INT. J. CONTROL, 1994, VOL. 59, NO. 5, 1191–1210

Properties of optimally robust controllers

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The robustness properties of controllers, which are optimally robust with respect to normalized coprime factor uncertainty, are analysed. It is shown that such optimally robust controllers admit perturbations which are, in size, larger than the robustness measure. It is shown how such perturbations can be constructed by solving Hankel norm approximation problems.

1. Introduction

The work in this paper was motivated by the results of Glover and McFarlane (1989) and their solution to the Normalized Coprime Factor Robust Stabilization Problem. Before introducing the actual topic of this paper we will review this problem.

The robust stabilization problem was first introduced by Vidyasagar and Kimura (1986) for the case of not necessarily normalized coprime factorizations. It is formulated in terms of unstructured additive perturbations on the normalized right coprime factors of the nominal system, G. A normalized right (left) coprime factorization NRCF (NLCF) of a plant G is a factorization $G = NM^{-1}$ ($G = \tilde{M}^{-1}\tilde{N}$) with $N, M \in \mathcal{H}_\infty$, ($\tilde{M}, \tilde{N} \in \mathcal{H}_\infty$), where $\mathcal{H}_\infty$ is the Hardy space of functions bounded and analytic in the open right half-plane (RHP). Moreover M ($\tilde{M}$) is required to be invertible with a proper inverse and the factors are coprime, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathcal{H}_\infty$ ($X, Y \in \mathcal{H}_\infty$) such that $-\tilde{X}N + \tilde{Y}M = I$ ($-\tilde{N}X + \tilde{M}Y = I$). That the factorization is normalized means that $N^*N + M^*M = I$ ($\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$).

Let the nominal $p \times m$ system, G, have NRCF (N, M) such that

$$G = NM^{-1}$$

Then any other system of the same input/output dimensions can be written in the form

$$G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1} \tag{1}$$

where $\Delta_N, \Delta_M \in \mathcal{H}_\infty$, i.e. are stable transfer functions. It is possible to define various families of systems by placing restrictions on the allowable perturbations Δ_N, Δ_M. The robust stabilization problem considered here is to stabilize the nominal system, G, with normalized right coprime factorization (r.c.f.) (N, M) and the family of systems $\mathcal{G}_\varepsilon$ defined by,

$$\mathcal{G}_\varepsilon := \left\{ (N + \Delta_N)(M + \Delta_M)^{-1} : \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{H}_\infty^{(p+m)\times m};\ \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < \varepsilon \right\} \tag{2}$$

using a dynamic feedback controller K.

Received 5 December 1991. Revised 27 October 1992. Communicated by Professor H. Kimura.

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The normalized coprime factor robust stabilization problem is now to find the maximal $\varepsilon_{\max}$ such that all plants in $\mathscr{G}_{\varepsilon_{\max}}$ can be stabilized by a controller K. This controller is called the optimally robust controller. Ober and Sefton (1991) showed that such a controller can also be interpreted as a maximally stabilizing controller. A maximally stabilizing controller is a controller that maximizes the minimum angle between the orthogonal complement of the graph of the nominal plant and the orthogonal complement of the transposed graph of a controller.

The following Lemma (see Vidyasagar and Kimura 1986) gives a necessary and sufficient condition for a controller K to stabilize all plants in $\mathscr{G}_\varepsilon$.

Lemma 1.1: *Let G be a plant and let K be a controller that internally stabilizes G. Let $G = NM^{-1}$ be a normalized right coprime factorization of G and let $K = \tilde{V}^{-1}\tilde{U}$ be a left coprime factorization of K, such that*

$$\tilde{V}M - \tilde{U}N = I$$

Then K internally stabilizes all $G_\Delta \in \mathscr{G}_\varepsilon$, $\varepsilon > 0$, if and only if

$$\varepsilon \leqslant \|[\tilde{V} \ \ \tilde{U}]\|_\infty^{-1} \tag{3}$$

Glover and McFarlane (1989) showed that the normalized right coprime factor robustness problem can be solved explicitly.

Theorem 1.2 (Glover and McFarlane 1989): *Given a $p \times m$ system G, with normalized r.c.f and l.c.f (N, M) and $(\tilde{N}, \tilde{M})$ respectively, then the controller K stabilizes all plants in $\mathscr{G}_\varepsilon$, $\varepsilon > 0$, if and only if K has a right coprime factorization $K = UV^{-1}$ for some $U, V \in \mathscr{H}_\infty$ satisfying the Nehari extension*

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty \leqslant (1 - \varepsilon^2)^{1/2} \tag{4}$$

or, equivalently, if and only if K has a left coprime factorization $K = \tilde{V}^{-1}\tilde{U}$ satisfying the Nehari extension

$$\left\| \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} \right\|_\infty \leqslant (1 - \varepsilon^2)^{1/2} \tag{5}$$

The maximum stability margin, $\varepsilon_{\max}$, is given by

$$\varepsilon_{\max} = \{1 - \|H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\|^2\}^{1/2} \tag{6}$$

The above result may at first seem surprising. However, further understanding of this problem can be gained by examining an equivalent problem.

The normalized right coprime factor robust stabilization problem can be shown to be equivalent to finding a minimum norm solution of a Bezout identity. This is demonstrated by substituting the Youla parametrization of all stabilizing controllers (see for example Vidyasagar 1985), for K in (3). Finding the optimally robust controller then amounts to finding the parameter Q that attains the infinimum in the expression

$$\inf_{Q \in \mathscr{H}_\infty} \|[-(\tilde{U} - Q\tilde{M}) \ \ (\tilde{V} - Q\tilde{N})]\|_\infty$$

The expression $[-\hat{U} \ \ \hat{V}] = [-(\tilde{U} - Q\tilde{M}) \ \ (\tilde{V} - Q\tilde{N})]$ parametrizes all solutions to the Bezout identity $\hat{V}M - \hat{U}N = I$. Therefore, the optimization problem is equivalent to finding the minimum norm solution to this Bezout identity. The following theorem links this problem to a Nehari Problem.

Theorem 1.3 (Sz-Nagy and Foias 1976, and Nikolskii 1986): *Let* $F_i \in \mathcal{H}_\infty$, $1 \leq i \leq n$, $\varepsilon > 0$. *The following statements are then equivalent*:

(1) *there exist functions* G_i, $G_i \in \mathcal{H}_\infty$, $1 \leq i \leq n$, *such that*

$$\sum_i F_i G_i = I, \quad \left\| \sum_i G_i^* G_i \right\|_\infty \leq \varepsilon^{-2}$$

(2) $\sum_i \|T_{F_i^*} x\|_2^2 \geq \varepsilon^2$ *for all* $x \in \mathcal{H}_2$, $(T_{F_i^*} x := P_+ F_i^* x)$.

If $F = \{F_i\}_{i=1}^n$ *is inner function, then statements* (1) *and* (2) *are equivalent to*

(3) $\inf_{Q \in \mathcal{H}_\infty} \|F^* - Q\| \leq (1 - \varepsilon^2)^{\frac{1}{2}}$.

The difficulty of the proof in Theorem 1.2 is showing that the suboptimal extension of $[\tilde{M} - \tilde{N}]^*$ is a coprime factorization of a robust controller. Theorem 2 of Georgiou and Smith (1990) offers an alternative proof of the relationship between the $\mathcal{H}_\infty$-norm of the minimal solution to the Bezout equation and the norm of the Hankel operator whose symbol is made up of the normalized coprime factors of the plant.

In this paper, we will consider a particular controller, the 'minimal Bezout controller', that solves the normalized right coprime factor robustness problem. In particular, we will introduce and study certain perturbations, so-called 'Bezout-perturbations' that bring a control system, which is controlled by a minimal Bezout controller, to the boundary of instability. The smallest such Bezout-perturbation has size $\varepsilon_{\max}$ and destabilizes the system. The other Bezout-perturbations are larger in size, than the first and hence larger than the maximal stability margin $\varepsilon_{\max}$. This shows that there are specific perturbations that are larger than the maximal stability margin and which do not destabilize the closed-loop system. The largest Bezout-perturbation, in fact, perturbs the nominal plant to the inverse of the controller.

The construction of the Bezout-perturbations is done by studying minimum norm solutions to a Bezout equation whose coefficients are the normalized coprime factors of the maximally robust controller. We will not only consider stable solutions but also solutions with a fixed number of unstable poles. These minimum norm solutions can be related to Hankel norm approximations of the Hankel operator whose symbol is given by the normalized coprime factors of the optimally robust controller. This, to an extent, generalizes the results of Theorem 1.3.

