Hankel-Norm Approximation and Control Systems

J. A. Sefton* and R. J. Ober
University of Texas at Dallas
Center for Engineering Mathematics
Richardson, Texas 75083-0688

Submitted by Paul A. Fuhrmann

ABSTRACT

Connections are established between Hankel-norm approximation, the problem of finding approximating subspaces in the Hilbert space $\mathscr{H}_2$, and stability and instability of control systems.

1. INTRODUCTION

Over the years it has become more and more evident that operator theory can be of great help in analyzing linear dynamical systems and in particular control systems (see e.g. [6]). This paper aims at establishing further connections between control theory and the theory of Hankel operators. We were motivated to do this work by results that interpreted robustness properties of control systems from the point of view of the geometry between the graph spaces of the plant and the controller (see e.g. [13, 1]). Let

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

$$y = Cx + Du$$

be a finite-dimensional linear continuous-time system, which we do not necessarily assume to be stable. By taking the Laplace transform and assum-

* Present address: Department of Applied Economics, Cambridge University, Sidegewick Av., Cambridge CB3 9DE, England.

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ing $x_0 = 0$, we obtain the transfer function $G(s) = C(sI - A)^{-1}B + D$, which is a matrix-valued proper rational function (see e.g. [10]). In the transform domain the linear system can be seen as acting as a multiplication operator on the Hardy space $\mathscr{H}_2$, which can be interpreted as the space of Laplace transforms of the space $L^2([0, \infty])$. Since we are dealing with multivariable systems, these are spaces of vector-valued functions. For convenience of notation we will however often drop the corresponding indices. With the transfer function G we will associate the graph $\mathscr{G}(G)$ of the multiplication operator with symbol G, i.e. the graph of the operator $M_G: \mathscr{H}_2 \to \mathscr{H}_2; f \mapsto Gf$. Clearly, if the system is not stable and therefore G has poles in the closed right half plane, then M_G will not be defined on the whole of $\mathscr{H}_2$.

In order to obtain a workable representation of the graph of M_G we need to introduce coprime factorizations (see e.g. [15]). The factorization $G = NM^{-1}$ ($G = \tilde{M}^{-1}\tilde{N}$) is a right (left) coprime factorization of G if $N, M \in \mathscr{R}\mathscr{H}_\infty$ ($\tilde{N}, \tilde{M} \in \mathscr{R}\mathscr{H}_\infty$), the space of proper real rational functions with poles only in the open left half plane, and N, M ($\tilde{N}, \tilde{M}$) are right (left) coprime, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathscr{R}\mathscr{H}_\infty$ ($X, Y \in \mathscr{R}\mathscr{H}_\infty$) such that $-\tilde{X}N + \tilde{Y}M = I$ ($-\tilde{N}X + \tilde{M}Y = I$). The factorization is called normalized if moreover $N^*N + M^*M = I$ ($\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$). Given a right coprime factorization $G = NM^{-1}$, the graph of M_G can be characterized as

$$\mathscr{G}(G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2.$$

In what follows we will make much use of the following geometric notions in a Hilbert space H (see e.g. [8, 12, 16]). In our case the Hilbert space H will be the space $\mathscr{H}_2 \times \mathscr{H}_2$. Let $A, B \subseteq H$ be two closed subspaces; then it is possible to define the minimal angle and the gap between these two spaces as follows:

$$\cos \theta_{\min}(A, B) = \sup_{u \in A,\, v \in B} \frac{|\langle u, v \rangle|}{\|u\| \, \|v\|},$$

$\theta_{\min}(A, B) \in [0, \pi/2]$, and

$$\operatorname{gap}(A, B) = \|P_A - P_B\|,$$

where P_C denotes the orthogonal projection onto the closed subspace C. Alternatively the sine of the minimal angle can be defined by

$$\sin \theta_{\min}(A, B) = \|P_{A\|B}\|^{-1},$$

where the skew projection $P_{A\|B}$ is defined by $P_{A\|B}: A + B \to A$, $u + v \mapsto u$, $u \in A$, $v \in B$. The skew projection is well defined on the Hilbert space H if $H = A + B$ and $A \cap B = \emptyset$. The skew projection is bounded if and only if $\theta_{\min}(A, B) > 0$. The following relationships hold:

$$\cos \theta_{\min}(A, B) = \|P_A P_B\| = \|P_B P_A\| = \sup_{u \in B, \|u\|=1} \operatorname{dist}(u, A^{\perp}),$$

where $\operatorname{dist}(u, A^{\perp}) = \inf_{v \in A^{\perp}} \|u - v\|$. The gap between two spaces can be characterized as follows:

$$\begin{aligned} \operatorname{gap}(A, B) &= \max\{\|P_A P_{B^{\perp}}\|, \|P_{A^{\perp}} P_B\|\} \\ &= \max\{\cos \theta_{\min}(A, B^{\perp}), \cos \theta_{\min}(B, A^{\perp})\} \\ &= \max\left\{ \sup_{u \in A, \|u\|=1} \operatorname{dist}(u, B), \sup_{v \in B, \|v\|=1} \operatorname{dist}(v, A)\right\}. \end{aligned}$$

If $\operatorname{gap}(A, B) < 1$ then $\|P_A P_{B^{\perp}}\| = \|P_{A^{\perp}} P_B\|$.

The central issue in the area of control theory is the stabilization of unstable systems by a controller K. With a controller K we associate the so-called transposed graph $\mathscr{G}^T(K)$ of the controller, i.e., if $K = UV^{-1}$ is a right coprime factorization of K, then

$$\mathscr{G}^T(K) := \begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2.$$

In [13] the following equivalent conditions were proved for a controller K to stabilize the plant G.

THEOREM 1.1. *Let $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be a right (left) coprime factorization of a $p \times m$ plant G, and let $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ be a right (left) coprime factorization of a controller K. Then the following statements are* *equivalent*:

(S0) *the control system* (G, K) *is internally stable, i.e.*

$$\begin{pmatrix} I & K \\ G & I \end{pmatrix} \in \mathcal{RH}_\infty,$$

(S1) *the function* $-\tilde{N}U + \tilde{M}V$ *is invertible in* $\mathcal{RH}_\infty$,
(S2) *the function* $-\tilde{U}N + \tilde{V}M$ *is invertible in* $\mathcal{RH}_\infty$,
(S3) $\mathcal{G}(G) + \mathcal{G}^T(K) = \mathcal{H}_2^{p+m}$,
(S4) $P_{[\mathcal{G}^T(K)]^\perp}\mathcal{G}(G) = [\mathcal{G}^T(K)]^\perp$,
(S5) $\theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp) > 0$,
(S6) $\operatorname{gap}(\mathcal{G}(G), [\mathcal{G}^T(K)]^\perp) < 1$.

Conditions (S1) and (S2) are the classical conditions for internal stability of a control system (see e.g. [15]). A substantial part of this paper will be devoted to an extension of these results to the case when the control system has a certain number of unstable poles.

In [13] it was argued that $\theta_{\min}([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp)$ is a good indicator for how far away a control system is from instability. When designing a controller for a given plant G it therefore appears natural to try to find the controller that maximizes this angle, i.e. to find a controller K_0 such that

$$\theta_{\min}\left([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K_0)]^\perp\right) = \sup_{K \text{ proper rational}} \theta_{\min}\left([\mathcal{G}(G)]^\perp, [\mathcal{G}^T(K)]^\perp\right)$$

$$= \sup_{K \text{ stabilizing}} \theta_{\min}\left(\mathcal{G}(G), \mathcal{G}^T(K)\right).$$

It was shown in [13, Section 6] that such a controller does exist and that it in fact coincides with the optimally robust controller with respect to normalized coprime factor uncertainty as studied in [11]. This controller is characterized through the solution of the following Nehari extension problem: $K_0 = U_0 V_0^{-1}$, where

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V_0 \\ U_0 \end{bmatrix} \right\|_\infty = \inf_{U, V \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \left\| H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} \right\|,$$

where $G = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization of G, and $H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}$ is the Hankel operator with symbol $\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}$.

Theorem 1.1 gives characterizations of stability in terms of the graph of the system and the transposed graph of the controller. It has been shown [13] that internal stability is equivalent to the minimal angle between the orthogonal complement of the graph associated with the system and the orthogonal complement of the transposed graph associated with the controller being greater than zero. Therefore, if the system is unstable, there exists an intersection between these subspaces. One of the aims of this paper is to characterize the intersection between these two subspaces. As this subspace is orthogonal to both the graph space associated with the system and the transposed graph space associated with the controller, the closed-loop system behaves as a stable system on this span of these graph spaces. This characterization enables most of the stability conditions in Theorem 1.1 to be generalized to unstable closed-loop systems with a finite number of poles in the open right half plane. Also, the angles between these subspaces can be calculated from expressions that involve the normalized coprime factors of the plant and the controller. Finally, one further condition for closed-loop stability in terms of the index of a Fredholm operator is briefly explored.

The notation used throughout this paper is standard in the control literature [6]. For a matrix $M \in \mathfrak{R}^{p \times m}$ or $\mathscr{C}^{p \times m}$, M^T denotes its transpose, M^* its conjugate transpose, $\sigma_{\max}(M)$ its maximum singular value, σ_i its ith singular value, and $\sigma_{\min}(M)$ its minimum singular value.

The Hardy spaces $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$ consist of all p-vector-valued functions f square-integrable on the imaginary axis with analytic continuations f_r and f_l into the right and the left plane, respectively, such that $\sup_{x>0} \int_{-\infty}^{\infty} \|f_r(x+iy)\|^2\, dy < \infty$ and $\sup_{x<0} \int_{-\infty}^{\infty} \|f_l(x+iy)\|^2\, dy < \infty$. The Hilbert space $\mathscr{L}_2^p$ is given by $\mathscr{L}_2^p = \mathscr{H}_2^p \oplus (\mathscr{H}_2^p)^\perp$, and the orthogonal projections P_+ and P_- map $\mathscr{L}_2^p$ onto $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$ respectively. The norm of a function $f \in \mathscr{H}_2^p$ is denoted $\|f\|_2$. The Hardy space $\mathscr{H}_\infty^{p\times m}$ consists of all $p \times m$ essentially bounded measurable functions f on the imaginary axis with analytic continuation f in the right half plane such that $\sup_{s\in \text{RHP}} \|f(s)\| < \infty$. It is a subspace of $\mathscr{L}_\infty^{p\times m}$ of all $p \times m$ bounded measurable functions on the imaginary axis. The $\mathscr{L}_\infty$ norm is defined by $\|G\|_\infty := \operatorname{ess\,sup}_{\omega\in\mathfrak{R}} \sigma_{\max}[G(i\omega)]$. The essential minimum on the imaginary axis is defined by $\tau(G) := \operatorname{ess\,inf}_{\omega\in\mathfrak{R}} \sigma_{\min}[G(i\omega)]$. For a system G, G^* denotes its complex conjugate transposed, i.e. $G(s)^* = \overline{G(-\bar{s})}^T$. The space $\mathscr{R}\mathscr{H}_2^p$ denotes the subspace of $\mathscr{H}_2^p$ containing the real rational functions; similar definitions apply to the other spaces. By $\mathscr{H}_{\infty,k}$ is meant the subset of $\mathscr{L}_\infty$ consisting of functions that can be written as the sum of a function in $\mathscr{H}_\infty$

plus a proper rational function that has at most k poles in the open right half plane. $\mathscr{B}_n$ denotes the set of rational square inner functions in $\mathscr{H}_\infty$ of McMillan degree at most n.

The domain and range of an operator Z are denoted by $\mathscr{D}(Z)$ and $\mathscr{R}(Z)$ respectively. The orthogonal projection operator onto a closed space $\mathscr{A}$ of $\mathscr{L}_2^p$ is denoted by P_A. Given a $p \times m$ symbol G, the multiplication operator $M_G: \mathscr{D}(M_G) \to \mathscr{H}_2^m$ is defined by $f \mapsto Gf$. If $G \in \mathscr{L}_\infty^{p\times m}$, then the Laurent operator $L_G: \mathscr{L}_2^m \to \mathscr{L}_2^p$, the Hankel operator $H_G: \mathscr{H}_2^m \to (\mathscr{H}_2^p)^\perp$, and the Toeplitz operator $T_G: \mathscr{H}_2^m \to \mathscr{H}_2^p$ with symbol G are defined by $f \mapsto Gf$, $f \mapsto P_{(\mathscr{H}_2^p)^\perp} Gf$, and $f \mapsto P_{\mathscr{H}_2^p} Gf$ respectively.

