{L}(U,Y)}(\partial\mathbb{D})$	$\{F \mid F : \partial\mathbb{D} \to \mathcal{L}(U,Y)$ measurable and essentially bounded on $\partial\mathbb{D}\}$,
P_+	the orthogonal projection of $L^2_Y(\partial\mathbb{D})$ onto $H^2_Y(\mathbb{D})$,
P_X	the orthogonal projection of $H^2_Y(\mathbb{D})$ onto $X \subseteq H^2_Y(\mathbb{D})$,
S	the forward shift: $(Sf)(z) = zf(z)$ for $f \in H^2_Y(\mathbb{D})$,
S^*	the backward shift: $(S^*f)(z) = z^{-1}[f(z) - f(0)]$ for $f \in H^2_Y(\mathbb{D})$,
$S(Q)$	$P_X S\|_X$, the compression of S to X, where $X = H^2_Y(\mathbb{D}) \ominus (QH^2_Y(\mathbb{D}))$,
$S(Q)^*$	$S^*\|_{H^2_Y(\mathbb{D}) \ominus (QH^2_Y(\mathbb{D}))}$, the restriction of S^* to $H^2_Y(\mathbb{D}) \ominus (QH^2_Y(\mathbb{D}))$,
$\sigma(A)$	the spectrum of an operator A,
$\sigma_p(A)$	the point spectrum of an operator A,
$\sigma(Q)$	the spectrum of an inner function $Q \in H^{\infty}_Y(\mathbb{D})$ (see §3),
$\sigma_s(G)$	the set of points in $\mathbb{C}$ where G has no analytic continuation (see Theorem 3.14),
$TLD^{U,Y}$	defined in §3,
$X \vee Y$	closed linear span of subsets X and Y of a Hilbert space,
$(F,G)_L = I_Y$	F and G are weakly left coprime (see §2),
$(F,G)_R = I_U$	F and G are weakly right coprime (see §2).

2. Hankel operators and shift realizations for discrete-time systems. Our results will be based on the analysis of restricted shift realizations whereby the shift realizations can be analyzed in terms of Hankel operators related to the transfer functions. Here we give a brief summary of some results on Hankel operators and the restricted shift realizations of discrete-time transfer functions. We start with some basic definitions.

Let U, X, and Y be separable Hilbert spaces. The linear systems considered in this paper are of the following form:

$$\left.\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}\right\} k = 0, 1, \ldots,$$

where $u_k \in U$, $x_k \in X$, and $y_k \in Y$. The system operators are assumed to be such that A is a contraction on X, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, and $D \in \mathcal{L}(U,Y)$. This system will be denoted by (A,B,C,D) and the set of all such systems $D_X^{U,Y}$. Unless otherwise stated, the spaces U, X, and Y are assumed to be infinite-dimensional.

For $(A,B,C,D) \in D_X^{U,Y}$, the function $G(z) = C(zI - A)^{-1}B + D$ is called the *transfer function* of (A,B,C,D) and (A,B,C,D) is called a *realization* of G. The *observability operator* $\mathcal{O} : D(\mathcal{O}) \to H_Y^2(\mathbb{D})$ of the system (A,B,C,D) is defined as

$$(\mathcal{O}x)(z) = \sum_{k\geq 0}(CA^k x)z^k$$

for $x \in D(\mathcal{O}) := \{x \in X \mid \sum_{k\geq 0}(CA^k x)z^k \in H_Y^2\}$. If $D(\mathcal{O}) = X$, $\mathcal{O}$ is bounded and $\operatorname{Ker}(\mathcal{O}) = \{0\}$, then the system (A,B,C,D) is said to be *observable*. The *dual system* of (A,B,C,D) is defined to be (A^*,C^*,B^*,D^*), which is, in fact, a realization of the transfer function $\tilde{G}(z) := (G(\bar{z}))^*$. The system (A,B,C,D) is said to be *reachable* if its dual system is observable, and the *reachability operator* $\mathcal{R}$ of (A,B,C,D) is defined to be the adjoint of the observability operator of the dual system. In fact, (A,B,C,D) is reachable if and only if the range of $\mathcal{R} : H_U^2(\mathbb{D}) \to X$ is dense in X, and in this case

$$\mathcal{R}\left(\sum_{k\geq 0} u_k z^k\right) = \sum_{k\geq 0} A^k B u_k, \quad \left(\sum_{k\geq 0} u_k z^k \in H_U^2\right).$$

Note that we define the observability operator $\mathcal{O}$ to have range in $H_Y^2(\mathbb{D})$ instead of l_Y^2. Accordingly the domain of the reachability operator $\mathcal{R}$ is in $H_U^2(\mathbb{D})$ instead of l_U^2. The definitions adapted here are found to be more convenient in our context.

We write $LD_X^{U,Y}$ for the class of reachable and observable systems with state space X. The set of $\mathcal{L}(U,Y)$-valued transfer functions that have reachable and observable realizations is denoted by $TLD^{U,Y}$. Note that $(A,B,C,D) \in LD_X^{U,Y}$ if and only if $(A^*,C^*,B^*,D^*) \in LD_X^{Y,U}$. Correspondingly, $G \in TLD^{U,Y}$ if and only if $\tilde{G} \in TLD^{Y,U}$, where $\tilde{G}(z) = (G(\overline{z}))^*$, $(z \in \mathbb{D}_e)$.

For an observable and reachable system (A,B,C,D) with observability operator $\mathcal{O}$ and reachability operator $\mathcal{R}$, the *observability gramian* is defined to be $\mathcal{M} := \mathcal{O}^*\mathcal{O} : X \to X$, and the *reachability gramian* is $\mathcal{W} := \mathcal{R}\mathcal{R}^* : X \to X$. If $\mathcal{M} = \mathcal{W}$, then the system is said to be *parbalanced*.

Let G be in $TLD^{U,Y}$; i.e., G has a reachable and observable realization $(A,B,C,D) \in LD_X^{U,Y}$ for some state space X. Let $\mathcal{R}$ be the reachability operator and $\mathcal{O}$ the observability operator of the realization. Hence the operator $\mathcal{O}\mathcal{R} : H_U^2(\mathbb{D}) \to H_Y^2(\mathbb{D})$ is bounded. By the fact that

$$G^{\perp}(z) = z^{-1}[G(z^{-1}) - G(\infty)] = C(I - zA)^{-1}B = \sum_{n\geq 0} CA^n Bz^n, \qquad z \in \mathbb{D},$$

it can be verified that, for any polynomial $f(z) = \sum_{k=0}^{n} u_k z^k$, $u_k \in U$,

$$\mathcal{O}\mathcal{R}f = P_+ G^{\perp} Jf,$$

where $(Jg)(z) = g(z^{-1})$ for any $g \in H^2_U(\mathbb{D})$. In this way $P_+G^{\perp}J : H^2_U(\mathbb{D}) \to H^2_Y(\mathbb{D})$ defines a bounded operator. It is called the *Hankel operator with symbol* $G^{\perp}$ and is denoted by $H_{G^{\perp}}$.

Conversely, let G be a $\mathcal{L}(U,Y)$-valued function, analytic on $\mathbb{D}_e$ and at infinity such that the Hankel operator $H_{G^{\perp}} = P_+G^{\perp}J : H^2_U(\mathbb{D}) \to H^2_Y(\mathbb{D})$ is defined for every polynomial $f(z) = \sum_{k=0}^{n} u_k z^k$, $(u_k \in U)$ and can be extended to a bounded operator. Then G has reachable and observable realizations. In fact, G has the restricted shift realization, which was first introduced by Fuhrmann [2] and Helton [6] (see also [13]).

