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INFINITE-DIMENSIONAL CONTINUOUS-TIME LINEAR SYSTEMS: STABILITY AND STRUCTURE ANALYSIS*

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Abstract. The question of exponential and asymptotic stability of infinite-dimensional continuous-time state-space systems is investigated. It is shown that every (par)balanced realization is asymptotically stable. Conditions are given for (par)balanced, input-normal, or output-normal realizations to be asymptotically and/or exponentially stable. The boundedness of the system operators is also studied. Examples of delay systems are given to illustrate the theory.

Key words. linear infinite-dimensional systems, balanced realizations, stability, Hankel operators, semigroups of operators

AMS subject classifications. 93B15, 93B20, 93B28, 93D20

1. Introduction. For a finite-dimensional linear system with transfer function G, there are standard ways to obtain a minimal, i.e., reachable and observable, state-space realization:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{cases}$$

This realization is unique in the sense that every other minimal realization is equivalent to it. The spectrum of the state propagation operator A is precisely the set of poles of the transfer function $G(s) = C(sI - A)^{-1}B + D$, which is proper rational. Hence the realization is exponentially stable if and only if the poles of G are all in the open left half plane. Furthermore, exponential stability of the system is equivalent to asymptotic stability.

This paper is concerned with the question of stability for infinite-dimensional systems. If the transfer function G is not rational, then we have an infinite-dimensional system of the above form, where the system operators A, B, and C are usually unbounded operators. In general, it is no longer true that all observable and reachable realizations are equivalent. The correspondence between the spectrum of the realization and the singularities of the transfer function does not necessarily hold. In general the exponential stability of a system cannot be determined by the location of the singularities of the transfer function (see, e.g., [18]). Also asymptotically stable systems are typically not exponentially stable.

There have been attempts to extend the results for finite-dimensional systems mentioned above to the infinite-dimensional case by restricting the transfer functions to a certain class. For example, Curtain [4], Yamamoto [29], and several other authors considered the equivalence between input/output stability and internal stability. We refer to [4] and [29] and the reference therein for the work in this direction. Inevitably, the stronger the results are, the smaller the class of transfer functions is.

Here we present another approach. Instead of putting too stringent restrictions on the class of transfer functions to be studied, we restrict the class of realizations to (par)balanced realizations and the closely related input-normal and output-normal realizations. These types of realizations have been advocated by several authors [20], [13], [14], [30]. They were introduced in the finite-dimensional case as a means to perform model reduction in an easy fashion

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[20]. Glover, Curtain, and Partington [14] derived infinite-dimensional continuous-time balanced realizations for a class of transfer functions with nuclear Hankel operators. Young [30] developed a general realization theory of balanced realizations of infinite-dimensional discrete-time systems. The results were generalized to the continuous-time case by Ober and Montgomery-Smith [23]. The results by Young were also used by the authors to conduct an analysis of the stability and structural properties for infinite-dimensional discrete-time systems in [24].

In this paper we extend our analysis in [24] to the continuous-time case. The exponential and asymptotic stability properties of parbalanced, input-normal, and output-normal realizations are studied in detail. It is shown that all parbalanced realizations are asymptotically stable. For a subclass of transfer functions—namely, strictly noncyclic functions—results that are reminiscent of the finite-dimensional case are obtained. For this class of transfer functions the location of the singularities of the transfer function determines the exponential stability properties of parbalanced systems. The stability properties of parbalanced realizations are studied without the explicit presentation of the realizations. Structural properties of the realizations are also analyzed. In particular the boundedness of the system operators of the input- and output-normal realizations is investigated.

Most of the results presented in this paper are in terms of the properties of the transfer functions and the Hankel operators with the transfer functions as symbols. This may therefore be regarded as expressing the internal properties of a system in terms of input/output properties. Related topics can be found in Dewilde [7], where systems with strictly noncyclic transfer functions are studied from an input/output point of view. We also refer to Baras, Brockett, and Fuhrmann [2], [3], [11]. For realization theory of nonrational transfer functions, Fuhrmann [11] and Helton [16] reference for transfer provide general references.