2. GRAPHS OF LINEAR SYSTEMS AND INSTABILITY

The first definition generalizes the usual definition of internal stability (see e.g. [15]) to include closed-loop systems with a finite number of poles in the open right half plane.

DEFINITION 2.1. Given a $p \times m$ system and an $m \times p$ controller with transfer functions G and K respectively, then the pair (G, K) is called *unstable to order* k, $k = 0, 1, \dots,$ if

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - GK)^{-1} & -(I - GK)^{-1}G \\ -K(I - GK)^{-1} & (I - KG)^{-1} \end{bmatrix} \in \mathscr{RH}_{\infty,k}^{(m+p)\times(m+p)}.$$

It is evident that a pair (G, K) is unstable to order k only if it is also unstable to order $k - 1$, and that a pair (G, K) is unstable to order zero if it is internally stable (see e.g. [15]).

It is noted that the set $\mathscr{RH}_{\infty,k}$ is not a linear space. However, the following statements hold for any transfer function $Z \in \mathscr{RH}_{\infty,k}$ and $Z \notin \mathscr{RH}_{\infty,k-1}$: $UZ \in \mathscr{RH}_{\infty,k}$ and $U + Z \in \mathscr{RH}_{\infty,k}$ for any $U \in \mathscr{RH}_\infty$ of appropriate dimensions, and also $UZ \notin \mathscr{RH}_{\infty,k-1}$ if U is a unit, i.e. $U \in \mathscr{RH}_\infty$ and $U^{-1} \in \mathscr{RH}_\infty$.

The following result is a generalization of well-known stability criteria (see e.g. [15]).

PROPOSITION 2.2. *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$, and the $m \times p$ transfer function K has a r.c.f.*

(U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$. Then the following statements are equivalent:

(U0) *The pair (G, K) is unstable to order k.*

(U1) *There exists a right inner-outer factorization,*

$$\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0,$$

where $\tilde{\Theta} \in \mathscr{RH}_\infty^{p\times p}$ is an inner function of McMillan degree less than or equal to k and $\tilde{S}_0 \in \mathscr{RH}_\infty^{p\times p}$ is a unit. The factors are unique up to right and a left multiplication, respectively by a constant unitary matrix.

(U2) *There exists a right inner-outer factorization*

$$\tilde{V}M - \tilde{U}N = \Theta S_0,$$

where $\Theta \in \mathscr{RH}_\infty^{m\times m}$ is an inner function of McMillan degree less than or equal to k and $S_0 \in \mathscr{RH}_\infty^{m\times m}$ is a unit. The factors are unique up to right and a left multiplication, respectively, by a constant unitary matrix.

Proof. First note that by using the coprime factorizations of G and K we obtain

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-GK)^{-1} & -(I-GK)^{-1}G \\ -K(I-GK)^{-1} & I+K(I-GK)^{-1}G \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} V \\ -U \end{bmatrix}(\tilde{M}V - \tilde{N}U)^{-1}\begin{bmatrix} \tilde{M} & -\tilde{N} \end{bmatrix}.$$

We now show that (U0) implies (U1). If the pair (G, K) is unstable to order k, this implies by the above equation that $[V^T \;\; -U^T]^T(\tilde{M}V - \tilde{N}U)^{-1}[\tilde{M} \;\; -\tilde{N}] \in \mathscr{RH}_{\infty,k}$. As $(\tilde{N}, \tilde{M})$ and (U, V) are left and right coprime, respectively, there exist $X, \tilde{X}, Y, \tilde{Y} \in \mathscr{RH}_\infty$ of appropriate dimensions such that $\tilde{M}Y - \tilde{N}X = I$ and $\tilde{Y}V - \tilde{X}U = I$. Hence

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix}\left(\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right)\begin{bmatrix} Y \\ X \end{bmatrix} = (\tilde{M}V - \tilde{N}U)^{-1} \in \mathscr{RH}_{\infty,k}.$$

As $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathscr{RH}_{\infty,k}$ and $(\tilde{M}V - \tilde{N}U) \in \mathscr{RH}_\infty$, the function $\tilde{M}V - \tilde{N}U$ has an inner-outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$ such that $\tilde{S}_0$ is a unit in $\mathscr{RH}_\infty$ and $\tilde{\Theta}$ has McMillan degree less than or equal than k (see e.g. [3, 9, 2]). Such a factorization is unique up to right respectively left multiplication by a constant unitary matrix.

To show that (U1) implies (U0), note that the factorization in (U1) implies that $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathscr{R}\mathscr{H}_{\infty,k}$ which implies by the above equation that

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} V \\ -U \end{bmatrix} (\tilde{M}V - \tilde{N}U)^{-1} \begin{bmatrix} \tilde{M} & -\tilde{N} \end{bmatrix}$$

$$\in \mathscr{R}\mathscr{H}_{\infty,k}^{(m+p)\times(m+p)}.$$

The equivalence of (U0) and (U2) is proved entirely analogously. ■

One of the aims of this section is to interpret the previous result from a geometric point of view by considering the graph of the plant and the graph of the controller. Before the main results of this section are stated, the following technical results are proved.

LEMMA 2.3. *Let $\mathscr{A}$ be a closed subspace of a Hilbert space $\mathscr{H}$. If Z is an invertible operator on $\mathscr{H}$, then the space $\mathscr{H}$ can be decomposed as*

(1) $\mathscr{H} = Z(\mathscr{A}) \oplus (Z^*)^{-1}(\mathscr{A}^\perp)$,
(2) $\mathscr{H} = \mathscr{A} + (Z^*Z)^{-1}(\mathscr{A}^\perp)$,
(3) $\mathscr{H} = (Z^*Z)(\mathscr{A}) + \mathscr{A}^\perp$.

Proof. (1): As Z is an invertible operator on $\mathscr{H}$, the space $Z(\mathscr{A})$ is closed. Now note that $\mathscr{H} \ominus Z(\mathscr{A}) = [\mathscr{R}(Z|_{\mathscr{A}})]^\perp = \operatorname{Ker}(P_{\mathscr{A}} Z^*|_{\mathscr{H}}) = (Z^*)^{-1}\mathscr{A}^\perp$. This therefore implies the first identity.

(2) and (3) follow immediately from the first decomposition on multiplying these expressions by Z^{-1} and Z^* respectively. ■

For compactness of notation the operator Z_G will be defined. Given a $p \times m$ transfer function G with normalized l.c.f. $(\tilde{N}, \tilde{M})$, the operator Z_G is defined as

$$Z_G := T^*_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} = T^*_{\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}} T_{\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}}.$$

It has been shown in [18] that this is a positive boundedly invertible operator. It therefore has a positive square root, denoted by $Z_G^{1/2}$, i.e. $Z_G^{1/2} Z_G^{1/2} = Z_G$ (see e.g. [16]). In [18] it was also shown that

$$\mathscr{R}\left(T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} \right) = [\mathscr{G}(G)]^\perp.$$

In the following lemma this orthogonal complement is further decomposed. For ease of presentation we introduce the following notation for subspaces of this orthogonal complement. For a $p \times p$ inner function $\tilde{\Theta}$ and $G = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization of G, set

$$I(G,\tilde{\Theta}) := T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1}\left(\tilde{\Theta}\mathscr{H}_2^p\right),$$

$$F(G,\tilde{\Theta}) := T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\left(\mathscr{H}_2^p \ominus \tilde{\Theta}\mathscr{H}_2^p\right).$$

We also need a similar notation that will be used to decompose the orthogonal complements of transposed graphs. For a $m \times m$ inner function Θ and $K = \tilde{V}^{-1}\tilde{U}$ a normalized left coprime factorization of K, set

$$I^T(K,\Theta) := T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}} Z_K^{-1}\left(\Theta\mathscr{H}_2^m\right),$$

$$F^T(K,\Theta) := T_{\begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix}}\left(\mathscr{H}_2^m \ominus \Theta\mathscr{H}_2^m\right).$$

LEMMA 2.4. *Given a $p \times m$ transfer function G, with normalized left coprime factorization $(\tilde{N}, \tilde{M})$, then the space $\mathscr{H}_2^p \times \mathscr{H}_2^m$ can be decomposed as*

$$\mathscr{H}_2^{p+m} = \mathscr{G}(G) \oplus I(G,\tilde{\Theta}) \oplus F(G,\tilde{\Theta})$$

for any inner function $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty$. Further, the orthogonal projection operator onto the closed subspace $I(G,\tilde{\Theta})$ can be written as

$$P_{I(G,\tilde{\Theta})} = T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}}\left[T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}\right]^{-1} T_{\tilde{\Theta}^*} Z_G^{-1} T_{[\tilde{M} \quad -\tilde{N}]}.$$

Proof. By the remark preceding the statement of the lemma and the fact that Z_G is boundedly invertible, we have that

$$\mathscr{H}_2^{p+m} = \mathscr{G}(G) \oplus T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1/2}\left(\mathscr{H}_2^p\right).$$

The orthogonal complement of the graph space of the system G can now be further decomposed as

$$T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1/2}(\mathscr{H}_2^p) = T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1/2}\left[Z_G^{-1/2}\left(\tilde{\Theta}\mathscr{H}_2^p\right)\right]$$

$$+ T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1/2}\left[\mathscr{H}_2^p \ominus \left(Z_G^{-1/2}\tilde{\Theta}\mathscr{H}_2^p\right)\right]$$

$$= T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1}\left(\tilde{\Theta}\mathscr{H}_2^p\right) + T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}\left(\mathscr{H}_2^p \ominus \tilde{\Theta}\mathscr{H}_2^p\right),$$

$$= I(G, \tilde{\Theta}) + F(G, \tilde{\Theta}),$$

where the second line follows from the first decomposition in Lemma 2.3. These spaces are also orthogonal, as for any $x \in \tilde{\Theta}\mathscr{H}_2^p$ and $y \in (\tilde{\Theta}\mathscr{H}_2^p)^\perp$,

$$\left\langle T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1}x, T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} y \right\rangle = \langle Z_G Z_G^{-1}x, y\rangle = \langle x, y\rangle = 0.$$

This completes the proof of the decomposition.

The previous lemma is now used to show that the operator $T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}$ has a bounded inverse. Let $\mathscr{H} = \mathscr{H}_2^p$, $Z = Z_G^{-1/2}$, and $\mathscr{A} = \tilde{\Theta}\mathscr{H}_2^p$; then from the third decomposition of Lemma 2.3,

$$\mathscr{H}_2^p = Z_G^{-1} T_{\tilde{\Theta}}(\mathscr{H}_2^p) + \left(\tilde{\Theta}\mathscr{H}_2^p\right)^\perp .$$

Since $T_{\tilde{\Theta}^*}\mathscr{H}_2^2 = \mathscr{H}_2^2$ and $\ker T_{\tilde{\Theta}^*} = \mathscr{H}_2^p \ominus \tilde{\Theta}\mathscr{H}_2^p$, this equation implies that

$$\mathscr{H}_2^p = T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}(\mathscr{H}_2^p).$$

As the operator $T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}$ is self-adjoint, this proves that it is bijective and that it therefore has bounded inverse; see e.g. Rudin [14].

Therefore the operator

$$Y := T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}}\left(T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}\right)^{-1} T_{\tilde{\Theta}^*} Z_G^{-1} T_{[\tilde{M} \quad -\tilde{N}]}$$

is bounded. Using the above decomposition it is straightforward to verify that $Y^2 = Y$, $YF(G,\tilde{\Theta}) = \{0\}$, $Y\mathscr{G}(G) = \{0\}$, and

$$YT_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}} x = T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}} x \qquad \text{for any} \quad x \in \mathscr{H}_2^p.$$

This implies that $Y = P_{I(G,\tilde{\Theta})}$. ■

The next proposition connects the unique right inner-outer factorizations of $\tilde{S} = \tilde{M}V - \tilde{N}U$ and $S = \tilde{V}M - \tilde{U}N$ directly to a decomposition of the space $\mathscr{H}_2^{p\times m}$ in terms of the graph space of the system G and the transposed graph of the controller K. Before proving the proposition we need to state a lemma that will also be of importance in later sections.

LEMMA 2.5 (see e.g. [12, p. 201]). *Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{A}$, $\mathscr{B}$ be closed subspaces of $\mathscr{H}$. Denote the orthogonal projection operator onto the space $\mathscr{A}$ as $P_{\mathscr{A}}: \mathscr{H} \to \mathscr{A}$, and the orthogonal projection onto $\mathscr{A}$ restricted to the subspace $\mathscr{B}$ as $P_{\mathscr{A}}|\mathscr{B}: \mathscr{B} \to \mathscr{A}$; and use analogous notation for similar operations onto the subspace $\mathscr{B}$. The following statements are equivalent*:

(i) $\operatorname{clos}(P_{\mathscr{B}}\mathscr{A}) = \mathscr{B}$,
(ii) $\mathscr{H} = \operatorname{clos}(\mathscr{B}^{\perp} + \mathscr{A})$,
(iii) $\mathscr{B} \cap \mathscr{A}^{\perp} = \{0\}$.