THEOREM 2.1. *Let G be a $\mathcal{L}(U,Y)$-valued function analytic on $\mathbb{D}_e$ and at infinity such that $H_{G^{\perp}} : H^2_U(\mathbb{D}) \to H^2_Y(\mathbb{D})$ defines a bounded operator. Then G has a state space realization (A,B,C,D) with state space X, i.e., for $z \in \mathbb{D}_e$,*

$$G(z) = C(zI - A)^{-1}B + D,$$

which is given in the following way:

The state space X is given by

$$X = \overline{\text{range}} H_{G^{\perp}} \subseteq H^2_Y(\mathbb{D}).$$

The state propagation operator $A :\ X \to X$, the input operator $B :\ U \to X$, the output operator $C :\ X \to Y$ and the feedthrough operator $D :\ U \to Y$ are given by the following, for $f \in X$ and $u \in U$:

$$(Af)(z) := (S^*f)(z) = \frac{f(z) - f(0)}{z},$$
$$(Bu)(z) := G^{\perp}(z)u,$$
$$Cf := f(0),$$
$$Du := G(\infty)u,$$

where S is the (forward) shift operator: $(Sf)(z) = zf(z)$, $f \in H^2_Y(\mathbb{D})$. The realization (A,B,C,D) is called the restricted shift realization *of the transfer function G.*

The following proposition shows that the restricted shift realization is reachable and observable.

PROPOSITION 2.2 (see [3] or [13]). *Assume the notation of Theorem 2.1. Then the system (A,B,C,D) is in $LD_X^{U,Y}$; i.e., it is observable and reachable. The observability operator $\mathcal{O}$ and reachability operator $\mathcal{R}$ of (A,B,C,D) are, respectively, given by*

$$\mathcal{O} = I_X : X \to H^2_Y(\mathbb{D}) \quad \textit{and} \quad \mathcal{R} = H_{G^{\perp}} : H^2_U(\mathbb{D}) \to X.$$

Therefore the class $TLD^{U,Y}$ of transfer functions can be characterized as the set of $\mathcal{L}(U,Y)$-valued functions analytic on $\mathbb{D}_e$ and at infinity such that the Hankel operator $H_{G^{\perp}}$ is bounded. For such transfer functions, the restricted shift realization exists. We emphasize these points by the following corollary.

COROLLARY 2.3. *The following statements are equivalent:*

1. *G is in $TLD^{U,Y}$, i.e. G has a reachable and observable realization in some state space X;*

2. *G has the restricted shift realization that is reachable and observable;*

3. *G is analytic on $\mathbb{D}_e$ and at infinity such that the Hankel operator $H_{G^\perp}$ is bounded.*

Note that, if $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$, then $G \in TLD^{U,Y}$, since, in this case, $H_{G^\perp}$ is bounded.

As a next step, we construct another realization, which is the dual realization of the restricted shift realization. Let G be in $TLD^{U,Y}$. Then $\tilde{G}(z) = (G(\bar{z}))^*$, $(z \in \mathbb{D}_e)$ is in $TLD^{Y,U}$. Moreover , if $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is the restricted shift realization of $\tilde{G}$, then the dual system $(\tilde{A}^*, \tilde{C}^*, \tilde{B}^*, \tilde{D}^*)$ is a realization of G, called the **-restricted shift realization* of G.

A concrete representation of the *-restricted shift realization can be obtained.

THEOREM 2.4. *Let G be in $TLD^{U,Y}$. The state space representation (A_*, B_*, C_*, D_*) of the *-restricted shift realization is given by the following:*

The state space X_* is $X_* = \overline{\text{range}} H_{\tilde{G}^\perp}$, where

$$\tilde{G}^\perp(z) = (G^\perp(\bar{z}))^*.$$

The operators A_*, B_*, C_*, and D_* are defined as

$$\begin{gathered} A_* = P_{X_*} S_{|X_*}, \\ B_* u = P_{X_*} u, \quad (u \in U) \\ C_* f = (H_{G^\perp} f)(0), \quad (f \in X_*), \\ D_* = G(\infty), \end{gathered}$$

where P_{X_*} is the orthogonal projection of $H^2_U(\mathbb{D})$ onto X_*, and the space U is considered as the subspace $\{u + 0z + 0z^2 + 0z^3 + \cdots \mid u \in U\}$ of $H^2_U(\mathbb{D})$.

The system (A^*, C^*, B^*, D^*) is observable and reachable. The reachability and observability operators $\mathcal{R}_*$ and $\mathcal{O}_*$ are, respectively, given by

$$\mathcal{R}_* = P_{X_*} : \ H^2_U(\mathbb{D}) \to X_* \quad \text{and} \quad \mathcal{O}_* = H^*_{\tilde{G}^\perp}|_{X_*} = H_{G^\perp}|_{X_*}.$$

Proof. Replacing G by $\tilde{G}$ in Theorem 2.1, we obtain the restricted shift realization of $\tilde{G}$, and the dual of this realization is the *-restricted shift realization stated in the theorem. Here we just verify the formula for the output operator C_*, which is the adjoint of the input operator $\tilde{B}$ of the restricted shift realization of $\tilde{G}$. Hence $C_* = \tilde{B}^*$. So by Theorem 2.1 we have

$$C_*^* y = \tilde{B} y = \tilde{G}^\perp y \in X_*, \qquad y \in Y.$$

From this, we obtain that, for $f \in X_* \subseteq H^2_U(\mathbb{D})$ and $y \in Y$,

$$\begin{aligned} \langle C_* f, \ y \rangle_Y &= \langle f, \ C_*^* y \rangle_{H^2_U(\mathbb{D})} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), \ \tilde{G}^\perp(e^{it}) y \rangle_U dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle \left(\tilde{G}^\perp(e^{it})\right)^* f(e^{it}), \ y \rangle_U dt \\ &= \langle \frac{1}{2\pi} \int_0^{2\pi} \left(\tilde{G}^\perp(e^{it})\right)^* f(e^{it}) dt, \ y \rangle_U. \end{aligned}$$

Hence, using a change of variable $z = e^{-it}$, we have

$$\begin{aligned} C_* f &= \frac{1}{2\pi} \int_0^{2\pi} \left(\tilde{G}^{\perp}(e^{it})\right)^* f(e^{it}) dt \\ &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \bar{z} (\tilde{G}^{\perp}(\bar{z}))^* f(\bar{z}) dz \\ &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \bar{z} G^{\perp}(z) f(\bar{z}) dz. \end{aligned}$$

Note that the last integral is the zeroth Fourier coefficient of $G^{\perp}(z)f(\bar{z})$, which is the same as the zeroth Fourier coefficient of $P_+ G^{\perp}(z) f(\bar{z}) = H_{G^{\perp}} f$. This is $(H_{G^{\perp}} f)(0)$, since $H_{G^{\perp}} f \in H^2_Y(\mathbb{D})$. □

From these results, we see that the state space for the restricted shift realization is given as the closed range of the Hankel operator whose symbol is the transfer function mapped to the unit disk. The state propagation operator is just the backward shift restricted to the state space. For the *-restricted shift realization, the state space is also the closed range of a Hankel operator, while the state propagation operator is the forward shift compressed to this state space. It is well known and readily verified that the closure of the range of a Hankel operator H_G is the orthogonal complement of a right invariant subspace of $H^2_Y(\mathbb{D})$ [3], [8]. A vector-valued version of Beurling's theorem (see, e.g., [3, Thm. 12.22, p. 186]) asserts that a right invariant space in $H^2_Y(\mathbb{D})$ can only be either the trivial space $\{0\}$ or $QH^2_Y(\mathbb{D})$, where $Q \in H^{\infty}_{\mathcal{L}(Y)}(\mathbb{D})$ is such that $\|Q\|_{\infty} \leq 1$ and $Q(e^{it})$ is for almost every $t \in [0, 2\pi)$ a partial isometry with a fixed nonzero initial space. Such a function Q is called a *rigid* function. A rigid function Q is called *inner* if $Q(e^{it})$ is a unitary operator for almost all $t \in [0, 2\pi)$.

This discussion leads to the cyclicity of functions defined as follows (see [3]).

DEFINITION 2.1. Let $G \in TLD^{U,Y}$. Then $G^{\perp}$ is called

1. *cyclic* if $(\text{range} H_{G^{\perp}})^{\perp} = \{0\}$,
2. *noncyclic* if $(\text{range} H_{G^{\perp}})^{\perp} = QH^2_Y(\mathbb{D})$ for some rigid function $Q \in H^{\infty}_{\mathcal{L}(Y)}(\mathbb{D})$,
3. *strictly noncyclic* if $(\text{range} H_{G^{\perp}})^{\perp} = QH^2_Y(\mathbb{D})$ for some inner function $Q \in H^{\infty}_{\mathcal{L}(Y)}(\mathbb{D})$.