Our main tool is a bilinear map that maps discrete-time systems to continuous-time systems. This bilinear map is routinely used for finite-dimensional systems to translate discrete-time results to continuous-time results and vice versa. In [23] properties of this bilinear map were studied for infinite-dimensional systems (see also [11]). Some continuous-time questions, however, such as exponential stability, cannot be directly answered by simply applying the bilinear transform to a discrete-time result. In such cases a more detailed study of the problem is necessary.

The contents of the paper can be summarized as follows. In §2 we review the settings of infinite-dimensional continuous-time systems we will deal with. We restrict ourselves to so-called admissible systems. We relate continuous-time systems to discrete-time systems in §3, using the above-mentioned bilinear map. As Hankel operators play an important role in our approach, we discuss Hankel operators in §4 in both the discrete- and the continuous-time case. Concrete constructions of the continuous time restricted and *-restricted shift realizations are given in §5. They respectively represent the classes of input-normal and output-normal realizations and are intimately related to Hankel operators and translation semigroups. In §6 we establish the asymptotic stability of all parbalanced continuous-time realizations. Conditions for input-normal or output-normal realizations to be asymptotically stable are also given in terms of the cyclicity of the transfer functions. The topic of §7 is exponential stability. Necessary and sufficient conditions are given for the input- and output-normal realizations to be exponentially stable. These conditions are based on the spectral properties of the transfer functions. They also hold for parbalanced realizations as long as the transfer functions are strictly noncyclic. In §8 we investigate when the system operators are bounded, and finally some examples are given in §9.

The following symbols are used:

$\mathbb{D}$	the open unit disk,
$\partial\mathbb{D}$	the unit circle,
$\mathbb{D}_e$	the complement of $(\partial\mathbb{D}) \cup \mathbb{D}$,
$D_X^{U,Y}$	admissible discrete-time systems (§3),
$C_X^{U,Y}$	admissible continuous-time systems (§3),
$D(A) \subseteq X$	the domain of an operator A on X,
$(D(A), \|\cdot\|_A)$	the space $D(A)$ equipped with norm $\|x\|_A^2 = \|x\|^2 + \|Ax\|^2$,
$(D(A)^{(\prime)}, \|\cdot\|')$	$\{f \mid f : (D(A), \|\cdot\|_A) \to \mathbb{C}$, antilinear, bounded$\}$,
$G_d^{\perp}(z)$	$\frac{1}{z}[G_d(\frac{1}{z}) - G_d(\infty)]$, $z \in \mathbb{D}$, for $G_d \in TLD^{U,Y}$,
$G_c(+\infty)$	$\lim_{\substack{r\in\mathbb{R} \\ r\to+\infty}} G_c(r)$,
H_K	the Hankel operator with symbol K,
$H^{\infty}_{L(U,Y)}(W)$	$\{F \mid F : W \to L(U,Y)$ analytic, $\sup_{z\in W}\|F(z)\| < \infty\}$; $W = \mathbb{D}$ or RHP,
$H_Y^2(\mathbb{D})$	$\{f \mid f : \mathbb{D} \to Y$ analytic on $\mathbb{D}$ and $\sup_{0<r<1}\int_0^{2\pi}\|f(re^{it})\|^2 dt < \infty\}$,
$H_Y^2(RHP)$	$\{f \mid f : RHP \to Y$ analytic on RHP and $\sup_{x>0}\int_{-\infty}^{\infty}\|f(x+iy)\|^2 dy < \infty\}$,
$\tilde{K}(z)$	$(K(\bar{z}))^*$,
$\mathcal{L}$	the Laplace transform (§3),
$L(U,Y)$	$\{A \mid A : U \to Y$ a bounded operator$\}$,
$L_Y^2(\Delta)$	$\{f \mid f : L \to Y$ square integrable on $\Delta\}$, $\Delta = \partial\mathbb{D}$ or $i\mathbb{R}$,
LHP	the open left half plane: $\{s \in \mathbb{C} : Re(s) < 0\}$,
P_+	The orthogonal projection of $L_Y^2(\Delta)$ onto $H_Y^2(W)$; $\Delta = \partial\mathbb{D}$, $W = \mathbb{D}$, or $\Delta = i\mathbb{R}$, $W = RHP$,
P_X	the orthogonal projection of $H_Y^2(W)$ onto $X \subseteq H_Y^2(W)$; $W = \mathbb{D}$ or RHP,
RHP	the open right half plane: $\{s \in \mathbb{C} : Re(s) > 0\}$,
S	the forward shift: $(Sf)(z) = zf(z)$ for $f \in H_Y^2(\mathbb{D})$,
S^*	the backward shift: $(S^*f)(z) = z^{-1}[f(z) - f(0)]$ for $f \in H_Y^2(\mathbb{D})$,
$S(Q)$	$P_X S\|_X$, the compression of S to X, where $X = H_Y^2(\mathbb{D}) \ominus (QH_Y^2(\mathbb{D}))$,
$S(Q)^*$	$S^*\|_{H_Y^2(\mathbb{D})\ominus(QH_Y^2(\mathbb{D}))}$, the restriction of S^* to $H_Y^2(\mathbb{D}) \ominus (QH_Y^2(\mathbb{D}))$,
$\sigma(A)$	the spectrum of an operator A,
$\sigma_p(A)$	the point spectrum of an operator A,
$\sigma(Q)$	the spectrum of an inner function $Q \in H_Y^{\infty}(W)$ (Lemma 7.3),
$\sigma_s(G)$	the set of points in $\mathbb{C}$ where G has no analytic continuation (§7),
$TLD^{U,Y}$	$\{G_d \mid G_d : \mathbb{D}_e \to L(U,Y)$ has a reachable and observable admissible realization$\}$,
$TLC^{U,Y}$	$\{G_c \mid G_c : RHP \to L(U,Y)$ has a reachable and observable admissible realization$\}$,
$X \vee Y$	closed linear span of subsets X and Y of a Hilbert space,
$(F,G)_L = I_Y$	F and G are weakly left coprime (§3),
$(F,G)_R = I_U$	F and G are weakly right coprime (§3).