The following statements are equivalent:

(i) $P_{\mathscr{B}}\mathscr{A} = \mathscr{B}$,
(ii) $\mathscr{H} = \mathscr{B}^{\perp} + \mathscr{A}$,
(iii) $\|P_{\mathscr{A}^{\perp}} P_{\mathscr{B}}\| < 1$.

The following statements are equivalent:

(i) $P_{\mathscr{B}}\mathscr{A} = \mathscr{B}$, $\mathscr{A} \cap \mathscr{B}^{\perp} = \{0\}$,
(ii) $\mathscr{H} = \mathscr{B}^{\perp} + \mathscr{A}$, $\mathscr{H} = \mathscr{A}^{\perp} + \mathscr{B}$,
(iii) $\|P_{\mathscr{A}^{\perp}} P_{\mathscr{B}}\| < 1$, $\|P_{\mathscr{B}^{\perp}} P_{\mathscr{A}}\| < 1$.

PROPOSITION 2.6. *Given the assumptions of Proposition* 2.2:

1. *Let $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ be an inner function. Then*

$$\left[\mathscr{G}(G) + \mathscr{G}^T(K)\right] \oplus F(G,\tilde{\Theta}) = \mathscr{H}_2^{p+m} \tag{1}$$

if and only if

$$\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$$

for some unit $\tilde{S}_0 \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$.

2. *Let* $\Theta \in \mathcal{RH}_\infty^{m\times m}$ *be an inner function. Then*

$$\left[\mathcal{G}(G) + \mathcal{G}^T(K)\right] \oplus F^T(K,\Theta) = \mathcal{H}_2^{p+m}$$

if and only if

$$\tilde{V}M - \tilde{U}N = \Theta S_0$$

for some unit $S_0 \in \mathcal{RH}_\infty^{m\times m}$.

Proof. 1: Assume that $\tilde{S} = \tilde{M}V - \tilde{N}U$ has the inner-outer factorization $\tilde{M}V - \tilde{N}U = \Theta\tilde{S}_0$, where $\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p}$ is an inner function and $\tilde{S}_0 \in \mathcal{RH}_\infty^{p\times p}$ is a unit. Using the expression for $P_{I(G,\tilde{\Theta})}$ in the previous lemma,

$$\begin{aligned}
P_{I(G,\tilde{\Theta})}\mathcal{G}^T(K) &= T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} Z_G^{-1}T_{\tilde{\Theta}}\left(T_{\tilde{\Theta}^*}Z_G^{-1}T_{\tilde{\Theta}}\right)^{-1}T_{\tilde{\Theta}^*}Z_G^{-1}T_{[\tilde{M}\ \ -\tilde{N}]}T_{\begin{bmatrix}V\\ U\end{bmatrix}}(\mathcal{H}_2^p)\\
&= T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} Z_G^{-1}T_{\tilde{\Theta}}\left(T_{\tilde{\Theta}^*}Z_G^{-1}T_{\tilde{\Theta}}\right)^{-1}T_{\tilde{\Theta}^*}Z_G^{-1}T_{\tilde{\Theta}}T_{\tilde{S}_0}(\mathcal{H}_2^p)\\
&= T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} Z_G^{-1}T_{\tilde{\Theta}}\left(T_{\tilde{\Theta}^*}Z_G^{-1}T_{\tilde{\Theta}}\right)^{-1}\left(T_{\tilde{\Theta}^*}Z_G^{-1}T_{\tilde{\Theta}}\right)(\mathcal{H}_2^p)\\
&= T_{\begin{bmatrix}\tilde{M}^*\\ -\tilde{N}^*\end{bmatrix}} Z_G^{-1}T_{\tilde{\Theta}}(\mathcal{H}_2^p)\\
&= I(G,\tilde{\Theta}),
\end{aligned}$$

as the operator $T_{\tilde{S}_0}$ is bijective on $\mathcal{H}_2^p$. It is now possible to apply Lemma 2.5 to show that this implies that

$$\mathcal{G}^T(K) + \left[I(G,\tilde{\Theta})\right]^\perp = \mathcal{H}_2^{p+m},$$

or equivalently, using the decomposition proved in Lemma 2.4.,

$$\mathcal{G}^T(K) + \left[F(G,\tilde{\Theta}) \oplus \mathcal{G}(G)\right] = \mathcal{H}_2^{p+m}.$$

It remains to show that the space $\mathscr{G}^T(K)$ is orthogonal to the space $F(G,\tilde{\Theta})$. Let $f \in \mathscr{G}^T(K)$ and $g \in F(G,\tilde{\Theta})$. Then

$$f = \begin{bmatrix} V \\ U \end{bmatrix} x \qquad \text{for some} \quad x \in \mathscr{H}_2^p,$$

and

$$g = T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} y \qquad \text{for some} \quad y \in \mathscr{H}_2^p - \tilde{\Theta}\mathscr{H}_2^p.$$

Hence,

$$\langle f, g\rangle = \left\langle \begin{bmatrix} V \\ U \end{bmatrix} x, T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} y \right\rangle = \left\langle (\tilde{M}V - \tilde{N}U)x, y \right\rangle = \langle \tilde{\Theta}\tilde{S}_0 x, y\rangle = 0.$$

Hence $\mathscr{G}^T(K) \perp F(G,\tilde{\Theta})$, and therefore

$$\left[\mathscr{G}(G) + \mathscr{G}^T(K)\right] \oplus F(G,\tilde{\Theta}) = \mathscr{H}_2^{p+m}.$$

To prove the converse, assume now that

$$\left[\mathscr{G}(G) + \mathscr{G}^T(K)\right] \oplus F(G,\tilde{\Theta}) = \mathscr{H}_2^{p+m}$$

for some inner function $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$. Therefore, by Lemma 2.4,

$$\mathscr{G}(G) + \mathscr{G}^T(K) = \mathscr{G}(G) \oplus I(G,\tilde{\Theta}).$$

This decomposition implies that

$$\begin{aligned} T_{\tilde{M}V-\tilde{N}U}(\mathscr{H}_2^p) &= T_{[\tilde{M}\ \ -\tilde{N}]}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^p + \begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right) \\ &= T_{[\tilde{M}\ \ -\tilde{N}]}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^p \oplus T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1}\left(\tilde{\Theta}\mathscr{H}_2^p\right)\right) \\ &= 0 + Z_G Z_G^{-1}\left(\tilde{\Theta}\mathscr{H}_2^p\right) \\ &= \tilde{\Theta}\mathscr{H}_2^p, \end{aligned}$$

and therefore $M_{\tilde{M}V-\tilde{N}U}(\mathscr{H}_2^p) = \Theta\mathscr{H}_2^p$, as $\tilde{M}V - \tilde{N}U \in \mathscr{R}\mathscr{H}_\infty$. The operator $M_{\tilde{\Theta}^*}:\tilde{\Theta}\mathscr{H}_2^p \to \mathscr{H}_2^p$ clearly satisfies $M_{\tilde{\Theta}^*}(\tilde{\Theta}\mathscr{H}_2^p) = \mathscr{H}_2^p$, and therefore

$$M_{\tilde{\Theta}^*} M_{\tilde{M}V-\tilde{N}U}(\mathscr{H}_2^p) = \tilde{\Theta}^*(\tilde{M}V - \tilde{N}U)(\mathscr{H}_2^p) = \mathscr{H}_2^p.$$

This implies that $\tilde{S}_0 := \tilde{\Theta}^*(\tilde{M}V - \tilde{N}U) \in \mathscr{R}\mathscr{H}_\infty$ is a unit, i.e., $(\tilde{M}V - \tilde{N}U)^{-1}\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty$. It has therefore been shown that $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$, where $\tilde{S}_0$ is a unit.

2: This is proved in an analogous fashion. ■

The following theorem summarizes the previous results. It gives further necessary and sufficient conditions for a control system to be unstable to order k. The importance of the result in our context is that the stability properties of the control system are characterized in terms of the graph of the plant and the transposed graph of the controller. This result generalizes the result on stable control systems given in [13].

THEOREM 2.7. *Given the assumptions of Proposition 2.2, the following statements are equivalent:*

(U0) *The pair (G, K) is unstable to order k.*

(U3) *There exists an inner transfer function $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ of McMillan degree less than or equal to k such that the space $\mathscr{H}_2^p \times \mathscr{H}_2^m$ can be decomposed as*

$$\mathscr{H}_2^{p+m} = [\mathscr{G}(G) + \mathscr{G}^T(K)] \oplus F(G, \tilde{\Theta}).$$

The inner function $\tilde{\Theta}$ is unique up to right multiplication by a constant unitary matrix.

(U4) *There exists an inner transfer function $\Theta \in \mathscr{R}\mathscr{H}_\infty^{m\times m}$ of McMillan degree less than or equal to k such that the space $\mathscr{H}_2^p \times \mathscr{H}_2^m$ can be decomposed as*

$$\mathscr{H}_2^{p+m} = [\mathscr{G}(G) + \mathscr{G}^T(K)] \oplus F^T(K, \Theta).$$

The inner function Θ is unique up to right multiplication by a constant unitary matrix.

Furthermore,

$$F(G, \tilde{\Theta}) = F(K, \Theta).$$

Proof. The proof follows immediately, as (U0) is equivalent to (U1) and (U2) by Proposition 2.2, and (U1) is equivalent to (U3), and (U2) to (U4), by Proposition 2.6. The final claim follows by comparing the decompositions in (U3) and (U4). ∎

To complete this section it is shown that it is possible to parametrize all controllers of a system G such that the closed-loop system is unstable to order k. The following theorem establishes the result in the framework of graphs of systems. As a corollary we obtain a generalization of the Kucera-Youla parametrization of all stabilizing controllers.

The proof will use geometric ideas related to skew projections. We therefore first have to summarize some related results. These will also be useful in later parts of the paper.

DEFINITION 2.8. Given two closed subspaces, $\mathscr{A}$ and $\mathscr{B}$, such that $\mathscr{B} \cap \mathscr{A} = \{0\}$, then the *skew projection onto $\mathscr{A}$ with kernel $\mathscr{B}$* is defined by

$$P_{\mathscr{A}\|\mathscr{B}} : \mathscr{A} + \mathscr{B} \to \mathscr{A}, \qquad P_{\mathscr{A}\|\mathscr{B}} : u + v \mapsto u, \quad u \in \mathscr{A},\ v \in \mathscr{B}.$$

What will be of interest to us is the skew projection onto the graph space of a system with kernel equal to the transposed graph space of the controller. The next lemma expresses this projection operator in terms of Toeplitz operators. Hence its norm is given in terms of the $\mathscr{L}_\infty$ norm of an $\mathscr{L}_\infty$ transfer function.

LEMMA 2.9. *Suppose the $p \times m$ transfer function G has a normalized r.c.f. and l.c.f (N, M) and $(\tilde{N}, \tilde{M})$, respectively, and an $m \times p$ stabilizing controller K has a normalized r.c.f. and l.c.f. (U, V) and $(\tilde{U}, \tilde{V})$. Then*

$$P_{\mathscr{G}(G)\|\mathscr{G}^T(K)} = T_{\begin{bmatrix} N \\ M \end{bmatrix}} T_{S^{-1}[-\tilde{U} \;\; \tilde{V}]},$$

$$P_{\mathscr{G}^T(K)\|\mathscr{G}(G)} = T_{\begin{bmatrix} V \\ U \end{bmatrix}\tilde{S}^{-1}} T_{[\tilde{M} \;\; -\tilde{N}]},$$

and further

$$\|P_{\mathscr{G}(G)\|\mathscr{G}^T(K)}\| = \|S^{-1}\|_\infty = \tau(S)^{-1},$$

$$\|P_{\mathscr{G}^T(K)\|\mathscr{G}(G)}\| = \|\tilde{S}^{-1}\|_\infty = \tau(\tilde{S})^{-1},$$

where $\tilde{S} = \tilde{M}V - \tilde{N}U$ and $S = \tilde{V}M - \tilde{U}N$.