It should be noted that the inner function Q in statement 3 of the above definition is unique up to right multiplication by a constant unitary operator on Y. Also, if $G^{\perp}$ is scalar, then $G^{\perp}$ is noncyclic if and only if it is strictly noncyclic. It is important to have characterizations for matrix-valued functions to be strictly noncyclic. To this end, we introduce some definitions. Let K be in $H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$. The function $\hat{K}$ defined on $\mathbb{D}_e$ with values in $\mathcal{L}(U,Y)$ is called a *meromorphic pseudocontinuation of bounded type* of K if $\hat{K}$ is *of bounded type*, i.e.,

$$\hat{K}(z) = \frac{F(z)}{h(z)}, \qquad z \in \mathbb{D}_e,$$

where F is a $\mathcal{L}(U,Y)$-valued function and h is a scalar-valued function, both bounded and analytic in $\mathbb{D}_e$; K and $\hat{K}$ have the same strong radial limits on $\partial\mathbb{D}$.

Let $F_1 \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$ and $F_2 \in H^{\infty}_{\mathcal{L}(Z,Y)}(\mathbb{D})$. We say that F_1 and F_2 are *left weakly coprime* and write

$$(F_1, F_2)_L = I_Y$$

if $F_1 H^2_U(\mathbb{D}) \vee F_2 H^2_Z(\mathbb{D}) = H^2_Y(\mathbb{D})$, where $\vee$ stands for the closed linear span.

Analogously, we say that $K_1 \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ and $K_2 \in H^\infty_{\mathcal{L}(U,Z)}(\mathbb{D})$ are *weakly right coprime* and write $(K_1, K_2)_R = I_U$ if $\tilde{K}_1$ and $\tilde{K}_2$ are weakly left coprime.

Using these notations, we have the following theorem ([3, Thm. 3.5, p. 254]).

THEOREM 2.5. *For $K \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ with U and Y finite-dimensional, the following statements are equivalent:*

1. *K is strictly noncyclic,*
2. *On $\partial\mathbb{D}$ the function K can be factored as*

$$K = Q_1(zF_1)^* = (zF_2)^*Q_2.$$

Q_1 and Q_2 are inner functions in $H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ and $H^\infty_{\mathcal{L}(U)}(\mathbb{D})$, respectively. The functions F_1 and F_2 are in $H^\infty_{\mathcal{L}(Y,U)}(\mathbb{D})$ and in $H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$, respectively, and the coprimeness conditions $(Q_1, F_1)_R = I_Y, (Q_2, F_2)_L = I_U$ hold. Here Q_1 (respectively, Q_2) is unique up to right (resepctively, left) multiplication by a constant unitary operator,

3. *K has a meromorphic pseudocontinuation of bounded type on $\mathbb{D}_e$.*

If statement 2 holds, then $Q_1 H^2_Y(\mathbb{D}) = (\text{range} H_K)^\perp$ and $\tilde{Q}_2 H^2_U(\mathbb{D}) = (\text{range} H_{\tilde{K}})^\perp$.

We will call the factorization of K in the theorem the Douglas–Shapiro–Shields factorization. In fact, this is the generalization due to Fuhrmann [3] of the result on scalar functions of Douglas, Shapiro, and Shields [1].

By Theorem 2.5, we immediately have the following corollary.

COROLLARY 2.6. *In the notation of the theorem with U and Y finite-dimensional, K is strictly noncyclic if and only if $\tilde{K}$ is strictly noncyclic.* □

From Theorems 2.1, 2.4, 2.5, and Definition 2.1, we see that the state space of a restricted shift realization of a transfer function G is the orthogonal complement of an invariant subspace, which is characterized by a rigid function Q. The state propagation operator A is the backward shift S^* restricted to the state space $(QH^2_Y(\mathbb{D}))^\perp$, i.e., $A = S^*_{|(QH^2_Y(\mathbb{D}))^\perp}$, which we will denote by $S(Q)^*$. One of the important points in our context is that the function Q can be determined from the transfer function G, if $G^\perp$ is strictly noncyclic.

For the *-restricted shift realization, the state space can be determined in a similar way to the derivation of the restricted shift realization. In this case, the state propagation operator is the forward shift operator S compressed to the orthogonal complement of an invariant subspace that is determined by a rigid function Q_*, i.e., $P_{(Q_* H^2_U(\mathbb{D}))^\perp} S_{|(Q_* H^2_U(\mathbb{D}))^\perp}$, which we denote by $S(Q_*)$.

We summarize these results in the following proposition.

PROPOSITION 2.7. *Let G be in $TLD^{U,Y}$ with U and Y finite-dimensional and let $(A, B, C, D) \in LD^{U,Y}_X$ be its restricted shift realization and $(A_*, B_*, C_*, D_*) \in LD^{U,Y}_{X_*}$ its *-restricted shift realization. Then*

1. *If $G^\perp$ is cyclic we have that* (a) *$A = S^*$ and $X = H^2_Y(\mathbb{D})$, and* (b) *$A_* = S$ and $X_* = H^2_U(\mathbb{D})$;*
2. *If $G^\perp$ is noncyclic, we have that* (a) *$A = S(Q)^*$, where $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ is a rigid function such that*

$$X = \overline{\text{range}} H_{G^\perp} = (QH^2_Y(\mathbb{D}))^\perp.$$

If $G^\perp$ is in $H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ and is strictly noncyclic with factorization $G^\perp = Q_1(zF_1)^*$, where $Q_1 \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ is inner and $F_1 \in H^\infty_{\mathcal{L}(Y,U)}(\mathbb{D})$ such that $(Q_1, F_1)_R = I_Y$, then $Q = Q_1 V_1$ for some unitary operator V_1 on Y, and (b) *$A_* = P_{X_*} S_{|X_*} = S(Q_*)$, where $Q_* \in H^\infty_{\mathcal{L}(U)}(\mathbb{D})$ is a rigid function such that*

$$X_* = \overline{\text{range}} H_{\tilde{G}^\perp} = (Q_* H^2_U(\mathbb{D}))^\perp.$$

If $G^{\perp}$ *is in* $H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$ *and is strictly noncyclic and* $\tilde{G}^{\perp}$ *has a factorization* $\tilde{G}^{\perp} = Q_2(zF_2)^*$, *where* $Q_2 \in H^{\infty}_{\mathcal{L}(U)}(\mathbb{D})$ *is inner and* $F_2 \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$ *such that* $(Q_2, F_2)_L = I_U$, *then* $Q_* = Q_2V_2$ *for some unitary operator* V_2 *on* U.

Proof. The proposition follows from Theorems 2.1, 2.4, 2.5, and Definition 2.1. □

3. Stability and spectral minimality of input normal and output normal realizations. In this section, we discuss the stability and questions of spectral minimality of input normal and output normal realizations using the results on restricted and *-restricted shift realizations studied in §2.

The following definition recalls the notion of an input normal and output normal system as defined by Moore [7] for finite-dimensional state space realizations. The definitions in the infinite-dimensional case are natural extensions of the finite-dimensional notions (see, e.g., [13]).

DEFINITION 3.1. Let (A, B, C, D) be in $LD_X^{U,Y}$. Then the system is

1. *output normal* if $\mathcal{M} = I$,
2. *input normal* if $\mathcal{W} = I$,
3. *parbalanced* if $\mathcal{M} = \mathcal{W}$,
4. *balanced* if $\mathcal{M} = \mathcal{W}$ and there is an orthonormal basis of the state space with respect to which $\mathcal{M}$ (and hence $\mathcal{W}$) has a diagonal matrix representation.

From our results on the restricted and the *-restricted shift realization we immediately have examples for input and output normal realizations.

PROPOSITION 3.1. *Let* $G \in TLD^{U,Y}$. *Then the restricted shift realization is output normal whereas the *-restricted shift realization is input normal.*

Proof. The proof follows from Proposition 2.2 and Theorem 2.4 □

Next, we quote a result that establishes a reachable output-normal realization of a transfer function is unitarily equivalent to its restricted shift realization

Two systems $(A_1, B_1, C_1, D_1) \in D_{X_1}^{U,Y}$ and $(A_2, B_2, C_2, D_2) \in D_{X_2}^{U,Y}$ are called *equivalent* (*unitarily equivalent*) if there exists a bounded and boundedly invertible operator (a unitary operator) V mapping the state space X_1 onto the state space X_2, such that

$$(A_1, B_1, C_1, D_1) = (V^{-1}A_2V, V^{-1}B_2, C_2V, D_2).$$

In this case, V is called an equivalence (unitary) transformation.