2. Notation

The notation used throughout this paper is standard in the control literature (Francis 1987). For a matrix $M \in \mathfrak{R}^{p \times m}$ or $\mathscr{C}^{p \times m}$, M^{T} denotes its transpose, M^* denotes its conjugate transposed, $\sigma_{\max}(M)$ denotes its maximum singular value, σ_i its ith singular value and $\sigma_{\min}(M)$ its minimum singular value.

The Hardy spaces $\mathcal{H}_2^p$ and $(\mathcal{H}_2^p)^\perp$, contain all p vector-valued rational functions square-integrable on the imaginary axis with analytic continuation into the right and left half-planes respectively. The Hilbert space $\mathcal{L}_2^p$ is given by $\mathcal{L}_2^p = \mathcal{H}_2^p \oplus (\mathcal{H}_2^p)^\perp$, and the orthogonal projections P_+ and P_- map $\mathcal{L}_2^p$ onto $\mathcal{H}_2^p$

and $(\mathscr{H}_2^p)^\perp$ respectively. The norm of a function $f \in \mathscr{H}_2^p$ is denoted $\|f\|_2$. The Hardy space $\mathscr{H}_\infty^{p\times m}$ consists of all $p \times m$ bounded functions on the imaginary axis with bounded analytic continuation in the right half-plane and is a subspace of $\mathscr{L}_\infty^{p\times m}$ of all $p \times m$ bounded functions on the imaginary axis. Clearly, these functions all have finite $\mathscr{L}_\infty$-norm defined by $\|G\|_\infty := \operatorname{ess\,sup}_{\omega\in\Re} \sigma_{\max}[G(j\omega)]$ and a minimum value on the imaginary axis defined by, $\tau(G) := \operatorname{ess\,inf}_{\omega\in\Re} \sigma_{\min}[G(j\omega)]$. For a system G, G^* denotes its complex conjugate transposed, i.e. $G(s)^* = \overline{G(-\bar{s})}^{\mathrm{T}}$. The symbol $\mathscr{R}\mathscr{H}_2^p$ denotes the subspace of $\mathscr{H}_2^p$ containing the real rational functions; similar definitions apply to the other spaces. By $\mathscr{H}_{\infty,k}$ is meant the subset of $\mathscr{L}_\infty$ consisting of functions that can be written as the sum of a function in $\mathscr{H}_\infty$ plus a rational function that has at most k poles in the open right half-lane. The set $\mathscr{B}_n$ denotes the set of rational square inner functions in $\mathscr{H}_\infty$ of McMillan degree at most n.

The domain and range of an operator Z is denoted by $\mathscr{D}(Z)$ and $\mathscr{R}(Z)$ respectively. The orthogonal projection operator onto a closed space, $\mathscr{A}$ of $\mathscr{L}_2^p$ is denoted by $P_{\mathscr{A}}$. Given a $p \times m$ symbol G the multiplication operator $M_G: \mathscr{D}(M_G) \to \mathscr{H}_2^m$ is defined by $f \mapsto Gf$. If $G \in \mathscr{L}_\infty^{p\times m}$ the Laurent operator $L_G: \mathscr{L}_2^m \to \mathscr{L}_2^p$, the Hankel operator H_G: $\mathscr{H}_2^m \to (\mathscr{H}_2^p)^\perp$ and the Toeplitz operator T_G: $\mathscr{H}_2^m \to \mathscr{H}_2^p$ with symbol G are defined by $f \mapsto Gf$, $f \mapsto P_{(\mathscr{H}_2^p)^\perp} Gf$ and $f \mapsto P_{\mathscr{H}_2^p} Gf$ respectively.

3. Minimal Bezout controller

This section starts by defining a particular controller that solves the normalized right coprime factorization robust stabilization problem. In the previous section it was shown that the coprime factors of this controller are the minimum norm solutions to a Bezout equation.

Definition 3.1: A rational stabilizing controller, K, will be called *a minimal Bezout controller* of the $p \times m$ system G if and only if

$$\sigma_i(M^{-1}(I - KG)^{-1}[-K \quad I])(j\omega) = \varepsilon_{\max}^{-1} \tag{7}$$

for all $i = 1, 2, \ldots, \min(m, p)$ and for all $\omega \in \Re$ and there exists no other stabilizing controller K of lower McMillan degree satisfying this condition. □

It follows from our discussion below that in the single-input single-output case the Bezout controller is, in fact, the unique optimally robust controller. In the multivariable case there is, in general, no unique optimally robust controller. Introducing the concept of a Bezout controller aims at picking certain optimally robust controllers that have properties that are easy to analyse. From a design point of view it might, however, be preferable to choose another optimally robust controller.

This section aims at establishing the existence of a minimal Bezout controller, and shows that the normalized coprime factors of this controller satisfy various identities which will be useful in the later development. Some technical results are needed for the Hankel and Toeplitz operators with symbol $[\widetilde{M} \quad -\widetilde{N}]^*$. For the first result, see Fuhrman and Ober (1993).

Proposition 3.2: *Given a $p \times m$ rational transfer function G, with NLCF $(\widetilde{N}, \widetilde{M})$, let $\Omega \in \mathscr{R}\mathscr{H}_\infty$ and $\widetilde{\Omega}$ be the smallest inner $m \times m$ and $p \times p$ functions* *respectively (with respect to the usual ordering of inner functions) such that*

$$\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix} \widetilde{\Omega} \in \mathcal{H}_\infty \quad and \quad \Omega\,[N^* \;\; M^*] \in \mathcal{H}_\infty \tag{8}$$

then

$$\operatorname{Ker}\left(H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}}\right) = \widetilde{\Omega}\mathcal{H}_2^p$$

and

$$\left[\mathcal{R}\left(H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}}\right)\right]^\perp = \left(\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix} \;\; \begin{bmatrix} N \\ M \end{bmatrix} \Omega^*\right)\begin{pmatrix} (\mathcal{H}_2^p)^\perp \\ (\mathcal{H}_2^p)^\perp \end{pmatrix}$$

It is possible to use the previous result in order to give a Schmidt decomposition of the self-adjoint operator $Z_G := T^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} : \mathcal{H}_2^p \to \mathcal{H}_2^p$.

Theorem 3.3: *Given a $p \times m$ rational transfer function G of McMillan degree n with NLCF $(\widetilde{N}, \widetilde{M})$, and let the Hankel operator with symbol $[\widetilde{M} \;\; -\widetilde{N}]^*$ have the singular value decomposition*

$$H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} h = \sum_{i=1}^{k} \sigma_i \sum_{j=1}^{r_i} < f_{(i,j)}, h > g_{(i,j)}$$

where $g_{(i,j)} \in (\mathcal{H}_2^{p+m})^\perp$, $f_{(i,j)} \in \mathcal{H}_2^p$ and $\sum r_i = n$ and $h \in \mathcal{H}_2^p$. Then the following identities hold,

(1) *the Schmidt vectors satisfy*

$$[\widetilde{M} \;\; -\widetilde{N}]\, g_{(i,j)} = \sigma_i f_{(i,j)}$$

(2) *the operator $T^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}}$ can be decomposed in the following manner*

$$T^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} h = T^*_{\Omega^*} T_{\Omega^*} h + \sum_{i=1}^{k} (1 - \sigma_i^2) \sum_{j=1}^{r_i} < f_{(i,j)}, h > f_{(i,j)}$$

where $\Omega \in \mathcal{H}_\infty^{p \times p}$ is an inner function such that $\operatorname{Ker}\left(H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}}\right) = \Omega\mathcal{H}_2^p$ *as defined in Proposition 3.2.*

Proof: The following Hankel and Toeplitz identity can be easily verified

$$H^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} + T^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} = I \tag{9}$$

Hence, for any input Schmidt vector $f_{(i,j)}$ we have

$$T^*_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} f_{(i,j)} = (1 - \sigma_i^2) f_{(i,j)} \tag{10}$$

The first identity then follows from noticing that

$$\begin{aligned} L_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} f_{(i,j)} - T_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} f_{(i,j)} &= H_{\begin{bmatrix} \widetilde{M}^* \\ -\widetilde{N}^* \end{bmatrix}} f_{(i,j)} \\ &= \sigma_i g_{(i,j)} \quad \in (\mathcal{H}_2^{p+m})^\perp \end{aligned}$$

which implies

$$\begin{aligned}\sigma_i L_{[\tilde{M}\ -\tilde{N}]} g_{(i,j)} &= L_{[\tilde{M}\ -\tilde{N}]}\left(L_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} f_{(i,j)} - T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} f_{(i,j)}\right)\\ &= f_{(i,j)} - T^*_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} f_{(i,j)}\\ &= \sigma_i^2 f_{(i,j)} \in \mathcal{H}_2^p\end{aligned}$$

In order to prove the second identity, it is first necessary to decompose the space $\mathcal{H}_2^p$ as $\mathcal{H}_2^p = \Omega\mathcal{H}_2^p \oplus (\mathcal{H}_2^p \ominus \Omega\mathcal{H}_2^p)$. This implies that for any $h \in \operatorname{Ker}(H_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}}) = \Omega\mathcal{H}_2^p$, we have

$$T^*_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} h = h \tag{11}$$

as a consequence of (9). The second identity is thus a combination of the orthogonal decomposition, noting that $f_{(i,j)} \in [\operatorname{Ker}(H_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}})]^\perp$, as well as (11) and (10). □

Identity (1) is basic for the following analysis. It states that

$$H^*_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} g_{(i,j)} = L^*_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} g_{(i,j)} = \sigma_i f_{(i,j)}$$

and therefore, in the study of the Hankel operators with an inner symbol, the projection operator does not complicate the analysis. In fact, the adjoint of the Hankel operator can be considered simply as a Laurent operator on the Schmidt vectors.