2. Admissible continuous-time state-space systems. The main aim of this section is to briefly set out the notation and introduce the most important system theoretic concepts for this paper. More details can be found in [11], [23], [27], and [6]. In the first subsection, admissible continuous-time systems are discussed. Input-normal, output-normal, and parbalanced realizations are defined in the second subsection. It is these classes of systems that are being analyzed in detail in later sections. What is meant by system equivalence for infinite-dimensional systems is defined in the third subsection.

2.1. Admissible continuous-time systems. It is well known that if A is the generator of a strongly continuous semigroup of operators $(e^{tA})_{t\geq 0}$ with domain of definition $D(A)$, then $D(A)$ is a Hilbert space with inner product induced by the graph norm

$$\|x\|_A^2 := \|x\|_X^2 + \|Ax\|_X^2, \quad x \in D(A).$$

Since $\|x\|_A \geq \|x\|$ for $x \in D(A)$, we can embed X in $D(A)^{(\prime)}$, the set of antilinear continuous functionals on $(D(A), \|\cdot\|_A)$, by

$$\begin{aligned} E: \quad & X \to D(A)^{(\prime)}, \\ & x \longmapsto (y \longmapsto \langle x, y\rangle). \end{aligned}$$

Note that $D(A)^{(\prime)}$ is a Hilbert space with norm $\|f\|' := \sup_{\|x\|_A\leq 1} |f(x)|$. Since $\langle\cdot,\cdot\rangle$ is linear in the first component, the embedding E is linear. By the above, we have the rigged structure

$$D(A) \subseteq X \subseteq D(A)^{(\prime)}.$$

If $(A, D(A))$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t\geq 0}$ on a Hilbert space, then the adjoint $(A^*, D(A^*))$ of $(A, D(A))$ is the generator of the adjoint semigroup $(e^{tA})^*_{t\geq 0}$ (see [26]). Hence, we have similarly that

$$D(A^*) \subseteq X \subseteq D(A^*)^{(\prime)}.$$

We are now in a position to define admissible continuous-time systems.