THEOREM 3.2 (see [13]). *Let* $(A_1, B_1, C_1, D_1) \in LD_{X_1}^{U,Y}$ *and* $(A_2, B_2, C_2, D_2) \in LD_{X_2}^{U,Y}$ *be two output normal realizations of a transfer function in* $TLD^{U,Y}$. *Then* (A_1, B_1, C_1, D_1) *and* (A_2, B_2, C_2, D_2) *are unitarily equivalent.*

By a duality argument, we have as a corollary that the same result holds for input normal realizations; i.e., an input normal realization is unitarily equivalent to the *-restricted shift realization.

COROLLARY 3.3. *Let* $(A_1, B_1, C_1, D_1) \in LD_{X_1}^{U,Y}$ *and* $(A_2, B_2, C_2, D_2) \in LD_{X_2}^{U,Y}$ *be two input normal realizations of a transfer function in* $TLD^{U,Y}$. *Then* (A_1, B_1, C_1, D_1) *and* (A_2, B_2, C_2, D_2) *are unitarily equivalent.*

We now turn to the study of stability. We introduce a classification of contractions according to their stability properties [12], which will simplify our notation.

DEFINITION 3.2. Let T be a contraction on the Hilbert space H. Then

1. $T \in C_{0\cdot}$ if $\lim_{n\to\infty} T^n h = 0$, for all $h \in H$,
2. $T \in C_{\cdot 0}$ if $\lim_{n\to\infty} (T^*)^n h = 0$, for all $h \in H$,
3. $T \in C_{1\cdot}$ if $\lim_{n\to\infty} T^n h \neq 0$, for all $h \in H$, $h \neq 0$,

4. $T \in C_{\cdot 1}$ if $\lim_{n\to\infty}(T^*)^n h \neq 0$, for all $h \in H$, $h \neq 0$.

We further set $C_{ij} = C_{i\cdot} \cap C_{\cdot j}$, $i, j = 0, 1$.

Now we define the two notions of stability we will consider in the remainder of the paper.

DEFINITION 3.3. A discrete time system $(A, B, C, D) \in D_X^{U,Y}$ or the state propagation operator A is called

1. *asymptotically stable* if for every $x \in X$,

$$A^k x \to 0 \quad \text{as } k \to \infty,$$

i.e., if A is of class $C_{0\cdot}$,

2. *power stable* if $r < 1$, where

$$r := \inf\{\bar{r} \mid \text{there is } M_{\bar{r}} > 0 \text{ such that } \|A^k\| \leq M_{\bar{r}}\bar{r}^k, k \geq 0\}.$$

The number r is called the *degree of power stability.*

It is easy to see that stability and observability, as well as reachability properties of discrete time systems, are preserved under equivalence transformations, whereas input and output normality are preserved under unitary equivalence. Moreover, two equivalent power stable systems have the same degree of power stability.

Therefore, by Theorem 3.2 and its corollary, we can establish all stability and other important results concerning input normal and output normal realizations by restricting ourselves to *-restricted and restricted shift realizations. Henceforth, when we prove statements about input normal or output normal reachable and observable realizations, we must only prove them in the case of restricted or *-restricted realization.

From Proposition 2.7, we can see that the study of stability and spectral properties of the restricted and *-restricted realizations reduces to the study of the operators $S(Q)^*$ and $S(Q_*)$, where Q and Q_* are rigid functions. We will need the following lemma (see [8, Cor., p. 43]).

LEMMA 3.4. *Let $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ be a rigid function. Denote by P_X the projection on $X := (QH^2_Y(\mathbb{D}))^\perp$. Then, for $f \in H^2_Y(\mathbb{D})$, $\lim_{n\to\infty} \|P_X S^n f\|^2 = \|f\|^2 - \|Q^* f\|^2$.*

The following theorem shows that an output normal realization of a transfer function in $TLD^{U,Y}$ is always asymptotically stable.

THEOREM 3.5. *Let $G \in TLD^{U,Y}$ and let (A, B, C, D) be an output normal reachable realization of G. Then*

1. *$A \in C_{0\cdot}$; i.e., A is asymptotically stable,*
2. *$A \in C_{00}$ if $G^\perp$ is strictly noncyclic,*
3. *$A \in C_{01}$ if $G^\perp$ is cyclic.*

Proof. By Proposition 3.1, we can assume without loss of generality that (A, B, C, D) is the restricted shift realization.

1. The state propagation operator A of the restricted shift realization is the restriction of the backward shift to a subspace of $H^2_Y(\mathbb{D})$. The backward shift S^* is such that for every $x_0 \in H^2_Y(\mathbb{D})$,

$$(S^*)^k x_0 \to 0, \quad \text{as } k \to \infty.$$

This immediately implies statement 1.

2. This follows from Proposition 2.7 and Lemma 3.4.

3. If $G^\perp$ is cyclic, then A is the backward shift S^* on the space $H^2_Y(\mathbb{D})$, and therefore $A \in C_{01}$. □

In the case of input normal realizations, the situation is, however, such that we cannot, in general, expect that the realization is asymptotically stable, since the state propagation operator of the *-restricted shift realization is the forward shift operator, compressed to a subspace of $H^2_U(\mathbb{D})$. The forward shift on $H^2_U(\mathbb{D})$ is not asymptotically stable. The following corollary states that, at least for an important class of transfer functions, input normal realizations are asymptotically stable.

COROLLARY 3.6. *Let $G \in TLD^{U,Y}$ and let (A, B, C, D) be an input normal observable realization of G. Then*

1. $A \in C_{\cdot 0}$,
2. $A \in C_{00}$ *if* $\tilde{G}^{\perp}$ *is strictly noncyclic,*
3. $A \in C_{10}$ *if* $\tilde{G}^{\perp}$ *is cyclic.*

Proof. Let (A, B, C, D) be the *-restricted realization of G. Recall that by definition (A, B, C, D) is the dual system of the restricted shift realization of $\tilde{G}$. Hence the result follows by duality from Theorem 3.5 □

We now proceed to power stability. The following result gives a characterization of power stability (see, e.g., Przyluski [11]).

PROPOSITION 3.7. *Let T be a contraction. Then the spectral radius $r(T)$ of T, i.e.,*

$$r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\},$$

is given by

$$r(T) = \inf\{0 \le \bar{r} \le 1 \mid \text{there exists } M_{\bar{r}} \ge 0 \text{ such that } \|T^k\| \le M_{\bar{r}}\bar{r}^k, k \ge 0\}.$$

Hence, if T is power-stable, then the degree of power stability equals the spectral radius.

Proof. The proof follows from an application of the well-known formula

$$\sup\{|\lambda| \mid \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}. \qquad \square$$

To establish whether the output normal and input normal realizations are power-stable, it is therefore important to determine the spectral radius of its state propagation operator. To this end, we must introduce C_0 contractions, which play an important role in the theory of contractive operators. C_0 contractions are defined via the H^∞ calculus for contractions and are a special class of completely nonunitary contractions (see [12]). Specifically, a contraction T on a Hilbert space H is *completely nonunitary* if there is no subspace $V \subseteq H$ such that $TV = V$ and $T|_V$ is unitary. For such T, the operator $u(T) := \lim_{\substack{r<1 \\ r\to 1}} u(rT)$ is a well-defined bounded operator for any $u \in H^\infty$ and satisfies $\|u(T)\| \le \|u\|_{H^\infty}$. In particular, $u(T)$ is a contraction if u is an inner function.

A completely nonunitary contraction is a C_0 *contraction* if there exists an inner function m such that $m(T) = 0$. The least common divisor of all such inner functions is called the *minimal function* m_T of T. For the minimal function m_T, we also have that $m_T(T) = 0$. Therefore the minimal function of a C_0 contraction can be seen to be a generalization of the minimal polynomials for matrices.