It is now possible to prove some results concerning the optimal Nehari extensions of co-inner functions and, in particular, of $[\tilde{M}\ -\tilde{N}]^*$. This is first done for the case when the system is single-input single-output (SISO), as the extensions in this case can be expressed in terms of the Schmidt vectors. In order to make this distinction clear in the notation, lower case letters are used to denote all SISO transfer functions. The following technical lemma will be required.

Lemma 3.4. (Coburn, see for example Nikolskii 1986): *If $\theta \in \mathcal{RL}_\infty$ is a SISO transfer function not almost everywhere zero, then either* $\operatorname{Ker}(T_\theta) = \{0\}$ *or* $\operatorname{Ker}(T_{\theta^*}) = \{0\}$.

The following Lemma will also be needed.

Lemma 3.5: *Suppose the $p \times m$ function G has a normalized r.c.f. and l.c.f (N, M) and $(\tilde{N}, \tilde{M})$ respectively and a $m \times p$ function, K, has a normalized r.c.f. and l.c.f (U, V) and $(\tilde{U}, \tilde{V})$. If*

$$\begin{aligned}\begin{bmatrix} C^* & S^* \\ \tilde{S} & -\tilde{C}\end{bmatrix} &:= \begin{bmatrix} N^* & M^* \\ \tilde{M} & -\tilde{N}\end{bmatrix}\begin{bmatrix} V & -\tilde{U}^* \\ U & \tilde{V}^*\end{bmatrix}\\ &= \begin{bmatrix} N^*V + M^*U & (\tilde{V}M - \tilde{U}N)^* \\ (\tilde{M}V - \tilde{N}U) & -(\tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*)^*\end{bmatrix}\end{aligned}$$

then

(1) $CC^* + \tilde{S}^*\tilde{S} = I_p$ (3) $\tilde{C}\tilde{C}^* + \tilde{S}\tilde{S}^* = I_p$
(2) $C^*C + S^*S = I_m$ (4) $\tilde{C}^*\tilde{C} + SS^* = I_m$

and if $\tilde{S}^{-1}, S^{-1} \in \mathscr{L}_\infty$,

(5) $C^*\tilde{S}^{-1} = S^{-1}\tilde{C}^*$ (6) $CS^{-1} = \tilde{S}^{-1}\tilde{C}$

and if $\tilde{S}^{-1}, S^{-1} \in \mathscr{RH}_\infty$

(7) $\tilde{S} = (I - CC^*)_{\mathrm{L}}^{1/2} = (I - \tilde{C}\tilde{C}^*)_{\mathrm{R}}^{1/2}$ (8) $S = (I - C^*C)_{\mathrm{L}}^{1/2} = (I - \tilde{C}^*\tilde{C})_{\mathrm{R}}^{1/2}$

where $F_{\mathrm{L}}^{1/2}(F_{\mathrm{L}}^{1/2})^* = (F_{\mathrm{R}}^{1/2})^*F_{\mathrm{R}}^{1/2} = F$.

Proof: The proof follows immediately from noticing that the function defined in this Lemma is simply the product of two all-pass functions, and is therefore all-pass itself. □

We can prove that for a SISO plant g there exists a minimal Bezout controller.

Theorem 3.6: *Given a SISO system g, with normalized coprime factors (n, m), and assuming that the Hankel operator with symbol $[m \ \ -n]^*$ has the Schmidt decomposition as defined in Theorem 3.3, there exists a unique minimal Bezout controller for the plant g. The minimal Bezout controller has a normalized coprime factorization (u, v) which satisfies the following identities*

(1) $\left\| \begin{bmatrix} m^* \\ -n^* \end{bmatrix} - (1 - \sigma_1^2)^{1/2} \begin{bmatrix} v \\ u \end{bmatrix} \right\|_\infty = \sigma_1$

(2) $mv - nu = (1 - \sigma_1^2)^{1/2}$

(3) $u^*u + v^*v = 1$

(4) If $c^* = m^*u + n^*v$ *then* $c^*c = \sigma_1^2$

(5) $\|H_{c^*}\| < \sigma_1$

Proof: The existence of a controller that satisfies the all-pass condition in (7) is proved by construction of its coprime factors from the Schmidt vectors of the Hankel operator with symbol $[m - n]^*$. It is shown that this coprime factorization satisfies identities (2) to (5). Finally, it is shown that if a stabilizing controller satisfies the all-pass condition in (7) then its normalized coprime factors satisfy these identities. The uniqueness of the minimal Bezout controller may then be deduced from the fact that the optimal Nehari extension $[m \ \ -n]^*$ is unique.

Adamyan *et al.* (1971) proved that in the SISO case an optimal Nehari extension of a transfer function can be calculated from the Schmidt vectors. Foias and Frazho (1990) showed that the same results also hold in the case of $n \times 1$ functions. Therefore, an extension of $[m \ \ -n]^*$ can be calculated as

$$(1 - \sigma_1^2)^{1/2} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} m^* \\ -n^* \end{bmatrix} - \sigma_1 \frac{g_{(1,1)}}{f_{(1,1)}}$$

and, further, this extension is unique. It is now shown that the transfer function $[v \ \ u]^{\mathrm{T}}$ satisfies the identities (2) to (5). The second identity follows directly from the expression for the extension and the first identity in Theorem 3.3

$$(1-\sigma_1^2)^{1/2}\,[m\quad -n]\begin{bmatrix} v \\ u \end{bmatrix} = 1-\sigma_1[m\quad -n]\,\frac{g_{(1,1)}}{f_{(1,1)}}$$
$$= 1-\sigma_1^2$$

This also implies that (u, v) is coprime.

The third identity is also a straightforward calculation, as

$$(1-\sigma_1^2)\,[v^*\quad u^*]\begin{bmatrix} v \\ u \end{bmatrix} = \left([m\quad -n]-\sigma_1\,\frac{g^*_{(1,1)}}{f^*_{(1,1)}}\right)\left(\begin{bmatrix} m^* \\ -n^* \end{bmatrix}-\sigma_1\,\frac{g_{(1,1)}}{f_{(1,1)}}\right)$$
$$= 1-\sigma_1^2-\sigma_1^2+\sigma_1^2\,\frac{g^*_{(1,1)}g_{(1,1)}}{f^*_{(1,1)}f_{(1,1)}}$$
$$= 1-\sigma_1^2$$

and $g^*_{(1,1)}g_{(1,1)} = f^*_{(1,1)}f_{(1,1)}$, see Foias and Frazho (1990). Identity (5) follows immediately from identity (4) of Lemma 3.5. To show the final identity, note that by construction

$$\left(\begin{bmatrix} m^* \\ -n^* \end{bmatrix}-(1-\sigma_1^2)^{1/2}\begin{bmatrix} v \\ u \end{bmatrix}\right) f_{(1,1)} = \sigma_1 g_{(1,1)}$$

which implies that

$$\begin{bmatrix} m & -n \\ n^* & m^* \end{bmatrix}\left(\begin{bmatrix} m^* \\ -n^* \end{bmatrix}-(1-\sigma_1^2)^{1/2}\begin{bmatrix} v \\ u \end{bmatrix}\right) f_{(1,1)} = \begin{bmatrix} \sigma_1^2 f_{(1,1)} \\ (1-\sigma_1^2)^{1/2}c^* f_{(1,1)} \end{bmatrix}$$
$$= \sigma_1\begin{bmatrix} m & -n \\ n^* & m^* \end{bmatrix} g_{(1,1)}$$