Proof. As the controller is stabilizing, we have that

$$\begin{bmatrix} N & V \\ M & U \end{bmatrix}^{-1} \in \mathscr{R}\mathscr{H}_\infty.$$

This implies that

$$\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m \cap \begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p = \mathscr{G}(G) \cap \mathscr{G}^T(K) = \{0\}.$$

Hence the spaces $\mathscr{G}(G)$, $\mathscr{G}^T(K)$ satisfy the first condition of Definition 2.8. It is also possible to write down a doubly coprime factorization.

$$\begin{bmatrix} S^{-1}[-\tilde{U} \quad \tilde{V}] \\ [\tilde{M} \quad -\tilde{N}] \end{bmatrix} \left[\begin{bmatrix} N \\ M \end{bmatrix} \quad \begin{bmatrix} V \\ U \end{bmatrix} \tilde{S}^{-1} \right] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

which implies that

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \left[\begin{bmatrix} N \\ M \end{bmatrix} \quad \begin{bmatrix} V \\ U \end{bmatrix} S^{-1} \right] \begin{bmatrix} S^{-1}[-\tilde{U} \quad \tilde{V}] \\ [\tilde{M} \quad -\tilde{N}] \end{bmatrix}$$

$$= \begin{bmatrix} N \\ M \end{bmatrix} S^{-1}[-\tilde{U} \quad \tilde{V}] + \begin{bmatrix} V \\ U \end{bmatrix} \tilde{S}^{-1}[\tilde{M} \quad -\tilde{N}].$$

Using this identity, one may verify that the expressions for the skew projection operators in the statement of this lemma satisfy the requirements for a skew projection in Definition 2.8. The expressions for the norms of the projection operators follow immediately, since

$$\|P_{\mathscr{G}(G)\|\mathscr{G}^T(K)}\| = \left\| T_{\begin{bmatrix} N \\ M \end{bmatrix} S^{-1}[-\tilde{U} \quad \tilde{V}]} \right\| = \left\| \begin{bmatrix} N \\ M \end{bmatrix} S^{-1}[-\tilde{U} \quad \tilde{V}] \right\|_\infty$$

$$= \|S^{-1}\|_\infty = \tau(S)^{-1},$$

as the norm of a Toeplitz operator is the $\mathscr{L}_\infty$ norm of its symbol, and the coprime factorizations are normalized. The expression for the norm of the other skew projection is proved analogously. ∎

We can now prove the theorem.

THEOREM 2.10. *Let G be a $p \times m$ transfer function with r.c.f. (N, M) and l.c.f. $(\tilde{N}, \tilde{M})$, and let K be a stabilizing controller with r.c.f. (U, V) and l. c. f. $(\tilde{U}, \tilde{V})$. Let $\hat{U}, \hat{V} \in \mathscr{R}\mathscr{H}_\infty$ of appropriate dimensions be right coprime and such that $\hat{V}$ is invertible. Let $k = 1, 2, \ldots$. Then the following statements are equivalent:*

1. *there exists an inner function $\Theta \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ of McMillan degree less than or equal to k, such that*

$$\left(\mathscr{G}(G) + \begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p\right) \oplus F(G, \tilde{\Theta}) = \mathscr{H}_2^{p\times m}; \tag{2}$$

2. *there exists an inner function $\tilde{\Theta}_1 \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ of McMillan degree less than or equal to k and $Q \in \mathscr{R}\mathscr{H}_\infty$ such that*

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p = \left(\begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}_1 + \begin{bmatrix} N \\ M \end{bmatrix} Q\right) \mathscr{H}_2^p. \tag{3}$$

Proof. It is first shown that any subspace of the form of

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p$$

in (3) satisfies the spanning condition in (2). This is proved in an almost identical manner to the first part of the proof of Proposition 2.6. Let $(\hat{S}_1, \tilde{\Theta}_1)$ be a right inner coprime factorization of $\tilde{\Theta}^*(\tilde{M}V - \tilde{N}U)$, i.e. $\tilde{\Theta}^*(\tilde{M}V - \tilde{N}U) = \hat{S}_1\tilde{\Theta}_1^*$. Note that $\hat{S} := \tilde{\Theta}^*(\tilde{M}V - \tilde{N}U)\tilde{\Theta}_1$ is a unit and $\tilde{\Theta}_1$ has McMillan degree less than or equal to k. Using the expression for the orthogonal projection operator $P_{I(G,\tilde{\Theta})}$ in Lemma 2.4, note that

$$\begin{aligned}
P_{I(G,\tilde{\Theta})}\left(\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p\right) &= T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}} \left(T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}\right)^{-1} T_{\tilde{\Theta}^*} Z_G^{-1} T_{(\tilde{M}V - \tilde{N}U)\tilde{\Theta}_1}(\mathscr{H}_2^p) \\
&= T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}} \left(T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}}\right)^{-1} T_{\tilde{\Theta}^*} Z_G^{-1} T_{\tilde{\Theta}\hat{S}}(\mathscr{H}_2^p) \\
&= T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} Z_G^{-1} T_{\tilde{\Theta}}(\mathscr{H}_2^p) = I(G, \tilde{\Theta}).
\end{aligned}$$

The rest of the argument is identical to the argument in Proposition 2.6.

The reverse direction is now proved. Given the spanning condition in (2), it is shown that $\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p$ must be of the form in (3). As $\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p$ satisfies (2), we have by Proposition 2.6 that $\tilde{M}\hat{V} - \tilde{N}\hat{U} = \Theta \tilde{S}_0$ for some unit $\tilde{S}_0$ and an inner function Θ of McMillan degree less than or equal to k. Since $I = P_{\mathscr{G}^T(K) \| \mathscr{G}(K)} + P_{\mathscr{G}(G) \| \mathscr{G}^T(K)}$, the space $\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p$ can be decomposed using the expressions for the skew projections in Lemma 2.9:

$$
\begin{aligned}
T_{\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix}}(\mathscr{H}_2^p) &= \left(P_{\mathscr{G}^T(K) \| \mathscr{G}(G)} + P_{\mathscr{G}(G) \| \mathscr{G}^T(K)}\right)\left(\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p\right) \\
&= \left(T_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\tilde{S}^{-1}(\tilde{M}\hat{V} - \tilde{N}\hat{U})} + T_{\begin{bmatrix} N \\ M \end{bmatrix}} T_{S^{-1}(\tilde{V}\hat{U} - \tilde{U}\hat{V})}\right)(\mathscr{H}_2^p) \\
&= \left(T_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\tilde{S}^{-1}\Theta \tilde{S}_0} + T_{\begin{bmatrix} N \\ M \end{bmatrix}} T_{S^{-1}(\tilde{V}\hat{U} - \tilde{U}\hat{V})}\right)(\mathscr{H}_2^p) \\
&= \left(T_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\tilde{\Theta}_1 \hat{S}^{-1} S_0} + T_{\begin{bmatrix} N \\ M \end{bmatrix}} T_{S^{-1}(\tilde{V}\hat{U} - \tilde{U}\hat{V})}\right)(\mathscr{H}_2^p) \\
&= \left(T_{\begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}_1 + \begin{bmatrix} N \\ M \end{bmatrix} S^{-1}(\tilde{V}\hat{U} - \tilde{U}\hat{V}) S_0^{-1} \hat{S}}\right) T_{\hat{S}^{-1} S_0}(\mathscr{H}_2^p) \\
&= \left(\begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}_1 + \begin{bmatrix} N \\ M \end{bmatrix} Q\right) \mathscr{H}_2^p,
\end{aligned}
$$

where $Q = S^{-1}(\tilde{V}\hat{U} - \tilde{U}\hat{V})S_0^{-1}\hat{S} \in \mathscr{RH}_\infty$, $S = \tilde{V}M - \tilde{U}N$, and $\tilde{S} = \tilde{M}V - \tilde{N}U$. ■

In the following corollary the previous theorem is reinterpreted to give a parametrization of controllers that lead to a closed-loop system that is unstable to order k. In case of $k = 0$ this is nothing else but the Kucera-Youla parametrization of all stabilizing controllers (see e.g. [15]).

COROLLARY 2.11. *Given the assumptions in Theorem 2.10, the following statements are equivalent*:

1. *The pair* $(G, \hat{K})$ *is unstable to order* k, *and* $\hat{K}$ *is proper.*
2. $\hat{K} = (U + NQ)(V + MQ)^{-1}$ *for a* $Q \in \mathcal{RH}_{\infty,k}$, *and* $\det(V + MQ)(i\infty) \neq 0$.
3. $\hat{K} = (\tilde{V} + \tilde{Q}\tilde{N})^{-1}(\tilde{U} + \tilde{Q}\tilde{M})$ *for a* $\tilde{Q} \in \mathcal{RH}_{\infty,k}$, *and* $\det(\tilde{V} + \tilde{Q}\tilde{N})(i\infty) \neq 0$.

Proof. $1 \Rightarrow 2$: If $(G, \hat{K})$ s unstable to order k, then given a r.c.f. $(\hat{U}, \hat{V})$ there exists an inner function $\tilde{\Theta} \in \mathcal{RH}_{\infty}^{p\times p}$ of McMillan degree less than or equal to k such that the spanning condition in (U3) is satisfied. Hence by the previous theorem there exists a $Q \in \mathcal{RH}_\infty$ and an inner function Θ_1 of McMillan degree less than or equal to k such that

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathcal{H}_2^p = \left(\begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}_1 + \begin{bmatrix} N \\ M \end{bmatrix} Q_0 \right) \mathcal{H}_2^p,$$

and $\tilde{\Theta}_1, Q_0$ are clearly coprime, as $\hat{U}, \hat{V}$ are coprime. This implies that $\hat{K} = (U\tilde{\Theta}_1 + NQ_0)(V\tilde{\Theta}_1 + MQ_0)^{-1} = (U + NQ_0\tilde{\Theta}_1^*)(V + MQ_0\tilde{\Theta}_1^*)^{-1} = (U + NQ)(V + MQ)^{-1}$, with $Q := Q_0\tilde{\Theta}_1^* \in \mathcal{RH}_{\infty,k}$. Further, if $\hat{K}$ is proper, this implies that $\det(V + MQ(i\infty) \neq 0$.

$2 \Rightarrow 1$: Let $\hat{K} = (U + NQ)(V + MQ)^{-1}$ for a $Q \in \mathcal{RH}_{\infty,k}$, and further let Q have a right inner coprime factorization $Q = Q_0\tilde{\Theta}_I^*$. Then the McMillan degree of $\tilde{\Theta}_1$ is less than or equal to k. As $\tilde{\Theta}_1, Q_0$ are right coprime, there exist $\tilde{X}, \tilde{Y} \in \mathcal{RH}_\infty$ such that $\tilde{X}\tilde{\Theta}_1 - \tilde{Y}Q_0 = I$. Therefore

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} = \begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}_1 + \begin{bmatrix} N \\ M \end{bmatrix} Q_0$$

is r.c.f. of $\hat{K}$ for

$$\begin{bmatrix} -\tilde{Y}S^{-1} & \tilde{X}\tilde{S}^{-1} \end{bmatrix} \begin{bmatrix} -\tilde{U} & \tilde{V} \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} = I,$$

where $S = \tilde{V}M - \tilde{U}N$ and $\tilde{S} = \tilde{M}V - \tilde{N}U$. By the previous theorem this implies that

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \mathscr{H}_2^p$$

satisfies the spanning condition, and therefore that the pair $(G, \hat{K})$ is unstable to order k by the equivalence of (U3) and (U0). Further, if $\det(V + MQ)(i\infty) \neq 0$, this implies that $\hat{K}$ is proper.

$1 \Leftrightarrow 3$: Note that (G, K) is unstable to order K if and only if (G^T, K^T) is unstable to order K. The result then follows from finding all controllers that stabilize G^T to order K with a r.c.f. $(\tilde{N}^T, \tilde{M}^T)$ as above and taking the transpose of each element in this set. ■

3. GAP BETWEEN GRAPH SPACES AND CONTROL SYSTEMS

In the previous section it was shown how order-k instability of a closed-loop system can be characterized in terms of spanning conditions of the graph of the plant and the transposed graph of the controller. In this section we are going to give further interpretations of these results in terms of the gap and minimum angle between certain graph spaces. These results generalize the results in [13], which were derived for stable closed-loop systems.

For ease of notation we define the following class of inner transfer functions:

$$\mathscr{B}_k^p := \left\{ \tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p} : \tilde{\Theta}^* \tilde{\Theta} = I_p ; \left(\text{McMillan degree of } \tilde{\Theta}\right) \leqslant k \right\}.$$

Therefore the class $\mathscr{B}_0^p = \{U \in \mathscr{C}^{p\times p} : U^*U = I_p\}$, and we define the class $\mathscr{B}_{-1}^p = \{0\}$. The following two technical results will be proved first.