As in the case of matrices, the spectrum of C_0 operators is given by the "zeros" of the minimal function in the following sense. We define the *spectrum* $\sigma(Q)$ of an inner function $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ to be

$$\sigma(Q) = \left\{ \lambda \in \overline{\mathbb{D}} \mid \lim_{\substack{\delta>0 \\ \delta\to 0}} \inf_{\substack{\xi\in\mathbb{D} \\ |\xi-\lambda|<\delta}} \inf_{\substack{\|y\|=1 \\ y\in Y}} \|Q(\xi)y\| = 0 \right\}.$$

Then we have the following proposition (see [8, p. 75]),

PROPOSITION 3.8. *If T is a C_0 operator, then*

$$\sigma(T) = \sigma(m_T) \quad \text{and} \quad \sigma_p(T) = \sigma(m_T) \cap \mathbb{D}.$$

Now we use these results to analyze the spectrum of the operators $S(Q)$ and $S(Q)^*$. First the following proposition [8, p. 73] shows when $S(Q)$ is a C_0 contraction.

PROPOSITION 3.9. *If $\dim(U) < \infty$ and $Q \in H^\infty_{\mathcal{L}(U)}(\mathbb{D})$ is a rigid function, then the determinant $d = \det(Q)$ is such that $d(S(Q)) = 0$. Therefore, when Q is an inner function and U has finite dimension, the operator $S(Q)$ is a C_0 contraction.*

In fact, $S(Q)$ and $S(Q)^*$ are both C_0 contractions when Q is inner and U is finite dimensional, as shown by the following result (see [3, Thm, 13.2, p. 191] or [8, p. 75]).

PROPOSITION 3.10. *For a given inner function $Q \in H^\infty_{\mathcal{L}(U)}(\mathbb{D})$, the operators $S(Q)$ and $S(\tilde{Q})^*$ are unitarily equivalent.*

More precisely, $S(Q) = \tau_Q^{-1} S(\tilde{Q})^* \tau_Q$, where the unitary operator τ_Q is given by

$$\tau_Q : \begin{array}{ccc} L^2_U(\partial\mathbb{D}) & \rightarrow & L^2_U(\partial\mathbb{D}) \\ f & \mapsto & e^{-it}\tilde{Q}Jf. \end{array}$$

One of the important results in the theory of the backward shift operator $S(Q)^*$ restricted to an invariant subspace is that its spectrum can be completely characterized by the associated inner function Q. Note that, if σ is a set of complex numbers, then σ^* is used to denote the set of the complex conjugates of the elements in σ.

THEOREM 3.11 (see [8, p. 75]). *The following statements hold:*

1. (a) *Let S^* be the backward shift on $H^2_Y(\mathbb{D})$. Then*

$$\sigma(S^*) = \overline{\mathbb{D}}, \qquad \sigma_p(S^*) = \mathbb{D},$$

(b) *Let S be the forward shift on $H^2_Y(\mathbb{D})$. Then*

$$\sigma(S) = \overline{\mathbb{D}}, \qquad \sigma_p(S) = \emptyset;$$

2. *Let Q be an inner function in $H^\infty_Y(\mathbb{D})$ with Y finite dimensional. Then*

(a)

$$\sigma(S(Q)^*) = \sigma(Q)^* = \sigma(m_{S(Q)^*}),$$

$$\sigma_p(S(Q)^*) = \sigma(S(Q)^*) \cap \mathbb{D} = \{\overline{\lambda} \in \mathbb{D} |\ \mathrm{Ker} Q(\lambda)^* \neq \{0\}\},$$

(b)

$$\sigma(S(Q)) = \sigma(Q) = \sigma(m_{S(Q)}),$$

$$\sigma_p(S(Q)) = \sigma(S(Q)) \cap \mathbb{D} = \{\lambda \in \mathbb{D} |\ \mathrm{Ker} Q(\lambda) \neq \{0\}\}.$$

The next result shows that we must only be concerned with inner functions if we are interested in the case when the spectral radius of the restricted backward shift is less than 1 (see [3, p. 194]).

THEOREM 3.12. *Let U be finite-dimensional and Q a rigid function that is not inner. Then $\sigma_p(S(Q)^*)$ is equal to the open unit disk $\mathbb{D}$.*

In terms of the restricted and *-restricted shift realizations, Theorems 3.11 and 3.12 can be translated into the following result.

PROPOSITION 3.13. *Let* (A, B, C, D) *and* (A_*, B_*, C_*, D_*) *be, respectively, the restricted and *-restricted shift realizations of a transfer function* $G \in TLD^{U,Y}$ *where* U *and* Y *have finite dimensions.*

1. *If* $G^\perp$ *is cyclic, then* $\sigma(A) = \overline{\mathbb{D}}$, $\sigma_p(A) = \mathbb{D}$ *and* $\sigma(A_*) = \overline{\mathbb{D}}$, *and* $\sigma_p(A) = \emptyset$.
2. *If* $G^\perp$ *is noncyclic but not strictly noncyclic, then* $\sigma(A) = \overline{\mathbb{D}}$, $\sigma(A_*) = \overline{\mathbb{D}}$.
3. *If* $G^\perp$ *is in* $H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ *and is strictly noncyclic with factorization* $G^\perp = Q_1(zF_1)^*$ *and* $\tilde{G}^\perp = Q_2(zF_2)^*$, *where* $Q_1 \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ *and* $Q_2 \in H^\infty_{\mathcal{L}(U)}(\mathbb{D})$ *are inner, and where* Q_1 *and* $F_1 \in H^\infty_{\mathcal{L}(Y,U)}(\mathbb{D})$ *are right weakly coprime, and* Q_2 *and* $F_2 \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ *are also right weakly coprime, then*

$$\sigma(A) = \sigma(Q_1)^* = \sigma(m_A),$$

$$\sigma_p(A) = \sigma(Q_1)^* \cap \mathbb{D} = \{\overline{\lambda} \in \mathbb{D} |\ \mathrm{Ker} Q_1(\lambda)^* \neq \{0\}\}$$

and

$$\sigma(A_*) = \sigma(Q_2) = \sigma(m_{A_*}),$$

$$\sigma_p(A_2) = \sigma(Q_2) \cap \mathbb{D} = \{\lambda \in \mathbb{D} |\ \mathrm{Ker} Q_2(\lambda) \neq \{0\}\}.$$

Proof. The proposition follows from Theorems 3.11 and 3.12 and Proposition 2.7. □

A very important property of finite-dimensional systems is that the eigenvalues of the state propagation matrix correspond exactly to the poles of the transfer function. For infinite-dimensional systems, it is desirable to have the analogous property. This was shown to be true for strictly noncyclic transfer functions by Fuhrmann ([3, Chap. III]).

DEFINITION 3.4. Let $G \in TLD^{U,Y}$ be such that $G^\perp$ has a meromorphic pseudocontinuation of bounded type on $\mathbb{D}_e$. Then we extend the definition of $G^\perp$ onto $\mathbb{D}_e$ to be this unique meromorphic pseudocontinuation and hence define G on $\mathbb{D}$. The set $\sigma_s(G)$ is defined to be the set of points z such that the extended G cannot be analytically continued to z.

A realization $(A, B, C, D) \in LD^{U,Y}_X$ of G is said to be *spectrally minimal* if $\sigma(A) = \sigma_s(G)$.

We note that a more general definition of spectral minimality can be made for a larger class of transfer functions (see [3]). However, the definition suffices for our discussion here. It turns out that, if $G \in TLD^{U,Y}$ is strictly noncyclic, then both the restricted and *-restricted shift realizations are spectrally minimal.

THEOREM 3.14. *Let* G *be in* $TLD^{U,Y}$, *where* U *and* Y *have finite dimensions. If* $G^\perp$ *is in* $H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ *and is strictly noncyclic, then*

1. *Every output normal realization* (A, B, C, D) *is spectrally minimal, i.e.,*

$$\sigma(A) = \sigma_s(G),$$

2. *Every input normal realization* (A_*, B_*, C_*, D_*) *is spectrally minimal.*

Proof. 1. See, [3, Thm. 4.11, p. 267] for the case of the restricted shift realization.