and that therefore $[m\quad -n]g_{(1,1)} = \sigma_1 f_{(1,1)}$ and $[n^*\quad m^*]g_{(1,1)} = (1-\sigma_1^2)^{1/2}\bar{g}_{(1,1)} \in \mathscr{H}_2^\perp$ where $\|\bar{g}_{(1,1)}\|_2 = 1$. Hence $c^* f_{(1,1)} = \sigma_1 \bar{g}_{(1,1)}$, implying that $\|H_{c^*}\| = \sigma_1$ as $\sigma_1 = \|c^*\|_\infty \geqslant \|H_{c^*}\| \geqslant \sigma_1$. Using the identity

$$H^*_{c^*}H_{c^*} + T^*_{c^*}T_{c^*} = \sigma_1^2$$

as c^*/σ_1 is all-pass, it is clear that $f_{(1,1)} \in \mathrm{Ker}(T_{c^*})$. By Lemma 3.4 this implies that $\mathrm{Ker}(T_c) = \{0\}$, and therefore $\|H_c\| < \sigma_1 = \|c\|_\infty$ as

$$H^*_c H_c + T^*_c T_c = \sigma_1^2$$

thus proving the last identity. Finally, note that $k = uv^{-1}$ satisfies the all-pass condition in (7) as for all $\omega \in \mathfrak{R}$

$$\sigma(m^{-1}(1-gk)^{-1}[-k\quad 1])(j\omega) = \sigma((mv-nu)^{-1}[v\quad u])(j\omega)$$
$$= (1-\sigma_1^2)^{-1/2}$$

and k is stabilizing as $(mv-nu)^{-1} \in \mathscr{RH}_\infty$. To complete the proof it is necessary to show that the normalized coprime factors (u, v) of every controller that satisfies the all-pass condition in (7) satisfy these identities. From the definition of such a controller

$$\begin{aligned}(1-\sigma_1^2)^{-1/2} &= \sigma(m^{-1}(1-gk)^{-1}[-k \quad 1](j\omega) \quad \forall\omega\in\Re \\ &= \sigma((mv-nu)^{-1}[-u \quad v])(j\omega) \quad \forall\omega\in\Re \\ &= \sigma((mv-nu)^{-1})(j\omega) \quad \forall\omega\in\Re\end{aligned}$$

As the controller is stabilizing, $mv - nu$ is a unit in $\mathcal{H}_\infty$. Since $mv - nu$ is also a scaled all-pass this implies that $mv - nu = (1-\sigma_1^2)^{1/2}$. Using identity (4) of Lemma 3.5 this implies that $c^* = mu^* + nv^*$ satisfies $c^*c = \sigma_1^2$. It is now clear that $(1-\sigma_1^2)^{1/2}\begin{bmatrix} v \\ u \end{bmatrix}$ is the optimal Nehari extension for

$$\sigma\left(\begin{bmatrix} m^* \\ -n^* \end{bmatrix} - (1-\sigma_1^2)^{1/2}\begin{bmatrix} v \\ u \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} m & -n \\ n^* & m^* \end{bmatrix}\left(\begin{bmatrix} m^* \\ -n^* \end{bmatrix} - (1-\sigma_1^2)^{1/2}\begin{bmatrix} v \\ u \end{bmatrix}\right)\right)$$

$$= \sigma\left(\begin{bmatrix} \sigma_1^2 \\ (1-\sigma_1^2)^{1/2}c^* \end{bmatrix}\right) = \sigma_1$$

As this extension is unique, the proof is thus completed. □

These results can be generalized from the SISO case to the multivariable case.

Theorem 3.7: *Given a $p \times m$, system G where $m \leq p$, with normalized left and right coprime factorization $(\tilde{N}, \tilde{M})$ and (N, M) respectively, then there exists a minimal Bezout controller, K. Further, every minimal Bezout controller has left coprime factors $(\tilde{U}, \tilde{V})$ that satisfy the following identities.*

(1) $\left\| \begin{bmatrix} N \\ M \end{bmatrix} - (1-\sigma_1^2)^{1/2}\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} \right\|_\infty = \sigma_1$

(2) $\tilde{V}M - \tilde{U}N = (1-\sigma_1^2)^{1/2} I_m$

(3) $\tilde{U}\tilde{U}^* + \tilde{V}\tilde{V}^* = I_m$

(4) *If* $\tilde{C}^* = \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*$ *then* $\tilde{C}^*\tilde{C} = \sigma_1^2 I_m$

(5) $\|H_{\tilde{C}}\| < \sigma_1$

where $\sigma_1 = \|H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\|$.

Proof: The existence of a controller satisfying the all-pass condition in (7) is proved by constructing the left coprime factors $(\tilde{U}, \tilde{V})$ that satisfy all the identities (1) to (5), from the parametrization of all solutions to the Bezout equation $\tilde{V}M - \tilde{U}N = I$, and then letting $K = \tilde{V}^{-1}\tilde{U}$. Then, it is shown that any controller satisfying the all-pass condition in (7) has normalized left coprime factors satisfying identities (1) to (4), and the one of minimal degree satisfies identity (5).

All solutions to the Bezout equation $\tilde{V}M - \tilde{U}N = I$ can be written $(\hat{U}, \hat{V}) = ((\tilde{U} - Q\tilde{M}), (\tilde{V} - Q\tilde{N}))$ where $(\tilde{U}, \tilde{V})$ is any solution to this Bezout equation. It is now possible to relate the singular values of $[-\hat{U} \quad \hat{V}]$ to the singular values of a $m \times p$ transfer function,

$$\begin{aligned}\sigma_i[-\hat{U}\ \ \hat{V}] &= \sigma_i\left([-\hat{U}\ \ \hat{V}]\begin{bmatrix}\widetilde{M}^* & N\\ -\widetilde{N}^* & M\end{bmatrix}\right)\\ &= \sigma_i([-(\hat{U}\widetilde{M}^* + \hat{V}\widetilde{N}^*)\ I])\\ &= (1+\sigma_i^2([\hat{U}\widetilde{M}^* + \hat{V}\widetilde{N}^*]))^{1/2}\\ &= (1+\sigma_i^2([\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^* - Q]))^{1/2} \qquad (12)\end{aligned}$$

By Nehari's Theorem there exists a $Q \in \mathfrak{R}\mathcal{H}_\infty$ which achieves the norm bound

$$\inf_{Q \in \mathfrak{R}\mathcal{H}_\infty} \|(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^*) + Q\|_\infty = \|H_{(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^*)}\|$$

Also, from Glover (1984), there exists a $Q \in \mathfrak{R}\mathcal{H}_\infty$ such that $(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^* + Q)$ is a scaled all-pass transfer function and

$$\|H_{((\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^*)+Q)^*}\| < \|H_{(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^*)}\|$$

Further, the transfer function $Q \in \mathfrak{R}\mathcal{H}_\infty$ satisfying the bounds in the above equations also minimizes the McMillan degree of the all-pass transfer function Q. (This follows from Lemma 5.2 of Glover 1989). This can be shown by padding the $m \times p$ transfer functions with zeros to make a square $p \times p$ transfer function

$$\begin{bmatrix}(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^*)\\ 0\end{bmatrix}$$

Theorem 6.3 of Glover (1984) implies that there exists an extension

$$Q = \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}$$

that makes the resultant sum a scaled all-pass transfer function. This, in turn, implies that $(\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^* + Q_1)\ (\widetilde{U}\widetilde{M}^* + \widetilde{V}\widetilde{N}^* + Q_1)^* = \gamma I_m$ for a $\gamma \in \mathfrak{R}$. This implies for such a Q_1 that $[-(\widetilde{U} - Q_1\widetilde{M})\ \ (\widetilde{V} - Q_1\widetilde{N})]$ is a scaled inner transfer function from (12). From now on this particular coprime factorization may be denoted as $[-\hat{U}\ \ \hat{V}]$. From McFarlane and Glover (1989) Theorem 1.3, the $\mathcal{H}_\infty$-norm of this factorization is

$$\|[-\hat{U}\ \ \hat{V}]\|_\infty = \frac{1}{(1-\sigma_1^2)^{1/2}}$$

as it is the coprime factorization which achieves the infimum in (4). Therefore, the coprime factorization $(1-\sigma_1^2)^{1/2}[-\hat{U}\ \ \hat{V}]$ clearly satisfies identities (2), (3), (4) and (5). It can be shown to satisfy identity (1) by noticing that

$$\begin{aligned}&\left\|\begin{bmatrix}N\\ M\end{bmatrix} - (1-\sigma_1^2)^{1/2}\begin{bmatrix}-\hat{U}^*\\ \hat{V}^*\end{bmatrix}\right\|_\infty\\ &\quad= \left\|\begin{bmatrix}N^* & M^*\\ \widetilde{M} & -\widetilde{N}\end{bmatrix}\left(\begin{bmatrix}N\\ M\end{bmatrix} - (1-\sigma_1^2)^{1/2}\begin{bmatrix}-\hat{U}^*\\ \hat{V}^*\end{bmatrix}\right)\right\|_\infty\\ &\quad= \left\|\begin{bmatrix}\sigma_1^2\\ -(1-\sigma_1^2)^{1/2}(\widetilde{M}\hat{U}^* + \widetilde{N}\hat{V}^*)\end{bmatrix}\right\|_\infty\\ &\quad= (\sigma_1^4 + \sigma_1^2(1-\sigma_1^2))^{1/2} = \sigma_1\end{aligned}$$