PROPOSITION 3.1. *Given the assumptions of Proposition 2.2. Assume there exists an inner function* $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ *of McMillan degree at most k such that*

$$\left[\mathscr{G}(G) + \mathscr{G}^T(K)\right] \oplus F(G, \tilde{\Theta}) = \mathscr{H}_2^{p+m}. \tag{4}$$

For the inner function $\Theta \in \mathscr{R}\mathscr{H}_\infty^{m\times m}$ *of McMillan degree at most k of* (U2) *and* (U4) *we have*

$$\left[I(G, \tilde{\Theta}) + I^T(K, \Theta)\right] \oplus F(G, \tilde{\Theta}) = \mathscr{H}_2^{p+m}.$$

Proof. First note that the decomposition in (4) implies, by Proposition 2.6, the existence of a unit $\tilde{S}_0 \in \mathcal{RH}_\infty$ such that $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$. By the equivalence of statements (U1) and (U2) this implies the existence of an inner-outer factorization $\tilde{V}M - \tilde{U}N = \Theta S$, where $\Theta \in \mathcal{RH}_\infty^{m\times m}$ is an inner function of McMillan degree at most k, which is unique up to right multiplication by a constant unitary function, and $S \in \mathcal{RH}_\infty^{m\times m}$ is a unit. Further, the following identity is also satisfied:

$$F(G,\tilde{\Theta}) = F^T(K,\Theta) \tag{5}$$

by Theorem 2.7. Consider,

$$P_{\mathcal{G}^T(K)}[\mathcal{G}(G)]^\perp = T_{\begin{bmatrix} V \\ U \end{bmatrix}}\left(T^*_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\begin{bmatrix} V \\ U \end{bmatrix}}\right)^{-1} T^*_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}(\mathcal{H}_2^p)$$

$$= T_{\begin{bmatrix} V \\ U \end{bmatrix}}\left(T^*_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\begin{bmatrix} V \\ U \end{bmatrix}}\right)^{-1} T_{\tilde{S}_0^*} T_{\tilde{\Theta}^*}(\mathcal{H}_2^p).$$

As the operator $T^*_{\tilde{\Theta}}: \mathcal{H}_2^p \to \mathcal{H}_2^p$ is surjective and both the operators

$$T^*_{\begin{bmatrix} V \\ U \end{bmatrix}} T_{\begin{bmatrix} V \\ U \end{bmatrix}} : \mathcal{H}_2^p \to \mathcal{H}_2^p \quad \text{and} \quad T_{\tilde{S}_0^*}: \mathcal{H}_2^p \to \mathcal{H}_2^p$$

are bijective, we have that

$$P_{\mathcal{G}^T(K)}[\mathcal{G}(G)]^\perp = \mathcal{G}^T(K).$$

Lemma 2.5 and Lemma 2.4 imply that this is equivalent to

$$\begin{aligned}\mathcal{H}_2^{p+m} &= [\mathcal{G}(G)]^\perp + [\mathcal{G}^T(K)]^\perp \\ &= \left[I(G,\tilde{\Theta}) \oplus F(G,\tilde{\Theta})\right] + \left[I^T(K,\Theta) \oplus F^T(K,\Theta)\right] \\ &= \left[I(G,\tilde{\Theta}) + I^T(K,\Theta)\right] \oplus F(G,\tilde{\Theta}).\end{aligned}$$

Combining this decomposition with Equation (5) gives the final result. ■

LEMMA 3.2. *Let G be a $p \times m$ transfer function, and let K be a $m \times p$ transfer function. Let $\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p}$ be inner. Then:*

1. $\|P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})}\| = \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]\perp} \left| \left[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta}) \right] \right. \right\|$;
2. *One has*

$$\left\| P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]\perp} \left| \mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta}) \right. \right\|$$
$$= \max\left\{ \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\|, \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp} \left| \left[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta}) \right] \right. \right\| \right\}.$$

Proof. 1: The statement follows immediately from the decomposition in Lemma 2.4, i.e.

$$\mathscr{H}_2^{p+m} = \mathscr{G}(G) \oplus I(G,\tilde{\Theta}) \oplus F(G,\tilde{\Theta}).$$

2: First note that the operators

$$\begin{bmatrix} P_{\mathscr{G}^T(K)} \\ P_{[\mathscr{G}^T(K)]^\perp} \end{bmatrix} : \mathscr{H}_2^{p+m} \to \mathscr{H}_2^{2(p+m)}, \qquad \begin{bmatrix} P_{\mathscr{G}(G)} \\ P_{[\mathscr{G}(G)]^\perp} \end{bmatrix} : \mathscr{H}_2^{p+m} \to \mathscr{H}_2^{2(p+m)}$$

are isometries. Therefore, using the decomposition in Lemma 2.4, we have that

$$\left\| \left(P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]^\perp} \right) \left| \left[\mathscr{H}_2^{p+m} \ominus F(G,\Theta) \right] \right. \right\|$$
$$= \left\| \left(P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]^T} \right) P_{\mathscr{G}(G) \oplus I(G,\tilde{\Theta})} \right\|$$
$$= \| P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G) \oplus I(G,\tilde{\Theta})} - P_{I(G,\tilde{\Theta})} \|$$
$$= \left\| \begin{bmatrix} P_{\mathscr{G}^T(K)} \\ P_{[\mathscr{G}(K)]^\perp} \end{bmatrix} \left(P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G) \oplus I(G,\tilde{\Theta})} - P_{I(G,\tilde{\Theta})} \right) \begin{bmatrix} P_{\mathscr{G}(G)} & P_{[\mathscr{G}(G)]^\perp} \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} P_{\mathscr{G}^T(K)} \\ P_{[\mathscr{G}^T(K)]^\perp} \end{bmatrix} \begin{bmatrix} P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)} & P_{\mathscr{G}^T(K)} P_{I(G,\tilde{\Theta})} - P_{I(G,\tilde{\Theta})} \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)} & 0 \\ 0 & P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})} \end{bmatrix} \right\|$$
$$= \max\left\{ \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\|, \|P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})}\| \right\}$$
$$= \max\left\{ \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\|, \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp} \left| \left[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta}) \right] \right. \right\| \right\}.$$

■

The main instability results are now stated and proved.

THEOREM 3.3. *Let G be a $p \times m$ transfer function, and let K be a $m \times p$ transfer function. Then the following statements are equivalent*:

(U0) *The pair (G, K) is unstable to order k.*
(U5) *There exists an inner transfer function $\tilde{\Theta} \in \mathscr{B}_k^p$ such that*

$$\left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp} \Big| \left[\mathscr{H}_2^{p+m} \ominus F(G, \tilde{\Theta}) \right] \right\| < 1.$$

(U6) *There exists an inner transfer function $\tilde{\Theta} \in \mathscr{B}_k^p$ such that*

$$\operatorname{gap}\left(\mathscr{G}^T(K), I(G, \tilde{\Theta}) \right) < 1.$$

(U7) *There exists an inner transfer function $\tilde{\Theta} \in \mathscr{B}_k^p$ such that*

$$\left\| P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]^\perp} \Big| \left[\mathscr{H}_2^{p+m} \ominus F(G, \tilde{\Theta}) \right] \right\| < 1.$$

Proof. (U0) ⇒ (U5): (U0) implies, by Theorem 2.7, that there exists a $\tilde{\Theta} \in \mathscr{B}_k^p$ such that the space $\mathscr{H}_2^p \times \mathscr{H}_2^m$ can be decomposed:

$$\left[\mathscr{G}(G) + \mathscr{G}^T(K) \right] \oplus F(G, \tilde{\Theta}) = \mathscr{H}_2^{m+p}.$$

This implies by Lemma 2.4 that

$$\mathscr{G}^T(K) + \left[I(G, \tilde{\Theta}) \right]^\perp = \mathscr{H}_2^{p+m},$$

and therefore by Lemma 2.5 and Lemma 3.2 that

$$1 > \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{I(G, \tilde{\Theta})} \right\| = \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp} \Big| \left[\mathscr{H}_2^{p+m} \ominus F(G, \tilde{\Theta}) \right] \right\|.$$

This gives the result by Lemma 3.2.

(U5) ⇒ (U0): We are going to show that (U5) implies (U1), which is equivalent to (U0). First note that Lemma 3.2 implies that

$$1 > \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp} \Big| \left[\mathscr{H}_2^{p+m} \ominus F(G, \tilde{\Theta}) \right] \right\| = \left\| P_{[\mathscr{G}^T(K)]^\perp} P_{I(G, \tilde{\Theta})} \right\|,$$

and by the equivalence relationships in Lemma 2.5,

$$\mathscr{G}^T(K) + \left[I(G,\tilde{\Theta})\right]^{\perp} = H_2^{p+m}.$$

Therefore by the decomposition in Lemma 2.4 we have that

$$\mathscr{G}^T(K) + \left[\mathscr{G}(G) \oplus F(G,\tilde{\Theta})\right] = \mathscr{H}_2^{p+m}.$$

As $(\tilde{N}, \tilde{M})$ are coprime, there exist $X, Y \in \mathscr{R}\mathscr{H}_\infty$ of appropriate dimensions such that $\tilde{N}Y - \tilde{M}X = I$. Hence

$$T_{[\tilde{M} \quad -\tilde{N}]}\begin{pmatrix} \mathscr{H}_2^p \\ \mathscr{H}_2^m \end{pmatrix} \supseteq T_{[\tilde{M} \quad -\tilde{N}]}\left(\begin{bmatrix} Y \\ X \end{bmatrix}\mathscr{H}_2^p\right) = \mathscr{H}_2^p.$$

But by the above identity we have

$$\mathscr{H}_2^p = T_{[\tilde{M} \quad -\tilde{N}]}\mathscr{H}_2^{p+m} = T_{[\tilde{M} \quad -\tilde{N}]}\left\{\mathscr{G}^T(K) + \left[\mathscr{G}(G) \oplus F(G,\tilde{\Theta})\right]\right\}$$

$$= T_{[\tilde{M} \quad -\tilde{N}]}\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p + \begin{bmatrix} M \\ N \end{bmatrix}\mathscr{H}_2^m + T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\left(\mathscr{H}_2^2 \ominus \tilde{\Theta}\mathscr{H}_2^p\right)\right)$$

$$= T_{\tilde{M}V-\tilde{N}U}\left(\mathscr{H}_2^p\right) + Z_G\left(\mathscr{H}_2^p \ominus \tilde{\Theta}\mathscr{H}_2^p\right).$$

Note that the space $\mathscr{H}_2^p \ominus \tilde{\Theta}\mathscr{H}_2^p$ has dimension equal to the McMillan degree of $\tilde{\Theta}$, which is at most k. As Z_G has a bounded inverse, this implies that the space $Z_G(\tilde{\Theta}\mathscr{H}_2^p)^{\perp}$ is a finite-dimensional space of dimension at most k. Therefore $M_{\tilde{M}V-\tilde{N}U}(\mathscr{H}_2^p)$ is a closed subspace of codimension at most k.

Let $\tilde{M}V - \tilde{N}U \in \mathscr{R}\mathscr{H}_\infty$ have inner-outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}_1\tilde{S}_0$ where $\tilde{\Theta}_1 \in \mathscr{R}\mathscr{H}_\infty$ is an inner function and $\tilde{S}_0 \in \mathscr{R}\mathscr{H}_\infty$ is an outer. As $M_{\tilde{\Theta}_1\tilde{S}_0}(\mathscr{H}_2^p)$ is closed, $\tilde{S}_0$ is a unit. This implies that $M_{\tilde{\Theta}_1\tilde{S}_0}(\mathscr{H}_2^p) = M_{\tilde{\Theta}_1}M_{\tilde{S}_0}(\mathscr{H}_2^p) = M_{\tilde{\Theta}_1}(\mathscr{H}_2^p)$. Therefore the codimension of $M_{\tilde{\Theta}_1}(\mathscr{H}_2^p)$ is at most k, and therefore $\tilde{\Theta}_1$ has McMillan degree at most k. This proves (U5) implies (U1), which is equivalent to (U0).