2. Without loss of generality, we assume (A, B, C, D) is the *-restricted shift realization of G.

Recall from Corollary 2.6 that $G^\perp$ is strictly noncyclic if and only if $\tilde{G}^\perp$ is strictly noncyclic. By statement 1 and the construction of the *-restricted shift realization, we have that $\sigma(A_*^*) = \sigma_s(\tilde{G})$. Since

$$\sigma_s(G) = (\sigma_s(\tilde{G}))^* = \sigma(A_*^*)^* = \sigma(A_*),$$

we have the spectral minimality of the *-restricted shift realization. □

We now show that an input normal or output normal system is power-stable if and only if it is finite-dimensional.

THEOREM 3.15. *Let G be in $TLD^{U,Y}$ such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ and let U and Y be finite-dimensional. Then an output normal (respectively, input normal) realization of G is power-stable if and only if G is rational.*

Proof. Let G be rational. Since $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$, there is a number $0 < r < 1$ such that the poles of G are contained in the set $\{\lambda : |\lambda| \leq r\}$. Being rational, $G^\perp$ has a meromorphic pseudocontinuation of bounded type and hence by Theorem 2.5 is strictly noncyclic. Theorem 3.14 then implies that the propagation operator of any output normal (respectively, input normal) realization of G has spectral radius less than 1. Now Proposition 3.7 shows that it is power-stable.

Conversely, assume that an output normal realization of G is power stable. Then the restricted shift realization is power stable. Let A be its propagation operator. We have by Proposition 3.7 that $r(A)$, the spectral radius of A, is less than 1. By Proposition 3.13, this implies that $G^\perp$ is strictly noncyclic. Then, by Theorem 3.14, any output normal realization of G is spectrally minimal, and hence G can be analytically continued across $\partial\mathbb{D}$. Being strictly noncyclic, $G^\perp$ has a meromorphic pseudocontinuation on $\mathbb{D}_e$, and thus G has a meromorphic pseudocontinuation on $\mathbb{D}$. Since a meromorphic pseudocontinuation is unique, the pseudocontinuation is an analytic continuation. Thus G is a meromorphic function on the extended complex plane. Hence it is rational.

If an input normal realization is assumed to be power-stable, a similar argument will also show that G is rational. □

4. Balanced realizations. This section is devoted to the study of the stability properties of balanced realizations with infinite-dimensional state space.

Balanced realizations of finite-dimensional systems have played an important role in model reduction and Hankel norm approximation of linear systems [7], [4]. In finite dimensions, it is straightforward to construct a balanced realization from input normal or output normal realizations. In infinite dimensions, it is not trivial to guarantee that this can be done, since the state space transformation that is involved, in general, has an unbounded inverse. That this is nevertheless possible was shown by Young [13]. Note that, in the following theorem, the subscripts o and i signify output normal and input normal realizations, respectively.

THEOREM 4.1. *Let $G \in TLD^{U,Y}$. Let (A_o, B_o, C_o, D_o) be the restricted shift realization of G with state space $X_o = \overline{\text{range}} H_{G^\perp}$ and let (A_i, B_i, C_i, D_i) be the *-restricted realization with state space $X_i = \overline{\text{range}} H_{\tilde{G}^\perp}$. Set*

$$\mathcal{W}_o = H_{G^\perp} H^*_{G^\perp}|_{X_o}, \qquad \mathcal{M}_i = H_{\tilde{G}^\perp} H^*_{\tilde{G}^\perp}|_{X_i}.$$

1. *There exist parbalanced realizations* $(A_{b1}, B_{b1}, C_{b1}, D_{b1}) \in D_{X_o}^{U,Y}$ *and* $(A_{b2}, B_{b2}, C_{b2}, D_{b2}) \in D_{X_i}^{U,Y}$ *of* G *that satisfy*

$$\mathcal{W}_o^{1/4} A_{b1} = A_o \mathcal{W}_o^{1/4}, \qquad \mathcal{W}_o^{1/4} B_{b1} = B_o,$$
$$C_{b1} = C_o \mathcal{W}_o^{1/4}, \qquad D_{b1} = D_o$$

and

$$A_{b2} \mathcal{M}_i^{1/4} = \mathcal{M}_i^{1/4} A_i, \qquad B_{b2} = \mathcal{M}_i^{1/4} B_i,$$
$$C_{b2} \mathcal{M}_i^{1/4} = C_i, \qquad D_{b2} = D_i.$$

2. *All parbalanced realizations of* G *are unitarily equivalent.*

3. *If* $G \in TLD^{U,Y}$ *is continuous on* $\partial\mathbb{D}$ *with values in the set of compact operators, then there exists a balanced realization whose state space is equal to the closure of the range of the Hankel operator with symbol* $G^{\perp}$. *The gramian has a matrix representation with respect to a basis such that its diagonal entries are the singular values of the Hankel operator with symbol* $G^{\perp}$.

Proof. Statements 2 and 3 and the existence of the first parbalanced realization of statement 1 can be found in [13]. The second realization of statement 1 can be obtained by taking the dual of the parbalanced realization of $\tilde{G}$ constructed by the method of the first realization. □

We have the following proposition concerning the transformation from the restricted (*-restricted) shift realization to the parbalanced realization in Theorem 4.1.

PROPOSITION 4.2. *In the notation of Theorem* 4.1, *the operators* $\mathcal{W}_o^{1/2}$ *and* $\mathcal{W}_o^{1/4}$ *are bounded positive definite with dense ranges in* X_o; *the operators* $\mathcal{M}_i^{1/2}$ *and* $\mathcal{M}_i^{1/4}$ *are bounded positive definite with dense ranges in* X_i.

Proof. Clearly, $\mathcal{W}_o$ is a bounded positive definite operator on X_o. Similarly, $\mathcal{M}_i$ is a positive definite operator on X_i. Since

$$\mathcal{W}_o^{1/2} (\mathcal{W}_o^{1/2})^* = H_{G^{\perp}} H_{G^{\perp}}^*$$

and $H_{G^{\perp}} H_{G^{\perp}}^* X_o$ is dense in X_o, $\mathcal{W}_o^{1/2}$ has dense range in X_o. Hence so does $\mathcal{W}_o^{1/4}$. Similarly, $\mathcal{M}_i^{1/2}$ and $\mathcal{M}_i^{1/4}$ also have dense ranges. □

Combining Theorem 4.1 and Proposition 4.2, we have, in the terminology of [12], that A_{b1} is a *quasi-affine transform* of A_o and A_i a quasi-affine transform of A_{b2}.

Theorem 4.1 has some by-products that may be of interest in their own right. First, since two parbalanced realizations are unitarily equivalent, their state spaces must be unitarily equivalent.

COROLLARY 4.3. *The spaces* $\overline{H_{G^{\perp}}(H_U^2)}$ *and* $\overline{H_{\tilde{G}^{\perp}}(H_Y^2)}$ *are unitarily equivalent, with a unitary transformation given by*

$$V = \mathcal{W}_o^{-1/4} H_{G^{\perp}} \mathcal{M}_i^{-1/4} : \overline{H_{G^{\perp}}(H_U^2)} \to \overline{H_{\tilde{G}^{\perp}}(H_Y^2)}.$$

Before stating the second consequence of Theorem 4.1, we quote from [3, p. 248] the following result regarding the closedness of the range of a Hankel operator.

PROPOSITION 4.4. *Let* $K \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$ *with* U *and* Y *finite-dimensional. Then* $H_K(H_U^2(\mathbb{D}))$ *is closed in* $H_Y^2(\mathbb{D})$ *if and only if there are functions* $Q \in H^{\infty}_{\mathcal{L}(Y)}(\mathbb{D})$, $F \in H^{\infty}_{\mathcal{L}(Y,U)}(\mathbb{D})$, $P_1 \in H^{\infty}_{\mathcal{L}(Y)}(\mathbb{D})$, *and* $P_2 \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbb{D})$ *such that, for almost all* $z \in \partial\mathbb{D}$,

$$K(z) = Q(z)(zF(z))^*, \qquad P_1(z)Q(z) + P_2(z)F(z) = I_Y$$

and Q is inner. Note that the last equality means that Q and F are strongly right coprime [3].