This also proves the existence of a controller satisfying the all-pass condition in (7). Let $K = \hat{V}^{-1}\hat{U}$, then

$$\sigma(M^{-1}(I - KG)^{-1}[-K \;\; I])(j\omega) = \sigma(M^{-1}(I - \hat{V}^{-1}\hat{U}NM^{-1})^{-1}[-K \;\; I])(j\omega)$$
$$= \sigma([-\hat{U} \;\; \hat{V}])(j\omega) = \frac{1}{(1-\sigma_1^2)^{1/2}} \quad \forall\omega \in \Re$$

and K is stabilizing as $(\hat{U}, \hat{V})$ was constructed from solutions to the Bezout equation $(\hat{V}M - \hat{U}N) = I$.

In an analogous manner, it will be shown that every minimal Bezout controller has a normalized coprime factorization $(\tilde{U}, \tilde{V})$ which satisfies identities (1) to (4). From the definition of the minimal Bezout controller the normalized coprime factors satisfy

$$\frac{1}{(1-\sigma_1^2)^{1/2}} = \sigma_i(M^{-1}(I - KG)^{-1}[-K \;\; I])(j\omega) \quad \forall\omega \in \Re, \quad \forall i = 1, 2, \ldots, m$$
$$= \sigma_i((\tilde{V}M - \tilde{U}N)^{-1}[-\tilde{U} \;\; \tilde{V}])(j\omega) \quad \forall\omega \in \Re, \quad \forall i = 1, 2, \ldots, m$$
$$= \sigma_i((\tilde{V}M - \tilde{U}N)^{-1})(j\omega) \quad \forall\omega \in \Re, \quad \forall i = 1, 2, \ldots, m$$

As $(\tilde{V}M - \tilde{U}N)$, $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathcal{RH}_\infty$ and $(1-\sigma_1^2)^{-1/2}(\tilde{V}M - \tilde{U}N)$ is all-pass, this implies that $(\tilde{V}M - \tilde{U}N) = (1-\sigma_1^2)^{1/2}I_m$. As $[-\tilde{U} \;\; \tilde{V}]$ is normalized and $(\tilde{V}M - \tilde{U}N) = (1-\sigma_1^2)^{1/2}I_m$, it is clear from identity (4) of Lemma 3.5 that $\tilde{C} = \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*$ satisfies $\tilde{C}^*\tilde{C} = \sigma_1^2 I_m$. Given these identities it can be shown that this coprime factorization satisfies identity (1) by the same argument as above. To complete the proof it is necessary to argue that of the controllers satisfying the all-pass condition in (7), it is the one of minimum degree that satisfies identity (5). Theorem 11.1 in Sefton and Ober (1991) (see also Sefton 1991) shows that the McMillan degree of $\tilde{C}^* = \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*$ equals the McMillan degree of the system plus the McMillan degree of the controller K. Hence, of all the stabilizing controllers satisfying the all-pass condition in (7) the one of minimum degree also minimizes the McMillan degree of the associated transfer function $\tilde{C}^* = \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*$. It was noted earlier that these are precisely the transfer functions that satisfy identity (5). This completes the proof. □

The next result is the analogous result for a $p \times m$ system G where $m \geq p$.

Corollary 3.8: *Given a $p \times m$ system G where $m \geq p$, with normalized left and right coprime factorization $(\tilde{N}, \tilde{M})$ and (N, M) respectively, then there exists a minimal Bezout controller, K. Further, every minimal Bezout controller has right coprime factors (U, V) that satisfy the following identities,*

(1) $\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - (1-\sigma_1^2)^{1/2} \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \sigma_1$

(2) $\tilde{M}V - \tilde{N}U = (1-\sigma_1^2)^{1/2} I_p$

(3) $U^*U + V^*V = I_p$

(4) *If* $C^* = M^*U + N^*V$ *then* $CC^* = \sigma_1^2 I_p$

(5) $\|H_C\| < \sigma_1$

where $\sigma_1 = \left\|H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\right\|$

Proof: This follows immediately by substituting $G^{\mathrm{T}} = (M^{\mathrm{T}})^{-1}N^{\mathrm{T}} = \tilde{N}^{\mathrm{T}}(\tilde{M}^{\mathrm{T}})^{-1}$ for G in the previous Theorem. Then, the transpose of the identities (2) to (5) derived from the transposed system, a right coprime factorization of the minimal Bezout controller $K = \tilde{U}^{\mathrm{T}}(\tilde{V}^{\mathrm{T}})^{-1} := UV^{-1}$ for the original system G is recovered. In a manner entirely analogous with the proof in the above Theorem, it can be shown that this factorization also satisfies identity (1). The final part of the proof can also be proved by substituting the transpose of the controller and the transpose of the system into the definition of the minimal Bezout controller, and then transposing the final closed-loop transfer function. □

These results are combined for square systems in the next corollary.

Corollary 3.9: *Given a square $m \times m$ system G, with normalized left and right coprime factorization $(\tilde{N}, \tilde{M})$ and (N, M) respectively, then there exists a minimal Bezout controller, K. Every minimal Bezout controller has left and right coprime factors $(\tilde{U}, \tilde{V})$ and (U, V) which satisfy the following*

(1 *a*) $\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - (1 - \sigma^1)^{1/2} \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \sigma_1$

(1 *b*) $\left\| \begin{bmatrix} N \\ M \end{bmatrix} - (1 - \sigma_1^2)^{1/2} \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} \right\|_\infty = \sigma_1$

(2 *a*) $\tilde{M}V - \tilde{N}U = (1 - \sigma_1^2)^{1/2} I_m$ (2 *b*) $\tilde{V}M - \tilde{U}N = (1 - \sigma_1^2)^{1/2} I_m$

(3 *a*) $U^*U + V^*V = I_m$ (3 *b*) $\tilde{U}\tilde{U}^* + \tilde{V}\tilde{V}^* = I_m$

(4 *a*) *If* $C^* = M^*U + N^*V$ *then* $C^*C = \sigma_1^2 I_m$ *and* (4 *b*) *If* $\tilde{C}^* = \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*$ *then* $\tilde{C}\tilde{C}^* = \sigma_1^2 I_m$ *and*

(5 *a*) $\|H_C\| < \sigma_1$ *where* $\sigma_1 = \|H_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\|$ (5 *b*) $\|H_{\tilde{C}}\| < \sigma_1$

Proof: Noting that given (2 *a*) and (2 *b*) then $C = \tilde{C}$ by Lemma 3.5. The result therefore follows immediately by combining the previous Theorem and Corollary. □

The final result in this section relates the Hankel singular values of the Hankel operator with symbol $[\tilde{M} \ \ -\tilde{N}]^*$ to those of the right coprime factorization (U, V) of the minimal Bezout controller satisyfing equations (1) to (4). It also establishes the McMillan degree of this controller. In order to prove the result the following Lemma is required.

Lemma 3.10. (Nikolskii 1986, p. 409): *Given $\theta \in \mathcal{RL}_\infty^{m\times m}$ a square all-pass transfer function, such that $\theta^*\theta = I_m$, then if* $[\text{Range } (T_\theta)]^\perp = \{0\}$

$$\sigma_k(H_{\theta^*}) = \sigma_{k+1}(H_\theta) \quad \forall k \geq 1$$

with the same multiplicity.