(U0) $\Rightarrow$ (U6): (U0) implies, by Theorem 2.7, that there exists a $\tilde{\Theta} \in \mathscr{B}_k^p$ such that the space $\mathscr{H}_2^p \times \mathscr{H}_2^m$ can be decomposed:

$$\mathscr{H}_2^{p+m} = \left[\mathscr{G}(G) + \mathscr{G}^T(K)\right] \oplus F(G,\tilde{\Theta}) \tag{6}$$

$$= \mathscr{G}^T(K) + \left[I(G,\tilde{\Theta})\right]^{\perp}, \tag{7}$$

by Lemma 2.4. Also, Equation (7) implies, by Proposition 3.1, that there exists an inner function $\Theta \in \mathscr{R}\mathscr{H}_\infty^{m\times m}$ of McMillan degree k such that

$$\mathscr{H}_2^{p+m} = \left[I(G,\tilde{\Theta}) + I^T(K,\Theta)\right] \oplus F(G,\tilde{\Theta})$$

$$= \left[I(G,\tilde{\Theta}) + I^T(K,\Theta)\right] \oplus F^T(K,\Theta). \tag{8}$$

By Theorem 2.7 we therefore have

$$\mathscr{H}_2^{p+m} = I(G,\tilde{\Theta}) + \left[\mathscr{G}^T(K)\right]^{\perp}. \tag{9}$$

The decompositions in Equations (7) and (9) imply, by the equivalence relations in Lemma 2.5, that $\|P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})}\| < 1$ and $\|P_{\mathscr{G}^T(K)} P_{[F(G,\tilde{\Theta})]^\perp}\| < 1$. These inequalities imply that

$$\operatorname{gap}\left(\mathscr{G}^T(K), I(G,\tilde{\Theta})\right) < 1.$$

Hence (U6) holds.

(U6) $\Rightarrow$ (U7): Since

$$\operatorname{gap}\left(\mathscr{G}^T(K), I(G,\Theta)\right) = \max\left\{\|P_{\mathscr{G}^T(K)} P_{[I(G,\tilde{\Theta})]^\perp}\|, \|P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})}\|\right\} < 1,$$

(U6) implies by Lemma 3.2 that

$$1 > \|P_{[\mathscr{G}^T(K)]^\perp} P_{I(G,\tilde{\Theta})}\| = \left\|P_{[\mathscr{G}^T(K)]^\perp} P_{[\mathscr{G}(G)]^\perp}\Big|\left[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta})\right]\right\|$$

and

$$1 > \|P_{\mathscr{G}^T(K)} P_{[I(G,\tilde{\Theta})]^\perp}\| \geq \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\|.$$

These two conditions imply (U7) by Lemma 3.2.

(U7) ⇒ (U5): This follows immediately from Lemma 3.2. ■

In the following theorem a number of quantities that appeared earlier are related to one another.

THEOREM 3.4. *Given the assumptions of Theorem 3.3, further assume that the pair (G, K) is stable to order k. Then*

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\big(\mathscr{G}^T(K), I(G,\tilde{\Theta})\big) = \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \Big\| P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]^\perp} \big| \big[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta})\big] \Big\|$$

$$= \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\| = \|M^*U + N^*V\|_\infty,$$

where $G = NM^{-1}$ and $K = UV^{-1}$ are normalized coprime factorizations.

Proof. We first show that

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \|P_{[I(G,\tilde{\Theta})]^\perp} P_{\mathscr{G}^T(K)}\| = \|P_{\mathscr{G}(G)} P_{\mathscr{G}^T(K)}\|.$$

To do this let $u \in \mathscr{H}_2^{p+m}$; then

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \|P_{[I(G,\tilde{\Theta})]^\perp} P_{\mathscr{G}^T(K)} u\|_2$$

$$= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \big\| \big(P_{\mathscr{G}(G)} + P_{F(G,\tilde{\Theta})}\big) P_{\mathscr{G}^T(K)} u \big\|_2$$

$$= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \Big(\|P_{\mathscr{G}(G)} P_{\mathscr{G}^T(K)} u\|_2^2 + \|P_{F(G,\tilde{\Theta})} P_{\mathscr{G}^T(K)} u\|_2^2 \Big)$$

$$= \Big(\|P_{\mathscr{G}(G)} P_{\mathscr{G}^T(K)} U\|_2^2 + \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \|P_{F(G,\tilde{\Theta})} P_{\mathscr{G}^T(K)} u\|_2^2 \Big)^{1/2}.$$

Let $\tilde{M}V - \tilde{N}U$ have inner-outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}_1 \tilde{S}_0$, where $\tilde{\Theta}_1$ is an inner function and $\tilde{S}_0 \in \mathscr{R}\mathscr{H}_\infty$ is a unit. As the pair (G, K) is unstable to order k, then by the equivalence of (U0) and (U1) this implies that the

McMillan degree of $\tilde{\Theta}_1$ is at most k. Now,

$$\begin{aligned}\|P_{F(G,\tilde{\Theta}_1)}P_{\mathscr{G}^T(K)}\| &= \|P_{\mathscr{G}^T(K)}P_{F(G,\tilde{\Theta}_1)}\| \\ &= \sup_{\substack{u\in(\tilde{\Theta}_1\mathscr{H}_2^p)^\perp \\ \|u\|=1}} \left\| P_{\mathscr{G}^T(K)}T_{\begin{bmatrix}\tilde{M}^* \\ -N^*\end{bmatrix}}u\right\|_2 \\ &= \sup_{\substack{u\in(\tilde{\Theta}_1\mathscr{H}_2^p)^\perp \\ \|u\|=1}} \left\| T_{\begin{bmatrix}V \\ U\end{bmatrix}}\left(T^*_{\begin{bmatrix}V \\ U\end{bmatrix}}T_{\begin{bmatrix}V \\ U\end{bmatrix}}\right)^{-1} T^*_{\begin{bmatrix}V \\ U\end{bmatrix}}T_{\begin{bmatrix}\tilde{M}^* \\ -\tilde{N}^*\end{bmatrix}}u\right\|_2 \\ &= \sup_{\substack{u\in(\tilde{\Theta}_1\mathscr{H}_2^p)^\perp \\ \|u\|=1}} \left\| T_{\begin{bmatrix}V \\ U\end{bmatrix}}\left(T^*_{\begin{bmatrix}V \\ U\end{bmatrix}}T_{\begin{bmatrix}V \\ U\end{bmatrix}}\right)^{-1} T_{\tilde{S}_0^*}T_{\tilde{\Theta}_1^*}u\right\| = 0.\end{aligned}$$

The final equality follows because for $u \in (\tilde{\Theta}_1\mathscr{H}_2^p)^\perp$ we have

$$\|T_{\tilde{\Theta}_1^*}u\| = \sup_{\substack{x\in\mathscr{H}_2^p \\ \|x\|=1}} \langle T_{\tilde{\Theta}_1^*}u, x\rangle = \sup_{\substack{x\in\mathscr{H}_2^p \\ \|x\|=1}} \langle u, \tilde{\Theta}_1 x\rangle = 0.$$

This implies that

$$\inf_{\tilde{\Theta}\in\mathscr{B}_k^p} \|P_{I(G,\tilde{\Theta})^\perp}P_{\mathscr{G}^T(K)}\| = \|P_{\mathscr{G}(G)}P_{\mathscr{G}^T(K)}\|,$$

where $\tilde{\Theta} = \tilde{\Theta}_1$ achieves the infimum.

We now show that for $\tilde{\Theta} = \tilde{\Theta}_1$ we have that

$$\|P_{[\mathscr{G}^T(K)]^\perp}P_{I(G,\tilde{\Theta}_1)}\| < 1.$$

This is the case, by Lemma 2.5, if

$$P_{I(G,\tilde{\Theta}_1)}\mathscr{G}^T(K) = I\big(G,\tilde{\Theta}_1\big).$$

Noting that $\tilde{M}V - \tilde{N}U = \tilde{\Theta}_1\tilde{S}_0$, it is straightforward to show that this condition is satisfied by using the expression for the orthogonal projection operator onto the space $I(G,\tilde{\Theta}_1)$ in Lemma 2.4.

Now we can summarize:

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\left(\mathscr{G}^T(K), I(G,\tilde{\Theta})^{\perp}\right)$$

$$= \|P_{\mathscr{G}^T(K)} P_{\mathscr{G}(G)}\|$$

$$= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \max\left\{\|P_{[\mathscr{G}^T(K)]^{\perp}} P_{I(G,\tilde{\Theta})}\|, \|P_{\mathscr{G}^T(K)} P_{[I(G,\tilde{\Theta})]^{\perp}}\|\right\}.$$

We have seen that for $\tilde{\Theta} = \tilde{\Theta}_1$,

$$\|P_{[\mathscr{G}^T(G)]^{\perp}} P_{I(G,\tilde{\Theta})}\| < 1.$$

Therefore,

$$\begin{aligned}\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\left(\mathscr{G}^T(K), I(G,\tilde{\Theta})\right) &= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \|P_{\mathscr{G}^T(K)} P_{[I(G,\tilde{\Theta})]^{\perp}}\| \\ &= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \|P_{[I(G,\tilde{\Theta})]^{\perp}} P_{\mathscr{G}^T(K)}\| \\ &= \|P_{\mathscr{G}(G)} P_{\mathscr{G}^T(K)}\| \\ &= \left\| T_{\begin{bmatrix} N \\ M \end{bmatrix}} T^*_{\begin{bmatrix} N \\ M \end{bmatrix}} T_{\begin{bmatrix} V \\ U \end{bmatrix}} T^*_{\begin{bmatrix} V \\ U \end{bmatrix}} \right\| \\ &= \|M^*U + N^*V\|_{\infty},\end{aligned}$$

where we have used in the last identities that the factorizations are normalized. That

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\left(\mathscr{G}^T(K), I(G,\tilde{\Theta})\right)$$

$$= \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \left\| P_{\mathscr{G}^T(K)} - P_{[\mathscr{G}(G)]^{\perp}} \left[\mathscr{H}_2^{p+m} \ominus F(G,\tilde{\Theta})\right] \right\|$$

follows from Lemma 3.2. ∎

4. INSTABILITY AND THE INDEX OF THE BEZOUT OPERATOR

In this section the stability of a control system will be discussed in a slightly different framework. It will be shown that the order of instability of the closed-loop system is equal to the index of an associated operator, which will be called the Bezout operator. This analysis will give an alternative outlook on the work in the previous sections and provide interesting links to a large body of work in operator theory on the index of a Fredhom operators; see Cordes and Labrousse [4], Douglas [5], and Nikolskii [12]. Here it is shown that the index of a Bezout operator possesses desirable geometrical properties.

A Fredholm operator is defined in the usual way.

DEFINITION 4.1. If Z is a linear operator on a Hilbert space H, then Z is a Fredholm operator if and only if its range is closed and $\dim \operatorname{Ker} Z$ and $\dim \operatorname{Ker} Z^*$ are finite. Further, if Z is Fredholm, its index $i(Z)$ is defined as

$$i(Z) := \dim \operatorname{Ker} Z - \dim \operatorname{Ker} Z^* .$$

It is now possible to state the main result of this section.

THEOREM 4.2. *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$, and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$. Then the following statements are equivalent:*

(U0) *The pair (G, K) is unstable to order k.*

(U8) *The Bezout operator $T_{V^*\tilde{M}^* - U^*\tilde{N}^*}$ is Fredholm, and its index $i(T_{V^*\tilde{M}^* - U^*\tilde{N}^*})$ is less than or equal to k.*

Proof. It is shown in this proof that (U1), which is equivalent to (U0), is equivalent to (U8). (U1) $\Rightarrow$ (U8): Let $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$ be an inner-outer factorization. The assumptions imply that $\tilde{\Theta}$ is an inner function of McMillan degree less than or equal than k and $\tilde{S}_0$ is a unit. First note that the Bezout operator $T_{V^*\tilde{M}^* - U^*\tilde{N}^*}$ has closed range, since $T_{V^*\tilde{M}^* - U^*\tilde{N}^*}(\mathscr{H}_2^p) = T_{\tilde{S}_0^*}T_{\tilde{\Theta}^*}(\mathscr{H}_2^p) = \mathscr{H}_2^p$, where we have used that $\tilde{S}_0$ is a unit. Also, $\operatorname{Ker} T_{V^*\tilde{M}^* - U^*\tilde{N}^*} = (\tilde{\Theta}\mathscr{H}_2^p)^\perp$ and $\operatorname{Ker} T^*_{V^*\tilde{M}^* - U^*\tilde{N}^*} = \{0\}$ are both finite-dimensional. Further, $i(T_{V^*\tilde{M}^* - U^*\tilde{N}^*}) = \dim(\tilde{\Theta}\mathscr{H}_2^p)^\perp$ is less than or equal to k, since the McMillan degree of $\tilde{\Theta}$ is less than or equal to k.