We have the following characterization of when the state space transformation in Theorem 4.1 that maps an input normal (respectively, output normal) realization to a (par-) balanced realization is an equivalence transformation.

PROPOSITION 4.5. *Let $G \in TLD^{U,Y}$. A parbalanced realization is equivalent to an output normal (or input normal) realization if and only if $H_{G^\perp}$ has closed range.*

Proof. If $H_{G^\perp}$ has closed range, then $\mathcal{W}_o$ in Theorem 4.1 has bounded inverse and $\mathcal{W}_o^{-1/4}$ is an equivalence transformation from the restricted shift realization to a parbalanced realization.

Conversely, if T is an equivalence transformation from a parbalanced realization to the restricted shift realization, then it will follow that $TT^*TT^* = \mathcal{W}_o$. Hence $\mathcal{W}_o$ has bounded inverse. Since $\mathcal{W}_o = H_{G^\perp}H_{G^\perp}^*$, $H_{G^\perp}$ must have closed range. Note that $H_{G^\perp}$ has closed range if and only if $H_{\tilde{G}^\perp}$ has closed range. This completes the proof. □

In fact, the state space isomorphism theorem holds when $H_{G^\perp}$ has closed range.

COROLLARY 4.6. *Let $G \in TLD^{U,Y}$. Then all reachable and observable realizations of G are equivalent if and only if $H_{G^\perp}$ has closed range.*

Proof. If all reachable and observable realizations of G are equivalent, then, in particular, the output normal and the parbalanced realizations are equivalent. By Proposition 4.5, $H_{G^\perp}$ has closed range.

Conversely, assume that $H_{G^\perp}$ has closed range. Let $(A, B, C, D) \in LD_X^{U,Y}$ be a reachable and observable realization of G with state space X. We show that it is equivalent to an output normal realization. Then, by Theorem 3.2, this shows that all reachable and observable realizations of G are equivalent.

Let $\mathcal{O}$ and $\mathcal{R}$ be, respectively, the observability and reachability operators of (A, B, C, D). It is easily verified that $H_{G^\perp} = \mathcal{OR} : H_U^2(\mathbb{D}) \to H_Y^2(\mathbb{D})$ (see the beginning of § 2). Hence $\mathcal{OR}(H_U^2(\mathbb{D}))$ is closed in $H_Y^2(\mathbb{D})$. By reachability, $\mathcal{R}(H_U^2(\mathbb{D})) \subseteq X$ is dense in X. Thus

$$\mathcal{O}(X) \subseteq \overline{\mathcal{OR}(H_U^2(\mathbb{D}))} = \mathcal{OR}(H_U^2(\mathbb{D})) \subseteq \mathcal{O}(X).$$

It follows that $\mathcal{O}(X) = \mathcal{OR}(H_U^2(\mathbb{D}))$, and hence $\mathcal{O}(X)$ is closed in $H_Y^2(\mathbb{D})$. Since by observability $\mathcal{O}$ is injective, the operator $\mathcal{O} : X \to \mathcal{O}(X)$ has bounded inverse. Consequently, the operator $\mathcal{O}^*\mathcal{O} : X \to X$ has bounded inverse on X. Now let $V = (\mathcal{O}^*\mathcal{O})^{-1/2}$, then V is bounded and is boundedly invertible. It is routine to verify that the realization $(V^{-1}AV, V^{-1}B, CV, D)$, which is equivalent to (A, B, C, D), is output normal. □

The main result in this section is that all parbalanced realizations are asymptotically stable. We need two lemmas in the proof.

LEMMA 4.7 (see [3, p. 124]). *Let $A : H_1 \to H$ and $B : H_2 \to H$ be two linear operators from Hilbert spaces H_1 and H_2, respectively, into a Hilbert space H. Then $AA^* \leq BB^*$ if and only if there exists a contraction $V : H_1 \to H_2$ such that $A = BV$. Moreover, $AA^* = BB^*$ if and only if V is a partial isometry with final space equal to $\overline{\text{range}}(B^*)$.*

LEMMA 4.8. *Let $G \in TLD^{U,Y}$. Let (A, B, C, D) be a realization of G and let (A^*, C^*, B^*, D^*) be its dual system. Then (A, B, C, D) is a parbalanced realization of G if and only if (A^*, C^*, B^*, D^*) is a parbalanced realization of $\tilde{G}$.*

THEOREM 4.9. *Let $G \in TLD^{U,Y}$ and let (A_b, B_b, C_b, D_b) be a parbalanced realization of G. Then $A_b \in C_{00}$.*

Proof. Here we use the notation and result of Theorem 4.1 and first prove that A_{b1} is asymptotically stable. Note that $A_o^* = P_{X_o}S|_{X_o}$ and $X_o^\perp \subseteq \ker(H_{G^\perp}^*)$. It is easy to verify that

$$H_{G^\perp}^* A_o^* = H_{G^\perp}^* S|_{X_o} = S^*|_{X_o} H_{G^\perp}^*|_{X_o} = A_o H_{G^\perp}^*|_{X_o}.$$

Hence we have

$$\begin{aligned}\langle A_o\mathcal{W}_oA_o^*x,\ x\rangle &= \langle A_oH_{G^\perp}H_{G^\perp}^*A_o^*x,\ x\rangle\\ &= \langle H_{G^\perp}^*A_o^*x,\ H_{G^\perp}^*A_o^*x\rangle\\ &= \langle A_oH_{G^\perp}^*x,\ A_oH_{G^\perp}^*x\rangle\\ &\le \langle H_{G^\perp}^*x,\ H_{G^\perp}^*x\rangle\\ &= \langle \mathcal{W}_ox,\ x\rangle,\end{aligned}$$

i.e., $A_o\mathcal{W}_oA_o^* \le \mathcal{W}_o$. Thus by Lemma 4.7 there exists a contraction V on X_o such that

$$A_o\mathcal{W}_o^{1/2} = \mathcal{W}_o^{1/2}V,$$

and hence for any positive integer n

$$A_o^n\mathcal{W}_o^{1/2} = \mathcal{W}_o^{1/2}V^n.$$

Let x be any element in $\mathcal{W}_o^{1/2}X_o$, i.e., $x = \mathcal{W}_o^{1/2}z$ for some $z \in X$. Then the element $y = \mathcal{W}_o^{1/4}z \in X_o$ is such that $y = \mathcal{W}_o^{-1/4}x$. The above equality applied to y yields

$$A_o^n\mathcal{W}_o^{1/4}x = \mathcal{W}_o^{1/2}V^ny.$$

Since the right-hand side of the last equality is in $\mathcal{W}_o^{1/2}X_o$, the operator $\mathcal{W}_o^{-1/4}$ can be applied to both sides to lead to

$$\mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x = \mathcal{W}_o^{1/4}V^ny.$$

Now, noting that $\mathcal{W}_o^{1/4}$ is selfadjoint and, from Theorem 4.1, $A_{b1}^nx = \mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x$, we have that

$$\begin{aligned}\|A_{b1}^nx\|^2 &= \langle A_{b1}^nx, A_{b1}^nx\rangle\\ &= \langle \mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x, \mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x\rangle\\ &= \langle \mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x, \mathcal{W}_o^{1/4}V^ny\rangle\\ &= \langle \mathcal{W}_o^{1/4}\mathcal{W}_o^{-1/4}A_o^n\mathcal{W}_o^{1/4}x, V^ny\rangle\\ &= \langle A_o^n\mathcal{W}_o^{1/4}x, V^ny\rangle\\ &\to 0\end{aligned}$$

as $A_o^nz \to 0$ for any z and $\|V^ny\| \le \|y\|$.