DEFINITION 2.1. *A quadruple of operators* (A_c, B_c, C_c, D_c) *is called an* admissible continuous-time system *with state space* X*, input space* U*, and output space* Y*, where* X*,* U*, and* Y *are separable Hilbert spaces, if*

1. $(A_c, D(A_c))$ *is the generator of a strongly continuous semigroup of contractions on* X;
2. $B_c : U \to (D(A_c^*)^{(\prime)}, \|\cdot\|')$ *is a bounded linear operator;*
3. $C_c : D(C_c) \to Y$ *is linear with* $D(C_c) = D(A_c) + (I - A_c)^{-1}B_cU$ *and* $C_{c|D(A_c)} : (D(A_c), \|\cdot\|_{A_c}) \to Y$ *is bounded;*
4. $C_c(I - A_c)^{-1}B_c \in L(U, Y)$;
5. A_c, B_c, *and* C_c *are such that* $\lim_{\substack{s\in\mathbb{R} \\ s\to+\infty}} C_c(sI - A_c)^{-1}B_c = 0$ *in the norm topology;*
6. $D_c \in L(U, Y)$.

We write $C_X^{U,Y}$ *for the set of admissible continuous-time systems with input space* U*, output space* Y*, and state space* X. □

By the resolvent identity, part 4 of the definition implies that $G_c(s) := C_c(sI - A_c)^{-1}B_c \in L(U, Y)$ for all $s \in RHP$ and G_c is analytic on the RHP. The function G_c is called the transfer function of the system, and (A_c, B_c, C_c, D_c) is called a realization of G_c.

2.2. Duality, observability, reachability, and parbalanced realizations. In order to define observability and reachability for continuous-time systems we need to introduce the notion of the dual system of an admissible continuous-time system.

DEFINITION 2.2. *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$. *Then the* dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *of* (A_c, B_c, C_c, D_c) *is given by*

1. $(\tilde{A}_c, D(\tilde{A}_c)) = (A_c^*, D(A_c^*))$, *the adjoint operator of* $(A_c, D(A_c))$;
2. $\tilde{B}_c : Y \to D(A_c)^{(\prime)}; y \longmapsto \tilde{B}_c(y)[\cdot] := \langle y, C_c(\cdot)\rangle$;

3. $\tilde{C}_c : D(\tilde{C}_c) \to U$, $D(\tilde{C}_c) = D(\tilde{A}_c) + (I - \tilde{A}_c)^{-1}\tilde{B}_c Y$, *where* $\tilde{C}_c x_0$ *is defined by*

$$\begin{cases} \langle u, \tilde{C}_c x_0\rangle = B_c(u)[x_0], & x_0 \in D(A_c^*), u \in U, \\ \langle \tilde{C}_c x_0, u\rangle = \langle y_0, C_c(I - A_c)^{-1}B_c u\rangle, & x_0 = (I - \tilde{A}_c)^{-1}\tilde{B}_c y_0,\ y_0 \in Y, u \in U; \end{cases}$$

4. $\tilde{D}_c := D_c^* : Y \to U$. □

It can be directly verified that the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ of an admissible continuous-time system (A_c, B_c, C_c, D_c) is admissible. If the continuous-time transfer function $G(s) : RHP \to L(U, Y)$ has an admissible realization (A_c, B_c, C_c, D_c), then the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a realization of the transfer function $\tilde{G}(s) := (G(\bar{s}))^*$, $s \in RHP$, i.e., for all $s \in RHP$,

$$\tilde{G}(s) = (G(\bar{s}))^* = \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c + \tilde{D}_c.$$

The definition of observability and reachability of admissible continuous-time systems is now given.