Theorem 3.11: *Given a square $m \times m$ system G, with normalized left and right coprime factorization $(\tilde{N}, \tilde{M})$ and (N, M) respectively, let K be a minimal Bezout controller of the system, G, with normalized right and left coprime factors (U, V) and $(\tilde{U}, \tilde{V})$, then the transfer function*

$$E = \frac{1}{\sigma_1}\left(\begin{bmatrix} \tilde{M}^* & 0 \\ -\tilde{N}^* & 0 \end{bmatrix} - \left[(1 - \sigma_1^2)^{1/2}\begin{bmatrix} V \\ U \end{bmatrix} \ \sigma_1 \begin{bmatrix} V \\ U \end{bmatrix}\right]\right)$$

is all-pass. Further, given the Schmidt decomposition of the Hankel operator with symbol $[\tilde{M} \; -\tilde{N}]^*$ *in Theorem 3.3 then*

$$\sigma_i(H_{[V^* \; U^*]}) = \sigma_i\left(H_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}}\right)$$
$$= \sigma_{i+1}\left(H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\right)$$

with multiplicity $r_{(i+r_1)}$ *and hence the McMillan degree of the minimal Bezout controller is equal to* $n - r_1$, *where* r_1 *is the multiplicity of the first singular value of*

$$H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}$$

Proof: It is straightforward to verify that E is an all-pass transfer function, as $\begin{bmatrix} V \\ U \end{bmatrix}$ satisfies the relationships in Corollary 3.9. In order to prove the second part of the Theorem, Lemma 3.10 is applied to the transfer function E. However, it is first necessary to check that $T_E(\mathcal{H}_2^{2m}) = (\mathcal{H}_2^{2m})$. First note that for any $y \in \mathcal{H}_2^m$ that

$$T_E \begin{bmatrix} 0 \\ y \end{bmatrix} = M_{\begin{bmatrix} V \\ U \end{bmatrix}} y$$

It is claimed that

$$P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}} T_E \begin{bmatrix} \mathcal{H}_2^m \\ 0 \end{bmatrix} = \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}$$

It is now shown that this implies the required result. For any $z \in \mathcal{H}_2^{2m}$, given the claim, there exists an $x \in \mathcal{H}_2^m$ such that

$$P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}} z = P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}} T_E \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Now let

$$M_{\begin{bmatrix} V \\ U \end{bmatrix}} y = P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)} \left(z - T_E \begin{bmatrix} x \\ 0 \end{bmatrix}\right),$$

then

$$T_E \begin{bmatrix} x \\ y \end{bmatrix} = P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)} \left(z - T_E \begin{bmatrix} x \\ 0 \end{bmatrix}\right) + T_E \begin{bmatrix} x \\ 0 \end{bmatrix}$$
$$= P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)} z + P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}} z = z$$

The claim is now proved, using an expression for the orthogonal projection operator (see for example Ober and Sefton 1991),

$$P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^m\right)^{\perp}} T_E \begin{bmatrix} \mathcal{H}_2^m \\ 0 \end{bmatrix} = T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} [T^*_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}}]^{-1} T^*_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} T_E \begin{bmatrix} \mathcal{H}_2^m \\ 0 \end{bmatrix}$$
$$= -\frac{1}{\sigma_1} T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} [T^*_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}}]^{-1} T_{(\tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*)}(\mathcal{H}_2^m)$$

implying the claim if $T_{(\tilde{U}\tilde{M}^*+\tilde{V}\tilde{N}^*)}(\mathcal{H}_2^m) = (\mathcal{H}_2^m)$. As $\tilde{C}^*/\sigma_1 = (\tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*)/\sigma_1$ is all-pass

$$T_{\tilde{C}^*}T_{\tilde{C}} + H_{\tilde{C}}^* H_{\tilde{C}} = \sigma_1^2 I_m$$

Therefore, a sufficient condition for $T_{(\tilde{U}\tilde{M}^*+\tilde{V}\tilde{N}^*)}(\mathcal{H}_2^m) = (\mathcal{H}_2^m)$ is that $\|H_{(\tilde{U}\tilde{M}^*+\tilde{V}\tilde{N}^*)^*}\| < \sigma_1$ which is satisfied by Theorem 3.7, and that $\mathcal{R}(T_{(\tilde{U}\tilde{M}^*+\tilde{V}\tilde{N}^*)})$ is closed.

Now, applying Lemma 3.10 to the transfer function E gives

$$\begin{aligned}\sigma_{i+r_1}(H_{\mathrm{E}}) &= \sigma_{i+r_1}\left(H_{\begin{bmatrix}\tilde{M}^* \\ -\tilde{N}^*\end{bmatrix}}\right) \\ &= \sigma_i(H_{E^*}) \\ &= \sigma_i\left(H_{\begin{bmatrix}(1-\sigma_1^2)^{1/2} \\ \sigma_1\end{bmatrix}[V^* \ U^*]}\right) \\ &= \sigma_i(H_{[V^* \ U^*]})\end{aligned}$$

This also shows the claim on the McMillan degree of $[V^{\mathrm{T}} \ U^{\mathrm{T}}]^{\mathrm{T}}$ since, by Kronecker's Theorem (Nikolskii 1986), the rank of a Hankel operator with rational symbol is equal to the number of unstable poles of the symbol. As the McMillan degree of the controller K is the McMillan degree of its minimum normalized coprime factors (Ober and McFarlane 1989), this established the final claim. The other equality is a known result (Ober and McFarlane 1989).

For an alternative proof of this result in the case of a scalar transfer function, see Fuhrmann and Ober (1993).

4. Bezout perturbations

As a consequence of the results stated in the introduction we know that the robustness measure of a control system in which the plant is stabilized by a minimal Bezout controller is given by $\sqrt{(1-\sigma_1^2)}$. Since no further structure is put on a perturbation other than the demand that it be norm-bounded, it is to be expected that such a control design will be conservative.

This section investigates this point for the particular case when the controller K is the minimal Bezout controller of the square system G. Certain perturbations called Bezout perturbations of the system G are defined for this case, and it is shown that these perturbations have some interesting properties. Further, it is shown that these perturbations can be calculated explicitly, as they are related to the optimal Hankel norm approximations of the normalized coprime factors of the controller.

In order to facilitate the discussion, the following terms will be defined. Given a square $m \times m$ system G with normalized r.c.f (N, M) and a minimal Bezout controller K, a perturbation

$$\begin{bmatrix}\Delta_N \\ \Delta_M\end{bmatrix} \in \mathcal{R}\mathcal{H}_\infty^{2m\times m}$$

is called stabilizing if (G_Δ, K) is internally stable and $((N+\Delta_N), (M+\Delta_M))$ is coprime where $G_\Delta = (N+\Delta_N)(M+\Delta_M)^{-1}$, otherwise it is called destabilizing. If the size of a perturbation refers to its $\mathcal{H}_\infty$-norm, a perturbation is called a minimum norm destabilizing perturbation, if there does not exist a perturbation of smaller size that also destabilizes the closed loop system (G, K).

The definition of the Bezout perturbations is motivated by the following observation, which shows that a minimum norm destabilizing perturbation can be calculated by solving a modified two-block $\mathcal{H}_\infty$ optimization problem. In this modified problem the search for the optimal transfer function is performed over the class $\mathcal{H}_{\infty,k}$ rather than $\mathcal{H}_\infty$. It was shown that this problem can be solved by an extension of the existing $\mathcal{H}_\infty$ optimization theory (Glover and Doyle 1989).

Proposition 4.1: *Given a $m \times m$ rational system G with normalized r.c.f. (N, M) and normalized l.c.f $(\tilde{N}, \tilde{M})$, and a minimal Bezout controller K with normalized r.c.f. (U, V) and normalized l.c.f. $(\tilde{U}, \tilde{V})$ satisfying the relations in Corollary 3.9 then*

$$\inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right\|_\infty = \inf_{\Theta \in \mathcal{B}_{n_k}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -\Theta \\ Q_2 \end{bmatrix} \right\|_\infty$$

$$= \inf_{Q \in \mathcal{H}_{\infty,n_k}} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty$$

where n_k is the McMillan degree of the minimal Bezout controller. Further there exists a Q^{opt} achieving the infimum in the final expression. If Q^{opt} has an inner r.c.f. $Q^{\text{opt}} = Q_2^{\text{opt}}(\Theta^{\text{opt}})^*$ where $\Theta \in \mathcal{B}_{n_k}$, then $Q_1 = \Theta^{\text{opt}}$ and $Q_2 = Q_2^{\text{opt}}$ achieve the infimum in the first expression. Hence

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} \Theta^{\text{opt}} \\ Q_2^{\text{opt}} \end{bmatrix}$$

is a minimum norm destabilizing perturbation of this closed-loop system.