(U8) $\Rightarrow$ (U1): Let $\tilde{M}V - \tilde{N}U$ have a right inner-outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$, where $\tilde{\Theta}$ is an inner function and $\tilde{S}_0$ is outer. If the Bezout operator is Fredholm, then its range is closed; this implies that $\operatorname{clos} T_{V^*\tilde{M}^* - U^*\tilde{N}^*}(\mathscr{H}_2^p) = T_{V^*\tilde{M}^* - U^*\tilde{N}^*}(\mathscr{H}_2^p)$ and therefore

$$\mathscr{H}_2^p = \operatorname{clos} T_{V^*\tilde{M}^* - U^*\tilde{N}^*}(\mathscr{H}_2^p) = T_{V^*\tilde{M}^* - U^*\tilde{N}^*}(\mathscr{H}_2^p) = T_{\tilde{S}_0^*}(\mathscr{H}_2^p),$$

implying that $\tilde{S}_0$ is a unit. Also by assumption $i(T_{V^*\tilde{M}^* - U^*\tilde{N}^*})$ is less than or equal to k. This implies that $k \geqslant \dim \operatorname{Ker} T_{V^*\tilde{M}^* - U^*\tilde{N}^*} - \dim \operatorname{Ker} T_{\tilde{M}V - \tilde{N}U} = \dim \operatorname{Ker} T_{\tilde{\Theta}^*} - 0 = \dim(\tilde{\Theta}\mathscr{H}_2^p)^{\perp}$. This implies that the McMillan degree of $\tilde{\Theta}$ is less than or equal to k. ■

5. OPTIMALLY UNSTABLE CONTROLLERS AND HANKEL-NORM APPROXIMATION

In [13] the concept of a maximally stabilizing controller was introduced (see also Section 1). The maximally stabilizing controller of a plant G is a stabilizing controller that maximizes the minimal angle between the graph of the plant and the transposed graph of the controller. Given the results in the previous sections, it is clearly possible to generalize this idea to the present framework. The optimal unstable controller to order k of a system G will be defined as the controller which maximizes the minimum angle between the graph of the system and the transposed graph of the controller, subject to the constraint that the closed-loop system is unstable to order k. Though this controller has little significance for design, the result is interesting in that it gives a geometrical interpretation of Hankel-norm approximation of non-square inner functions. For this reason the analysis is pursued in this section.

DEFINITION 5.1. Given a $p \times m$ system G, the *optimal minimal angle to order k*, $(\theta_{\min}^{\text{opt}})_k$, is defined by

$$\cos\left(\theta_{\min}^{\text{opt}}\right)_k := \inf_{K \in \mathscr{K}_G} \cos \theta_{\min}\left(\mathscr{G}(G), \mathscr{G}^T(K)\right),$$

where $\mathscr{K}_G := \{m \times p$ transfer functions K s.t. (G, K) is unstable to order $k\}$. Further, a controller achieving this infimum is called an *optimal unstable controller to order k.*

The following technical lemma will be needed. Similar results appear in Nikolskii [12], though not directly relating the McMillan degree of inner functions to the invertibility of Toeplitz operators with all-pass symbols.

LEMMA 5.2. *Given two inner transfer functions* $\tilde{\Theta}_1, \tilde{\Theta}_2 \in \mathcal{RH}_\infty^{p\times p}$, *where the McMillan degree of* $\tilde{\Theta}_1$ *is strictly greater than that of* $\tilde{\Theta}_2$, *then*

1. $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) \neq \mathcal{H}_2^p$,
2. $\inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \|\tilde{\Theta}_2 - \tilde{\Theta}_1 Q\|_\infty = 1$.

Proof. 1: Assume that the McMillan degree of the inner function $\tilde{\Theta}_1$ is strictly greater than that of the inner function $\tilde{\Theta}_2$. Also assume that $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) = \mathcal{H}_2^p$. It is shown that this second assumption contradicts the first. If $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) = \mathcal{H}_2^p$ then $T_{\tilde{\Theta}_2}T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) = T_{\tilde{\Theta}_2}(\mathcal{H}_2^p)$. Note that $T_{\tilde{\Theta}_2}T_{\tilde{\Theta}_2^*} = P_{\tilde{\Theta}_2\mathcal{H}_2^p}$ implies $P_{\tilde{\Theta}_2\mathcal{H}_2^p}(\tilde{\Theta}_1\mathcal{H}_2^p) = \tilde{\Theta}_2\mathcal{H}_2^p$. Hence from the equivalence relations in Lemma 2.5 we have $\mathcal{H}_2^p = (\tilde{\Theta}_2\mathcal{H}_2^p)^\perp + \tilde{\Theta}_1\mathcal{H}_2^p$, implying in turn that $P_{(\tilde{\Theta}_1\mathcal{H}_2^p)^\perp}(\tilde{\Theta}_2\mathcal{H}_2^p)^\perp = (\tilde{\Theta}_1\mathcal{H}_2^p)^\perp$. Since the dimension of $(\tilde{\Theta}_i\mathcal{H}_2^p)^\perp$ is equal to the McMillan degree of $\tilde{\Theta}_i$ $i = 1,2$, the identity $P_{(\tilde{\Theta}_1\mathcal{H}_2^p)^\perp}(\tilde{\Theta}_2\mathcal{H}_2^p)^\perp = (\tilde{\Theta}_1\mathcal{H}_2^p)^\perp$ implies that the McMillan degree of the inner function $\tilde{\Theta}_2$ is greater than or equal to that of the inner function $\tilde{\Theta}_1$. This contradicts the first assumption.

2: This is proved by showing that $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) \neq \mathcal{H}_2^p$ implies $\inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \|\tilde{\Theta}_2 - \tilde{\Theta}_1 Q\|_\infty = 1$. Note that $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) + H_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) = \tilde{\Theta}_2^*\tilde{\Theta}_1(\mathcal{H}_2^p)$. Since $\tilde{\Theta}_2^*\tilde{\Theta}_1(\mathcal{H}_2^p)$ is closed and $H_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p)$ is finite-dimensional, this identity implies that $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p)$ is also closed. As $T_{\tilde{\Theta}_2^*\tilde{\Theta}_1}(\mathcal{H}_2^p) \neq \mathcal{H}_2^p$, then $\operatorname{Ker} T_{\tilde{\Theta}_1^*\tilde{\Theta}_2} = [\mathcal{R}(T_{\tilde{\Theta}_2^*\tilde{\Theta}_1})]^\perp \neq \{0\}$. This fact and the relationship

$$T^*_{\tilde{\Theta}_1^*\tilde{\Theta}_2}T_{\tilde{\Theta}_1^*\tilde{\Theta}_2} + H_{(\tilde{\Theta}_1^*\tilde{\Theta}_2)^*}H_{\tilde{\Theta}_1^*\tilde{\Theta}_2} = I$$

imply that there exists a $v \in \mathcal{H}_2$ such that $\|H_{\tilde{\Theta}_1^*\tilde{\Theta}_2}v\|_2 = 1$. This implies that $\|H_{\tilde{\Theta}_1^*\tilde{\Theta}_2}\| = 1$, since $1 \geqslant \|\tilde{\Theta}_1^*\tilde{\Theta}_2\|_\infty \geqslant \|H_{\tilde{\Theta}_1^*\tilde{\Theta}_2}\| \geqslant 1$. This gives the result from Nehari's theorem,

$$1 = \|H_{\tilde{\Theta}_1^*\tilde{\Theta}_2}\| = \inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \|\tilde{\Theta}_1^*\tilde{\Theta}_2 - Q\|_\infty = \inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \|\tilde{\Theta}_2 - \tilde{\Theta}_1 Q\|_\infty.$$

■

PROPOSITION 5.3. *Suppose the* $p \times m$ *transfer function* G *has a* normalized *r.c.f.* (N, M) *and a* normalized *l.c.f.* $(\tilde{N}, \tilde{M})$, *and the* $m \times p$ *transfer function* K *has a* normalized *r.c.f.* (U, V) *and a* normalized *l.c.f.* $(\tilde{U}, \tilde{V})$.

Then

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\left(\mathscr{G}^T(K), I(G, \tilde{\Theta})\right)$$

$$= \inf_{Q \in \mathscr{H}_{\infty,k}^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$= \inf_{Q \in \mathscr{H}_{\infty,k}^{m \times m}} \left\| \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right\|_\infty .$$

Proof. The proof that the first expression equals the third expression can be split into two parts. The first is to assume that

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \| P_{\mathscr{G}^T(K)} - P_{I(G, \tilde{\Theta})} \| = 1$$

and to show that this implies

$$\inf_{Q \in \mathscr{H}_{\infty,k}^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = 1.$$

The second part is to assume that

$$\inf_{\Theta \in \mathscr{B}_k^p} \| P_{\mathscr{G}^T(K)} - P_{I(G, \tilde{\Theta})} \| < 1$$

and to show that under this assumption the two expressions are equal. To prove the first part assume

$$\inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \| P_{\mathscr{G}^T(K)} - P_{I(G, \tilde{\Theta})} \| = 1;$$

then by the equivalence of (U6) and (U1) this implies that $(\tilde{M}V - \tilde{N}U)^{-1} \notin \mathscr{R}\mathscr{H}_{\infty,k}$. It is first also assumed that $(\tilde{M}V - \tilde{N}U)^{-1} \notin \mathscr{R}\mathscr{L}_\infty$, and it is shown that this implies

$$\inf_{Q \in \mathscr{H}_{\infty,k}^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = 1.$$

Then it is assumed that $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathcal{RL}_\infty$, and it is shown that this again implies the same result. Assume $(\tilde{M}V - \tilde{N}U)^{-1} \notin \mathcal{RL}_\infty$; then there exists an $\omega_0 \in \mathfrak{R}$ and a $v_0 \in \mathcal{C}^p$, where $v_0^* v_0 = 1$, such that $v_0^*(\tilde{M}V - \tilde{N}U)(i\omega_0) = 0$. For all $Q \in \mathcal{H}_{\infty,k}^{p\times p}$ there exists an m such that $\|Q\|_\infty \leqslant m$ almost everywhere. Note that $\tilde{M}V - \tilde{N}U \in \mathcal{RH}_\infty$, and therefore $\omega \mapsto (\tilde{M}V - \tilde{N}U)(i\omega)$ is a continuous function. Hence for all ε such that $1/m > \varepsilon > 0$ there exists $\delta > 0$ such that $\sigma_{\max}(v_0^*(\tilde{M}V - \tilde{N}U)(i\omega)) < \varepsilon$ for all $\omega \in]\omega_0 - \varepsilon, \omega_0 + \delta[$. Now

$$\inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$\geqslant \inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$= \inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| I - (\tilde{M}V - \tilde{N}U)Q \right\|_\infty .$$

Therefore $\sigma_{\max}(v_0^*(\tilde{M}V - \tilde{N}U)(i\omega)Q(i\omega)) < m\epsilon < 1$ almost everywhere on the interval $\omega \in]\omega_0 - \delta, \omega_0 + \delta[$. Finally, almost everywhere on $\omega \in]\omega_0 - \delta, \omega_0 + \delta[$ we have that $\sigma_{\max}(v_0^*[I - (\tilde{M}V - \tilde{N}U)(i\omega)Q(i\omega)]) \geqslant \sigma_{\max}(v_0^*) - \sigma_{\max}(v_0^*(\tilde{M}V - \tilde{N}U)(i\omega)Q(i\omega)) > 1 - m\epsilon$. Since this inequality holds for arbitrary small $\epsilon > 0$, this implies that $\|I - (\tilde{M}V - \tilde{N}U)Q\|_\infty = 1$ for every $Q \in \mathcal{H}_{\infty,k}$, as for any transfer function $A \in \mathcal{H}_\infty$ we have $\|A\|_\infty = \operatorname{ess\,sup}_{\omega\in\mathfrak{R}} \sup_{v\in\mathcal{C}^p,\ \|v\|=1} \sigma_{\max}(v^*A)(i\omega)$. Now assume $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathcal{RL}_\infty$ but $(\tilde{M}V - \tilde{N}U)^{-1} \notin \mathcal{RH}_{\infty,k}$. There exists an inner-outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$, where $\tilde{S}_0$ is a unit and $\tilde{\Theta}$ is an inner function. By the equivalence of (U6) and (U0) the McMillan degree of $\tilde{\Theta}$ is greater than k. Now

$$\inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty \geqslant \inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| I - (\tilde{M}V - \tilde{N}U)Q \right\|_\infty$$

$$= \inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| I - \tilde{\Theta}\tilde{S}_0 Q \right\|_\infty$$

$$= \inf_{Q\in\mathcal{H}_{\infty,k}^{p\times p}} \left\| I - \tilde{\Theta} Q \right\|_\infty .$$

For all $Q \in \mathscr{H}_{\infty,k}$ there exists a right inner coprime factorization $Q = Q_N \tilde{\Theta}_M^*$ where $Q_N \in \mathscr{H}_\infty$ and $\tilde{\Theta}_M$ is an inner function of McMillan degree less than or equal to k. Therefore for all $Q \in \mathscr{H}_{\infty,k}$,

$$\|I - \tilde{\Theta}Q\|_\infty = \|I - \tilde{\Theta}Q_N\tilde{\Theta}_M^*\|_\infty = \|\tilde{\Theta}_M - \tilde{\Theta}Q_N\| \geqslant 1$$

by Lemma 5.2. This completes the proof of the first part.