DEFINITION 2.3. *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$; *then the operator*

$$\begin{aligned} \mathcal{O}_c : D(\mathcal{O}_c) &\to L_Y^2([0,\infty)), \\ x &\longmapsto (C_c e^{tA_c}x)_{t\geq 0} \end{aligned}$$

is called the observability operator *of the system* (A_c, B_c, C_c, D_c), *where*

$$D(\mathcal{O}_c) = \{x \in X \mid C_c e^{tA_c}x \text{ exists for almost all } t \in [0,\infty), \text{ and } C_c e^{tA_c}x \in L_Y^2([0,\infty))\}.$$

We say that (A_c, B_c, C_c, D_c) *has a bounded observability operator if* $D(A_c) \subseteq D(\mathcal{O}_c)$ *and* $\mathcal{O}_c$ *extends to a bounded operator on* X. *This extension will also be denoted by* $\mathcal{O}_c$.

If (A_c, B_c, C_c, D_c) *has a bounded observability operator* $\mathcal{O}_c$ *such that* $\mathrm{Ker}(\mathcal{O}_c) = \{0\}$, *then the system* (A_c, B_c, C_c, D_c) *is called* observable.

Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *be the dual system of* (A_c, B_c, C_c, D_c). *If the observability operator* $\tilde{\mathcal{O}}_c$ *of* $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *is a bounded operator on* X, *the adjoint of* $\tilde{\mathcal{O}}_c$ *is called the* reachability operator, *denoted by* $\mathcal{R}_c$, *of* (A_c, B_c, C_c, D_c), *i.e.*,

$$\mathcal{R}_c := \tilde{\mathcal{O}}_c^*.$$

If $\mathcal{R}_c$ *exists and* range($\mathcal{R}_c$) *is dense in* X, *the system* (A_c, B_c, C_c, D_c) *is said to be* reachable. □

The set of all reachable and observable continuous-time systems with input space U, output space Y, and state space X is denoted by $LC_X^{U,Y}$. We mainly deal with this set of systems.

The *reachability Gramian* $\mathcal{W}_c$ and the *observability Gramian* $\mathcal{M}_c$ of a continuous-time system with bounded reachability operator $\mathcal{R}_c$ and bounded observability operator $\mathcal{O}_c$ are defined to be

$$\begin{aligned} \mathcal{W}_c &:= \mathcal{R}_c\mathcal{R}_c^* : X \to X, \\ \mathcal{M}_c &:= \mathcal{O}_c^*\mathcal{O}_c : X \to X. \end{aligned}$$

When $\mathcal{W}_c = \mathcal{M}_c$ and the admissible system is observable and reachable, we say that the system is *parbalanced*. A reachable and observable admissible system is said to be *balanced* if $\mathcal{W}_c = \mathcal{M}_c$ and $\mathcal{W}_c$ has a diagonal representation with respect to an orthonormal basis of the state space. If $\mathcal{W}_c = I$, then a reachable and observable admissible system is called *input-normal*. If $\mathcal{M}_c = I$, then a reachable and observable admissible system is called *output-normal*.

2.3. System equivalence. The concept of an equivalent state-space transformation of an admissible continuous-time system is slightly more complicated than in the discrete time case as the system operators are in general unbounded.

Two systems $(A_c^i, B_c^i, C_c^i, D_c^i) \in C_{X_i}^{U,Y}$, $i = 1, 2$, are called *equivalent* if there exists a boundedly invertible operator $V \in L(X_1, X_2)$ such that

$$((A_c^2, D(A_c^2)), B_c^2, (C_c^2, D(C_c^2)), D_c^2) =$$
$$((VA_c^1V^{-1}, VD(A_c^1)), VB_c^1, (C_c^1V^{-1}, VD(C_c^1)), D_c^1),$$

where

$$B_c^2 = (VB_c^1) : U \to \left(\left(D(A_c^{2*})\right)^{(\prime)}, \|\cdot\|'\right)$$

is given by

$$[B_c^2(u)](x) = (VB_c^1)(u)[x] := B_c^1(u)[V^*x], \quad u \in U, \ x \in D(A_c^{2*}) = (V^*)^{-1}D(A_c^{1*}).$$

If V is a unitary operator, then the two systems are said to be *unitarily equivalent*.