Proof: This observation follows from noticing that

$$\inf_{\Theta \in \mathcal{B}_{n_k}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -\Theta \\ Q_2 \end{bmatrix} \right\|_\infty = \inf_{\Theta \in \mathcal{B}_{n_k}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q_2 \Theta^* \end{bmatrix} \Theta \right\|_\infty$$

$$= \inf_{\Theta \in \mathcal{B}_{n_k}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q_2 \Theta^* \end{bmatrix} \right\|_\infty$$

Now, as $Q_2\Theta^* \in \mathcal{H}_{\infty,n_k}$, and further any $Q \in \mathcal{H}_{\infty,n_k}$ has an inner r.c.f $Q = Q_2\Theta^*$ where $Q_2 \in \mathcal{H}_\infty$ and $\Theta \in \mathcal{B}_{n_k}$, then

$$\inf_{\Theta \in \mathcal{B}_{n_k}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -\Theta \\ Q_2 \end{bmatrix} \right\|_\infty = \inf_{Q \in \mathcal{H}_{\infty,n_k}} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty$$

Since for any $\Theta \in \mathcal{B}_{n_n}$, $(I - \Theta)^{-1} \notin \mathcal{RH}_\infty$, it remains to show that the norm of the first expression and the norm of the third are equal. This can be proved using the relationship in Theorem 3.7. Now

$$\inf_{Q \in \mathcal{H}_{\infty,n_k}} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty = \inf_{Q \in \mathcal{H}_{\infty,n_k}} \left\| \begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix} \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty$$

$$= \inf_{Q \in \mathcal{H}_{\infty,n_k}} \left\| \begin{bmatrix} V^*N + U^*M + Q \\ (1 - \sigma_1^2)^{1/2} \end{bmatrix} \right\|_\infty = (1 - \sigma_1^2)^{1/2}$$

as $V^*N + U^*M \in \mathcal{H}_{\infty,n_k}$ and $Q^{\text{opt}} = V^*N + U^*M$ achieves the infimum. By Proposition 6.4. in Sefton an Ober (1991)

$$\inf_{\substack{Q_1 \in \mathcal{RH}_\infty \\ (I+Q_1)^{-1} \notin \mathcal{RH}_\infty}} \inf_{Q_2 \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right\|_\infty = \tau(\tilde{M}V - \tilde{N}U)$$

$$= (1 - \|N^*V + M^*U\|_\infty^2)^{1/2} = (1 - \sigma_1^2)^{1/2}$$

where the second equality is a consequence of Lemma 3.5. Since

$$\left\|\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}\right\|_\infty = \sqrt{(1-\sigma_1^2)}$$

and

$$[-\widetilde{U} \quad \widetilde{V}]\begin{bmatrix}N+\Delta_N\\ M+\Delta_M\end{bmatrix} = I + \Theta^{\text{opt}}$$

where $(I+\Theta^{\text{opt}})^{-1} \neq \mathcal{H}_\infty$ we have that

$$\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}$$

is a destabilizing perturbation. □

The above argument motivates the definition of the Bezout perturbations. First recall the following result from the preliminary section.

Lemma 4.2: *Given a square $m \times m$ system, G with normalized r.c.f (N, M) and normalized l.c.f $(\widetilde{N}, \widetilde{M})$, and then let $(g_{g(i,j)}, f_{g(i,j)})$ for $j = 1, 2, \ldots r_i$ be a set of orthogonal (normalized) Schmidt pairs which span the r_i^{th} dimensional eigenspace corresponding to singular value, σ_i of the Hankel operator with symbol $[-\widetilde{N} \quad \widetilde{M}]^*$, that is for $h \in \mathcal{H}_2^m$*

$$H_{\begin{bmatrix}\widetilde{M}^*\\ -\widetilde{N}^*\end{bmatrix}} h = \sum_{i=1}^{k} \sigma_i \sum_{j=1}^{r_i} < f_{g(i,j)}, h > g_{g(i,j)}$$

where $g_{g(i,j)} \in (\mathcal{H}_2^{p+m})^\perp$ $f_{g(i,j)} \in \mathcal{H}_2^m$ and $\sum r_i = n$, the McMillan degree of the system G. Also given a minimal Bezout controller with left and right normalized coprime factors $(\widetilde{U}, \widetilde{V})$ and (U, V) respectively then the Schmidt decomposition of the Hankel operator with symbol $[-\widetilde{U} \quad \widetilde{V}]^*$ can be written

$$H_{\begin{bmatrix}-\widetilde{U}^*\\ \widetilde{V}^*\end{bmatrix}} h = \sum_{i=2}^{k} \sigma_i \sum_{j=1}^{r_i} < f_{k(i-1,j)}, h > g_{k(i-1,j)}$$

where $g_{k(i-1,j)} \in (\mathcal{H}_2^{p+m})^\perp$, $f_{k(i-1,j)} \in \mathcal{H}_2^m$ are a Schmidt pair corresponding to singular value σ_i.

Proof: The proof follows immediately from Theorem 3.11.

Definition 4.3: Given the assumptions in Lemma 4.2 then

$$\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}_i \in \mathcal{R}\mathcal{H}_\infty^{2m\times m}$$

is a Bezout perturbation in the ith direction $i = 1, 2, \ldots, k$ where k is the number of distinct singular values of

$$H_{\begin{bmatrix}-\widetilde{M}^*\\ -\widetilde{N}^*\end{bmatrix}}$$

if and only if

$$\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}_i = \begin{bmatrix}N & V\\ M & U\end{bmatrix}\begin{bmatrix}-\Theta_i\\ Q_i\end{bmatrix}$$

where (Θ_i, Q_i) is an inner r.c.f of a $Q_i^{\text{opt}} = Q_i\Theta_i^*$ that achieves the infimum in the expression

$$\inf_{Q\in\mathcal{H}_{\infty,n_i}} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty$$

where $n_i = \sum_{j=2}^{i} r_j$, $n_1 = 0$. □

It is apparent from the formulation of the definition that

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_k \right\|_\infty \leqslant \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_{k-1} \right\|_\infty \leqslant \cdots \leqslant \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_2 \right\|_\infty \leqslant \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_1 \right\|_\infty$$

It will be shown in the next theorem that these are, in fact, strict inequalities, and that the norm of these perturbations can be calculated explicitly in terms of the Hankel singular values σ_i.

The Bezout perturbations in the kth direction is clearly a minimum norm destabilizing perturbation from Proposition 4.1. The Bezout perturbation in the first direction perturbs the system G to the inverse of the controller. Note that

$$\begin{bmatrix} N \\ M \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_1 = \begin{bmatrix} N \\ M \end{bmatrix} + \left(- \begin{bmatrix} N \\ M \end{bmatrix} + \begin{bmatrix} V \\ U \end{bmatrix} Q_1 \right) = \begin{bmatrix} V \\ U \end{bmatrix} Q_1$$

and therefore $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1} = VU^{-1} = K^{-1}$ and $(I - G_\Delta K) = 0$. It could be argued that this is the worst or most destructive perturbation as, in this case, every input signal into the closed-loop system (G_Δ, K) will result in an unbounded output signal. The Bezout perturbations in the other directions are also destabilizing. These Bezout perturbations can be seen to lie between the minimum norm destabilizing perturbation and the worst case perturbation in the first direction in degree of severity.

However, any perturbation

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = \delta \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i, \quad i = 1, 2, \ldots, k$$

is not destabilizing if $\delta < 1$. Therefore, the Bezout perturbations can be seen as describing alternative perturbation directions where the size of the perturbations that preserve closed-loop stability can be larger than the stability margin. There is one further perturbation direction of interest, of the form

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = \begin{bmatrix} V \\ U \end{bmatrix}$$

which is called the harmless direction. Any perturbation in this direction is not destabilizing whatever its size.

The next theorem proves the earlier claim that the Bezout perturbations are optimal Hankel norm approximations of the normalized coprime factors of the controller.