To prove the second part assume

$$\sigma := \inf_{\tilde{\Theta} \in \mathscr{B}_k^p} \operatorname{gap}\left(\mathscr{G}^T(K), I(G,\tilde{\Theta}\right) < 1.$$

Then by the equivalence of (U6) and (U1) this implies that $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathscr{H}_{\infty,k}$ and $\sigma = \|M^*U + N^*V\|_\infty$ by Theorem 3.4. Using $Q = (1 - \sigma^2)(\tilde{M}V - \tilde{N}U)^{-1}$, then

$$\inf_{Q \in \mathscr{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$\leqslant \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - (1-\sigma^2)\begin{bmatrix} V \\ U \end{bmatrix}(\tilde{M}V - \tilde{N}U)^{-1} \right\|_\infty$$

$$= \left\| \begin{bmatrix} \tilde{M} & -\tilde{N} \\ N^* & M^* \end{bmatrix} \left(\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - (1-\sigma^2)\begin{bmatrix} V \\ U \end{bmatrix}(\tilde{M}V - \tilde{N}U)^{-1} \right) \right\|_\infty$$

$$= \left\| \begin{bmatrix} \sigma^2 \\ -(1-\sigma^2)(M^*U + N^*V)(\tilde{M}V - \tilde{N}U)^{-1} \end{bmatrix} \right\|_\infty .$$

As (U, V) is a normalized coprime factorization and $\sigma = \|M^*U + N^*V\|_\infty$, this implies, by Lemma 6.2 in [13], that $\|(M^*U + N^*V)(\tilde{M}V - \tilde{N}U)^{-1}\|_\infty = \sigma(1 - \sigma^2)^{-1/2}$. Hence,

$$\inf_{Q \in \mathscr{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$\leqslant \left\| \begin{bmatrix} \sigma^2 \\ -(1-\sigma^2)(M^*U + N^*V)(\tilde{M}V - \tilde{N}U)^{-1} \end{bmatrix} \right\|_\infty$$

$$= \left[\sigma^4 + \sigma^2(1-\sigma^2)\right]^{1/2} = \sigma .$$

Therefore it has been established that

$$\inf_{\tilde{\Theta}\in\mathscr{B}_k^p} \|P_{\mathscr{G}^T(K)} - P_{I(G,\tilde{\Theta})}\| \leqslant \inf_{Q\in\mathscr{H}_{\infty,k}^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

whenever $\inf_{\tilde{\Theta}\in\mathscr{B}_k^p}\|P_{\mathscr{G}^T(K)} - P_{I(G,\tilde{\Theta})}\| < 1$. The proof of the reverse equality for this case is analogous to the proof of Proposition 6.3 in [13].

The final equality can be proved analogously. ■

In Definition 5.1 an optimal unstable controller to order k was defined. That such a controller exists is shown in the following theorem. This theorem generalizes the results for stabilizing controllers in [13, 11].

THEOREM 5.4. *Suppose the* $p \times m$ *transfer function* G *has a r.c.f.* (N, M) *and a l.c.f.* $(\tilde{N}, \tilde{M})$, *and let* σ_j *with multiplicity* r_j *be the* jth *singular value of the Hankel operator with symbol* $[\tilde{M} \ \ -\tilde{N}]^*$, *where* $\sigma_1 > \sigma_2 > \cdots > \sigma_j > \cdots > 0$. *Then*

$$\cos\left(\theta_{\min}^{\text{opt}}\right)_k := \inf_{K\in\mathscr{K}_G} \cos\theta_{\min}(\mathscr{G}(G), \mathscr{G}^T(K)) = \sigma_i,$$

where $\sum_{j=1}^{i-1} r_j \leqslant k < \sum_{j=1}^{i} r_j$, *and* $\mathscr{K}_G = \{K | K$ *is an* $m \times p$ *proper rational function s.t.* (G, K) *is unstable to order* $k\}$. *An optimal unstable controller to order* k *exists, and every optimal unstable controller has a right coprime factorization* (U, V) *such that*

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}^* \right\|_\infty = \sigma_i,$$

where $\tilde{\Theta} \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ *is an inner function such that* $(\tilde{M}V - \tilde{N}U)\tilde{\Theta}^*$ *is a unit.*

Proof. By the basic properties of minimal angles between subspaces we have

$$1 > \cos\left(\theta_{\min}^{\text{opt}}\right)_k := \inf_{K\in\mathscr{K}_G} \cos\theta_{\min}(\mathscr{G}(G), \mathscr{G}^T(K))$$

$$= \inf_{\substack{K=UV^{-1}\in\mathscr{K}_G \\ U,V \text{ coprime}}} \|P_{\mathscr{G}(G)} P_{G^T(K)}\|,$$

where the infimum is less that one, as there exists a stabilizing controller. Now if $K \in \mathscr{K}_G$ then $\inf_{\tilde{\Theta} \in \mathscr{B}_k} \operatorname{gap}(\mathscr{G}^T(K), I(G, \tilde{\Theta})) = \|P_{\mathscr{G}(G)} P_{\mathscr{G}^T(K)}\| < 1$. If $K \notin \mathscr{K}_G$ then $\inf_{\Theta \in \mathscr{B}_k} \operatorname{gap}(\mathscr{G}^T(K), I(G, \tilde{\Theta})) = 1$. Hence the above optimization can be rephrased as

$$1 > \cos\left(\theta_{\min}^{\text{opt}}\right)_k := \inf_{\substack{U, V \in \mathscr{H}_\infty \\ \text{coprime}}} \inf_{\Theta \in \mathscr{B}_k} \operatorname{gap}\left(\mathscr{G}^T(K), I(G, \tilde{\Theta})\right)$$

$$= \inf_{\substack{U, V \in \mathscr{H}_\infty \\ \text{coprime}}} \inf_{Q \in \mathscr{H}_{\infty,k}^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

by the previous proposition. First note the inequality

$$1 > \inf_{\substack{U, V \in \mathscr{H}_\infty \\ \text{coprime}}} \inf_{Q \in \mathscr{H}_{\infty,k}^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$\geqslant \inf_{\begin{bmatrix} V \\ U \end{bmatrix} \in \mathscr{RH}_{\infty,k}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty .$$

The reverse inequality will now be shown, proving that the above two expressions are in fact equal.

Let

$$\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \in \mathscr{RH}_{\infty,k}$$

achieve the infimum in the expression on the right; its existence is proved for example in Glover [7]. Now

$$\tau\left(\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}}\right) > 0;$$

otherwise it would be possible to construct a sequence $v_i \in \mathscr{H}_2^\perp$, $\|v_i\| = 1$, $i = 1, 2, \ldots,$ such that

$$\lim_{i \to \infty} \left\| \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} v_i \right\|_2 = 0.$$

This would imply

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \right\|_\infty \geqslant \lim_{i\to\infty} \left\| \left(\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \right) v_i \right\|_2$$

$$= \lim_{i\to\infty} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} v_i \right\|_2 = 1,$$

establishing a contradiction. Let

$$\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \in \mathcal{RH}_{\infty,k}$$

have a right coprime factorization

$$\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} = \begin{bmatrix} V \\ U \end{bmatrix}_N^{\text{opt}} Q_M^{-1},$$

where

$$\begin{bmatrix} V \\ U \end{bmatrix}_N^{\text{opt}} \in \mathcal{RH}_\infty \quad \text{and} \quad Q_M^{-1} \in \mathcal{RH}_{\infty,k}.$$

As

$$\tau\left(\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \right) > 0,$$

this implies that

$$\tau\left(\begin{bmatrix} V \\ U \end{bmatrix}_N^{\text{opt}} \right) > 0$$

and there exists an inner-outer factorization of

$$\begin{bmatrix} V \\ U \end{bmatrix}_N^{\text{opt}} = \begin{bmatrix} V \\ U \end{bmatrix}_{\text{cp}}^{\text{opt}} \Theta$$

where (U_{cp}, V_{cp}) are coprime and Θ is inner; see for example [17, 2]. Combining these factorizations,

$$\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} = \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}}_{\text{cp}} (\Theta Q_M^{-1}),$$

where $\begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}}_{\text{cp}}$ is coprime and $\Theta Q_M^{-1} \in \mathscr{R}\mathscr{H}_{\infty,k}$. Hence

$$\begin{aligned} &\inf_{\begin{bmatrix} V \\ U \end{bmatrix} \in \mathscr{H}_{\infty,k}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty \\ &= \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} (\Theta Q_M^{-1}) \right\|_\infty \\ &\geqslant \inf_{\substack{U, V \in \mathscr{H}_\infty \\ \text{coprime}}} \; \inf_{Q \in \mathscr{H}^{p \times p}_{\infty,k}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty . \end{aligned}$$

It has therefore been proved that

$$\cos\left(\theta_{\min}^{\text{opt}}\right)_k = \inf_{\begin{bmatrix} V \\ U \end{bmatrix} \in \mathscr{H}_{\infty,k}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}^{\text{opt}} \right\|_\infty ,$$

where the fact that

$$\inf_{\begin{bmatrix} V \\ U \end{bmatrix} \in \mathscr{H}_{\infty,k}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \sigma_i$$

is a standard result from Hankel-norm approximation theory [7], and further, $K = U^{\text{opt}}(V^{\text{opt}})^{-1}$ is an optimal unstable controller to order k.

The final claim can now be proved, in a manner almost identical to the second part of the previous proposition. Let K be an optimal unstable controller to order k; then it has a normalized right coprime factorization (U, V) such that $\sigma_i = \|P_{\mathscr{G}(G)} P_{\mathscr{G}^T_{\sim}(K)}\| = \|M^* V + N^* U\|_\infty$, and $\tilde{M}V - \tilde{N}U$ has an inner outer factorization $\tilde{M}V - \tilde{N}U = \tilde{\Theta}\tilde{S}_0$, where $\tilde{\Theta} \in \mathscr{B}_k$ and $\tilde{S}_0$ is a unit. Let

$$\begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} = (1 - \sigma_i^2) \begin{bmatrix} V \\ U \end{bmatrix} \tilde{S}_0^{-1}$$

be another factorization of this optimal controller. The proof is completed in a manner identical to the second part of the previous proposition, to show

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} \hat{V} \\ \hat{U} \end{bmatrix} \tilde{\Theta}^* \right\|_\infty = \sigma_i .$$

∎

This section has proved that it is possible to construct a controller K for a system G that maximizes the minimum angle $\cos\theta_{\min}(\mathscr{G}(G), \mathscr{G}^T(K))$ subject to the constraint that the control system is unstable to order k. These results give an interesting interpretation to the problem of Hankel-norm approximation of nonsquare coinner or inner transfer functions.

Given a nonsquare coinner transfer function $\tilde{\Theta}_{\mathrm{ci}} \in \mathscr{RH}_\infty^{l\times n}$, then there exists a "unique" minimal degree unitary dilation of $\tilde{\Theta}_{\mathrm{ci}}^*$ such that $[\tilde{\Theta}_{\mathrm{ci}}^* \;\; \Theta_{\mathrm{i}}]$ is a square all-pass transfer function and $\Theta_{\mathrm{i}} \in \mathscr{RH}_\infty^{n\times(n-l)}$. This implies that Θ_{i} must be a nonsquare inner transfer function. The result in this section prove that the problem of finding a kth-order Hankel-norm approximation of $\tilde{\Theta}_{\mathrm{ci}}^*$ is equivalent to the problem of finding an inner function $\Theta_{\mathrm{app}} \in \mathscr{RH}_\infty^{n\times(n-l)}$ that maximizes the angle $\cos\theta_{\min}(\Theta_{\mathrm{i}}\mathscr{H}_2^{(n-l)}, \Theta_{\mathrm{app}}\mathscr{H}_2^{(n-l)})$ subject to the condition that the codimension of the closed space $\Theta_{\mathrm{i}}\mathscr{H}_2^{(n-l)} + \Theta_{\mathrm{app}}\mathscr{H}_2^{(n-l)}$ in $\mathscr{H}_2^n$ is less than or equal to k.

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