We have the following results concerning equivalent systems.

PROPOSITION 2.4. *Let $(A_c^i, B_c^i, C_c^i, D_c^i) \in C_{X_i}^{U,Y}$, $i = 1, 2$, be two equivalent systems such that*

$$((A_c^2, D(A_c^2)), B_c^2, (C_c^2, D(C_c^2)), D_c^2) =$$
$$((VA_c^1V^{-1}, VD(A_c^1)), VB_c^1, (C_c^1V^{-1}, VD(C_c^1)), D_c^1)$$

with $V \in L(X_1, X_2)$ a boundedly invertible operator. Then

1. *both $(A_c^1, B_c^1, C_c^1, D_c^1)$ and $(A_c^2, B_c^2, C_c^2, D_c^2)$ realize the same transfer function.*
2. *if $(A_c^1, B_c^1, C_c^1, D_c^1) \in C_{X_1}^{U,Y}$ has observability operator $\mathcal{O}$ and reachability operator $\mathcal{R}$, then the observability and reachability operators of $(A_c^2, B_c^2, C_c^2, D_c^2) \in C_{X_2}^{U,Y}$ are respectively*

$$\mathcal{O}V^{-1} \quad and \quad V\mathcal{R}.$$

Proof. The proof is straightforward. □

Thus equivalent systems have the same transfer function as well as the same observability and reachability properties. Moreover, it can be seen that unitary equivalent systems have the same Gramians. Hence unitary equivalence preserves parbalancing.

We point out that for an admissible system $((A_c^1, D(A_c^1)), B_c^1, C_c^1, D_c^1) \in C_{X_1}^{U,Y}$ and a unitary operator $V : X_1 \to X_2$, the system

$$((A_c^2, D(A_c^2)), B_c^2, (C_c^2, D(C_c^2)), D_c^2) =$$
$$((VA_c^1V^{-1}, VD(A_c^1)), VB_c^1, (C_c^1V^{-1}, VD(C_c^1)), D_c^1)$$

is also admissible, where VB_c^1 is defined as above. Therefore $((A_c^1, D(A_c^1)), B_c^1, C_c^1, D_c^1)$ and $(A_c^2, B_c^2, C_c^2, D_c^2)$ are unitarily equivalent.

The class of continuous-time transfer functions that we are interested in are those that have reachable and observable continuous time realizations on some state space X, where X is a separable Hilbert space. This class will be denoted by $TLC^{U,Y}$, where U and Y are the input and output spaces, respectively. We characterize those transfer functions in terms of their Hankel operators in §4.

3. Connection between continuous- and discrete-time systems. What is essential in our development is to relate discrete-time systems to continuous-time systems using a generalization of the well-known bilinear transformation for finite-dimensional systems. Thereby it is possible to carry some of the results in [24] for discrete-time systems over to continuous-time systems. It should be noted, however, that not all results of discrete-time systems can be translated to the continuous-time case in this way. For example, under this bilinear map an exponentially stable continuous-time system does not necessarily correspond to a power stable discrete-time system.

3.1. Admissible discrete-time systems. We recall [24] that an *admissible discrete-time system* with input space U, output space Y, and state space X, with U, X, and Y being separable Hilbert spaces, is a quadruple of operators (A_d, B_d, C_d, D_d) that satisfy the following:

1. $A_d \in L(X)$ is a contraction and $-1 \notin \sigma_p(A_d)$;
2. $B_d \in L(U, X)$, $C_d \in L(X, Y)$ and $D_d \in L(U, Y)$;
3. the limit $\lim_{r>1, r\to 1} C_d(rI + A_d)^{-1}B_d$ exists in the norm topology.