Theorem 4.4: *Given the assumptions in Lemma 4.2 and the notation of Definition 4.3 then*

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \right\|_\infty = \frac{(1 - \sigma_1^2)^{1/2}}{(1 - \sigma_{i+1}^2)^{1/2}} \quad i = 1, 2, \ldots, k$$

where $\sigma_{k+1} := 0$. *Further, the Bezout perturbations satisfy the extension*

$$\sigma_{i+1} = \left\| \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \frac{(1-\sigma_{i+1}^2)}{(1-\sigma_1^2)^{1/2}} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \Theta_i^* \right\|_\infty \qquad i = 1, 2, \ldots, k$$

Proof: First note that the r.c.f (U, V) satisfy the identities in Corollary 3.9. Let $(\hat{N}, \hat{N}) = ((1-\sigma_1^2)^{-1/2}N, (1-\sigma_1^2)^{-1/2}M)$, then $[\hat{N} \;\; \hat{M}]$ satisfy the Bezout identity $\tilde{V}\hat{M} - \tilde{U}\hat{N} = I$. In Fuhrmann and Ober (1993) it is shown that the singular values of the Hankel oprator with symbol $H_{(V^*\hat{N}+U^*\hat{M})}$ are invariants of the closed-loop system, and that

$$\sigma_i(H_{(V^*\hat{N}+U^*\hat{M})}) = (\sigma_i(H_{[V^* \; U^*]}))/(1 - \sigma_i^2(H_{[V^* \; U^*]}))^{1/2}$$

In Theorem 3.11 the Hankel singular values of $H_{[V^* \; U^*]}$ are expressed in terms of the Hankel singular values of $H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}$. Using this information it is clear that

$$(1-\sigma_1^2)^{-1/2}\sigma_i(H_{(V^*N+U^*M)}) = \sigma_i(H_{(V^*\hat{N}+U^*+\hat{M})}) = \frac{\sigma_{i+1}}{(1-\sigma_{i+1}^2)^{1/2}}$$

with multiplicity r_{i+1}. Therefore, by the theory of Hankel norm approximation, (see for example Glover 1984), this implies that

$$\inf_{Q \in \mathcal{H}_{\infty,n_i}} \|V^*N + U^*M + Q\|_\infty = \frac{(1-\sigma_1^2)^{1/2}\sigma_{i+1}}{(1-\sigma_{i+1}^2)^{1/2}}$$

where $n_i = \sum_{j=2}^{i} r_j$. The norm of the Bezout perturbations follows from this fact as

$$\begin{aligned}
\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \right\|_\infty &= \inf_{Q \in \mathcal{H}_{\infty,n_i}} \left\| \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty \\
&= \inf_{Q \in \mathcal{H}_{\infty,n_i}} \left\| \begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix} \begin{bmatrix} N & V \\ M & U \end{bmatrix} \begin{bmatrix} -I \\ Q \end{bmatrix} \right\|_\infty \\
&= \inf_{Q \in \mathcal{H}_{\infty,n_i}} \left\| \begin{bmatrix} V^*N + U^*M + Q \\ (1-\sigma_1^2)^{1/2} \end{bmatrix} \right\|_\infty \\
&= \left\{ (1-\sigma_1^2) + \inf_{Q \in \mathcal{H}_{\infty,n_i}} \|V^*N + U^*M + Q\|_\infty^2 \right\}^{1/2}
\end{aligned}$$

where $n_i = \sum_{j=2}^{i} r_j$. Therefore, if $i \leq k-1$

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \right\|_\infty = \left\{ (1-\sigma_1^2) + (1-\sigma_1^2)\frac{\sigma_{i+1}}{(1-\sigma_{i+1}^2)^{1/2}} \right\}^{1/2} = \frac{(1-\sigma_1^2)^{1/2}}{(1-\sigma_{i+1}^2)^{1/2}}$$

and if $i = k$

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_k \right\|_\infty = (1-\sigma_1^2)^{1/2}$$

as $V^*N + U^*M \in \mathcal{H}_{\infty,n_k}$. It is now shown that

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i$$

also satisfies the extension in the statement of the theorem. Note from the definition of the Bezout peturbations that, $[-\tilde{U} \;\; \tilde{V}][\Delta_N^{\mathrm{T}} \;\; \Delta_M^{\mathrm{T}}]^{\mathrm{T}} = (1-\sigma_1^2)^{1/2}\Theta_i$.

This implies that

$$\frac{(1-\sigma_1^2)^{1/2}}{(1-\sigma_{i+1}^2)^{1/2}} = \left\| \begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \right\|_\infty$$

$$= \left\| \begin{bmatrix} V^*(\Delta_N)_i + U^*(\Delta_M)_i \\ (1-\sigma_1^2)^{1/2}\Theta_i \end{bmatrix} \right\|_\infty$$

$$= ((1-\sigma_1^2)^{1/2} + \|V^*(\Delta_N)_i + U^*(\Delta_M)_i\|_\infty^2)^{1/2}$$

and hence

$$\|V^*(\Delta_N)_i + U^*(\Delta_M)_i\|_\infty = \frac{\sigma_{i+1}(1-\sigma_1^2)^{1/2}}{(1-\sigma_{i+1}^2)^{1/2}}$$

It is now clear that

$$\left\| \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \frac{(1-\sigma_{i+1}^2)}{(1-\sigma_1^2)^{1/2}} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \Theta_i^* \right\|_\infty$$

$$= \left\| \begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix} \left(\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \frac{(1-\sigma_{i+1}^2)}{(1-\sigma_1^2)^{1/2}} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_i \Theta_i^* \right) \right\|_\infty$$

$$= \left\| \begin{bmatrix} \dfrac{(1-\sigma_{i+1}^2)}{(1-\sigma_1^2)^{1/2}} (V^*(\Delta_N)_i + U^*(\Delta_M)_i)\Theta_i^* \\ I - (1-\sigma_{i+1}^2)I \end{bmatrix} \right\|_\infty$$

$$= (\sigma_{i+1}^4 + \sigma_{i+1}^2(1-\sigma_{i+1}^2))^{1/2} = \sigma_{i+1} \qquad \square$$

This theorem shows that it is possible to calculate explicitly the size of the Bezout perturbations and that they are, in fact, scaled optimal Hankel-norm approximations of the normalized coprime factors of the controller. If the system G is single-input single-ouput then the Hankel norm approximations are unique and therefore the Bezout perturbations are the unique optimal Hankel norm approximations of the coprime factors of the controller. In the multivariable case, it would be necessary to construct a particular optimal all-pass approximation, using an argument similar to the one used in Theorem 3.7. It is also apparent from this theorem that as

$$0 = \left\| \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \frac{1}{(1-\sigma_1^2)^{1/2}} \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_k \Theta_k^* \right\|_\infty .$$

a minimum norm destabilizing perturbation of this closed-loop system is

$$\begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix}_k = (1-\sigma_1^2)^{1/2} \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} \Theta_k$$

where Θ_k is the inner function of minimal degree such that

$$\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} \Theta_k \in \mathcal{RH}_\infty$$

References

Adamyan, V., Arov, D. Z., and Krein, M., 1971, Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem. *Math USSR Sbornick*, **15**, 31–73.

Foias, C., and Frazho, A., 1990, *The Commutant Lifting Approach to Interpolation Problems* (Basel: Birkhäuser).

Francis, B., 1987, *A Course in* $\mathcal{H}_\infty$ *Control Theory*. Lecture Notes in Control and Information Science, Vol. 88 (Berlin: Springer-Verlag).

Fuhrmann, P., and Ober, R., 1993, A functional approach to LQG balancing. *International Journal of Control*, **57**, 627–741.

Georgiou, T., and Smith, M., 1990, Optimal robustness in the gap metric. *IEEE Transactions on Automatic Control*, **35**, 673–686.

Glover, K., 1984, All optimal Hankel-norm approximations of linear multivariable systems and their $\mathcal{L}_\infty$ error bounds. *International Journal of Control*, **39** 1115–1193; 1989, *A Tutorial on Hankel Norm Approximation* (Springer-Verlag).

Glover, K., and Doyle, J., 1989, General distance problem in H-infinity (k). *Proceedings of the Symposium on Mathematical Theory of Networks and Systems*, Amsterdam, June.

McFarlane, D., and Glover, K., 1989, *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*. Lecture Notes in Control and Information Sciences, Vol 110 (Springer-Verlag).

Nikolskii, N., 1986, *Treatise on the Shift Operator*. Grundlehren der mathematischen Wissenschaft (Berlin: Springer-Verlag).

Ober, R., and McFarlane, D., 1989, Balanced canonical forms for minimal systems: A normalized coprime factor approach. *Linear Algebra and its Applications*, **122–124**, 23–64.

Ober, R., and Sefton, J., 1991, Stability of control systems and graphs of linear systems. *Systems and Control Letters*, **16**, 265–280.

Sefton, J., 1991, A Geometrical Approach to Feedback Stability. Ph.D. thesis, Department of Engineering, University of Cambridge, U.K.

Sefton, J., and Ober, R., 1993, On the gap metric and coprime factor perturbations. Technical Report 208, Programs in Mathematical Sciences, 1991. *Automatica*, **29**, 723–734; 1993, Uncertainty in the weighted gap metric: a geometric Technical Report 214, Programs in Mathematical Sciences, 1991. *Ibid*., **29**, 1079–1100.

Sz-Nagy, B., and Foias, C., 1976, On contractions similar to isometries and Toeplitz operators. *Annales Academiae Scientiarum Fennicae, Series A. I. Mathematica*, **2**, 553–564.

Vidyasagar, M., 1985, *Control System Synthesis: a Factorization Approach* (Cambridge, Mass: MIT Press).

Vidyasagar, M., and Kimura, H., 1986, Robust controllers for uncertain linear multivariable systems. *Automatica*, **22**, 85–94.