We thus have proved $\|A_{b1}^nx\| \to 0$ for any $x \in \mathcal{W}_o^{1/2}X_o$. Let $z \in X_o$ and $\epsilon > 0$. Since $\mathcal{W}_o^{1/2}X_o$ is dense in X_o, there exists $x \in \mathcal{W}_o^{1/2}X_o$ such that $\|z - x\| < \epsilon/2$. Choosing N such that $\|A_{b1}^nx\| < \epsilon/2$ whenever $n \ge N$ and using the fact that A_{b1} is a contraction, we obtain, for $n \ge N$

$$\|A_{b1}^nz\| \le \|A_{b1}^n(z-x)\| + \|A_{b1}^nx\| < \|A_{b1}^n\|\|z-x\| + \epsilon/2 \le \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that A_{b1} is asymptotically stable. Since, by statement 2 of Theorem 4.1, all observable and reachable parbalanced realizations are unitarily equivalent, A_b must be asymptotically stable. Hence any parbalanced realization of any transfer function in $TLD^{U,Y}$ is asymptotically stable. Since by Lemma 4.8 $(A_b^*, C_b^*, B_b^*, D_b^*)$ is a parbalanced realization of the transfer function $\tilde{G} \in TLD^{Y,U}$, we can apply this statement to $(A_b^*, C_b^*, B_b^*, D_b^*)$ to get the asymptotic stability of A_b^*. Therefore we have proved that $A_b \in C_{00}$. □

We discuss the spectral properties of a parbalanced realization and relate these properties to the characterization of power-stability of parbalanced realizations.

PROPOSITION 4.10. *Let (A_b, B_b, C_b, D_b), (A_i, B_i, C_i, D_i), and (A_o, B_o, C_o, D_o) be, respectively, a parbalanced, an input normal and an output normal realization of $G \in TLD^{U,Y}$ with U and Y finite-dimensional. If $G^\perp$ is in $H^\infty_{\mathcal{L}(U,Y)}$ and is strictly noncyclic, then A_b, A_i, A_o are all C_0 operators. Moreover, they have the same minimal function.*

Proof. By Theorem 3.5, Corollary 3.6, and Theorem 4.9, the assumption in the proposition implies that A_i, A_o, and A_b are all in C_{00}. Hence they are all completely nonunitary (see [12] or [8]). Furthermore, as noted after Proposition 4.2, A_{b1} is a quasi-affine transform of A_o and A_i a quasi-affine transform of A_{b2}. The result now follows from Proposition 3.13 and [12, Prop. 4.6, p. 125], which shows the following: For two completely nonunitary operators A and B on a Hilbert space H, if there is a bounded injective operator C on H with dense range in H such that $AC = CB$ (i.e., B is a quasi-affine transform of A), then A is a C_0 operator if and only if B is, and in this case they both have the same minimal function. □

For the spectrum of the state propagation operators, we obtain the following result.

COROLLARY 4.11. *Under the assumption of Proposition 4.10, we have*

$$\sigma(A_b) = \sigma(A_i) = \sigma(A_o) \quad \textit{and} \quad \sigma_p(A_b) = \sigma_p(A_i) = \sigma_p(A_o).$$

Proof. The proof is an immediate consequence of Propositions 4.10 and 3.8. □

For the question of the spectral minimality, we have the same result as for input normal and output normal realizations in the case of finite-dimensional U and Y.

COROLLARY 4.12. *Under the assumption of Proposition 4.10, the systems $(A_b$, B_b, C_b, $C_b)$, $(A_i$, B_i, C_i, $D_i)$, and (A_o, B_o, C_o, D_o) are spectrally minimal, i.e.,*

$$\sigma_s(G) = \sigma(A_b) = \sigma(A_i) = \sigma(A_o).$$

Proof. Combining Theorems 3.14 with 3.2, we have that

$$\sigma_s(G) = \sigma(A_i) = \sigma(A_o).$$

Corollary 4.11 now implies the result. □

The criteria for power-stability are also identical to those in the input normal and output normal case if $G^\perp$ is strictly noncyclic.

COROLLARY 4.13. *Let (A_b, B_b, C_b, D_b) be a parbalanced realization of $G \in TLD^{U,Y}$ with U and Y finite-dimensional. Assume that $G^\perp$ is in $H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ and is strictly noncyclic. Then A_b is power-stable if and only if G is rational.*

Proof. The proof follows from Corollary 4.11 and Theorem 3.15. □

This corollary shows that a parbalanced realization of $G \in TLD^{U,Y}$, with $G^\perp$ nonrational and strictly noncyclic, cannot be power-stable. When $G^\perp$ is not strictly noncyclic the situation is complicated. Here we give an example of a power-stable parbalanced realization of a cyclic function with l_2 as its state space.

Example. Let S and S^* be the right and left shifts on the space l_2. Let $A = \frac{1}{5}(I + S + S^*)$. Clearly, $\|A\| \leq \frac{3}{5}$. Define $B : \mathbb{C} \to l_2$ as

$$B(\lambda) = (\lambda, 0, 0, \cdots)^T, \qquad \lambda \in \mathbb{C}$$

and $C : l_2 \to \mathbb{C}$ as

$$C(x_k)_{k\geq 1} = x_1, \qquad (x_k)_{k\geq 1} \in l_2.$$

We take D to be zero. We have $\|B\| = 1$ and $\|C\| = 1$. Now consider $e_i = (\delta_{ij})_{j\geq 1}$, where δ_{ij} is the Kronecker delta. Then $\{e_i\}_{i\geq 1}$ forms a basis of l_2. With respect to this basis, we have the following matrix representations of A, B, and C:

$$A = \frac{1}{5}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \text{and} \quad C = [\, 1 \quad 0 \quad 0 \quad 0 \quad \cdots \,].$$

We show that (A, B, C, D) is an observable and reachable system. Let $\mathcal{O}$ and $\mathcal{R}$ be, respectively, the observability and reachability operators. For $x = (x_k)_{k\geq 0} \in l^2$, we have

$$\|\mathcal{O}x\|^2 = \|(CA^k x)_{k\geq 0} z^k\|^2 = \sum_{k\geq 0} \|CA^k x\|^2 \leq \sum_{k\geq 0} \|A\|^{2k}\ \|x\|^2 \leq \sum_{k\geq 0} (\tfrac{3}{5})^{2k}\ \|x\|^2.$$

Hence $\mathcal{O}$ is bounded. Let $x = (x_k)_{k\geq 0} \in l^2$ be such that $\mathcal{O}x = 0$, i.e., $CA^k x = 0$ for $k = 0, 1, \cdots$. Then it follows that $x_1 = CA^0 x = 0$, and hence $x_2 = 0$ because $0 = CAx = (x_1 + x_2)/5$, and so on. So we have $x = 0$. This shows that the system is observable. Note that $\mathcal{R} = \mathcal{O}^*$. Hence the system is reachable. It is obviously parbalanced. Also, the transfer function $g(z) = C(zI - A)^{-1}B$ is such that $g^\perp \in H^\infty$ due to the fact that $\|A\| < 1$. Since this is a power-stable realization, by Corollary 4.13, g must be cyclic. Thus there exists a cyclic transfer function that has power-stable parbalanced realizations.

5. Concluding remarks. We have shown the asymptotic stability of parbalanced realizations and have given conditions for an input normal or output normal realization to be asymptotically stable. An input normal or output normal realization cannot be power-stable unless the transfer function is rational. This is also true for parbalanced realizations when the transfer functions are assumed to be strictly noncyclic. If the transfer function is cyclic, the problem of finding a full characterization for power stability of parbalanced realizations remains open.

Concluding the paper, we point out that the results here can be translated to continuous-time systems by the bilinear mapping defined in [10]. However, to use that mapping, we restrict the discrete-time transfer functions to be *admissible.* A function G is said to be an admissible discrete-time transfer function if G is in $TLD^{U,Y}$ and the limit

$$\lim_{\substack{\lambda < -1,\ \lambda \to -1 \\ \lambda \in \mathbb{R}}} G(\lambda)$$

exists in the norm topology. Correspondingly, the discrete-time linear systems (A, B, C, D) must be *admissible*, also; that is, in addition to A being contractive, B, C, and D being bounded, the limit

$$\lim_{\substack{\lambda \to 1,\ \lambda > 1 \\ \lambda \in \mathbb{R}}} C(\lambda I + A)^{-1} B$$

must exist in the norm topology and $-1 \notin \sigma_p(A)$. It can be easily verified that the restricted and *-restricted shift realizations of admissible transfer functions are admissible systems. Moreover, the dual system of an admissible system is admissible, and any reachable and observable parbalanced realization of an admissible transfer function is an admissible system. Since the class of admissible transfer functions (linear systems) is smaller than the class of transfer functions (linear systems) considered in this paper, all the results of this paper are also valid for the smaller class.

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