The set of all such systems is denoted by $D_X^{U,Y}$. For $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, the function

$$G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d : \mathbb{D}_e \to L(U, Y)$$

is called the *transfer function* of (A_d, B_d, C_d, D_d) and (A_d, B_d, C_d, D_d) is called a *realization* of G_d. Evidently, the transfer function G_d is analytic on $\mathbb{D}_e$ and at infinity.

For $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, its *observability operator* $\mathcal{O}_d : D(\mathcal{O}_d) \to H_Y^2$ is defined as

$$(\mathcal{O}_d x)(z) = \sum_{n\geq 0}(C_d A_d^n x)z^n, \qquad x \in D(\mathcal{O}_d) := \left\{x \mid \sum_{n\geq 0}(C_d A_d^n x)z^n \in H_Y^2\right\}.$$

If $D(\mathcal{O}_d) = X$, $\mathcal{O}_d$ is bounded and $\operatorname{Ker}(\mathcal{O}_d) = \{0\}$, then the system (A_d, B_d, C_d, D_d) is said to be *observable*. The system (A_d, B_d, C_d, D_d) is said to be *reachable* if its *reachability operator* $\mathcal{R}_d : D(\mathcal{R}_d) \to X$ defined by

$$\mathcal{R}_d\left(\sum_{n\geq 0} u_n z^n\right) = \sum_{n\geq 0} A_d^n B_d u_n \quad \left(\sum_{n\geq 0} u_n z^n \in D(\mathcal{R}_d)\right),$$

where $D(\mathcal{R}_d) = \{\sum_{n=0}^{N} u_n z^n \mid N = 0, 1, \ldots, \; u_n \in U\}$ can be extended to a bounded operator with range dense in X. The set of all reachable and observable discrete-time admissible systems with input space U, output space Y, and state space X is denoted by $LD_X^{U,Y}$. The set of all discrete-time transfer functions that have realizations $(A_d, B_d, C_d, D_d) \in LD_X^{U,Y}$ for some state space X is denoted by $TLD^{U,Y}$. A characterization will be given of this class of transfer functions in the next section.

For $(A_d, B_d, C_d, D_d) \in LD_X^{U,Y}$, we define its *reachability Gramian* $\mathcal{W}_d : X \to X$ as

$$\mathcal{W}_d x = \mathcal{R}_d \mathcal{R}_d^* x, \quad x \in X,$$

and its *observability Gramian* $\mathcal{M}_d : X \to X$ as

$$\mathcal{M}_d x = \mathcal{O}_d^* \mathcal{O}_d x, \quad x \in X.$$

If $\mathcal{W}_d = \mathcal{M}_d$ and (A_d, B_d, C_d, D_d) is reachable and observable, then (A_d, B_d, C_d, D_d) is said to be a *parbalanced* realization. If the Gramian of a parbalanced realization has a diagonal representation with respect to an orthonormal basis, the realization is said to be *balanced.* If $\mathcal{W}_d = I$, then the reachable and observable admissible system is called *input-normal.* If $\mathcal{M}_d = I$, then the reachable and observable admissible system is called *output-normal.*

3.2. Bilinear transform. In the following theorems (see [23]) we introduce the map $T : D_X^{U,Y} \to C_X^{U,Y}$, which transforms discrete-time systems to continuous-time systems. Throughout the rest of this paper T will denote this map.

THEOREM 3.1. *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$; *then* $T((A_d, B_d, C_d, D_d)) := (A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, *where the operators* A_c, B_c, C_c, *and* D_c *are defined as follows:*

1. $A_c := (I + A_d)^{-1}(A_d - I) = (A_d - I)(I + A_d)^{-1}$, $D(A_c) := D((I + A_d)^{-1})$. *It generates a strongly continuous semigroup of contractions on* X *given by* $\varphi_t(A_d)$, $t \geq 0$, *with* $\varphi_t(z) = e^{t\frac{z-1}{z+1}}$.

2. *The operator* B_c *is given by*

$$\begin{aligned} B_c := \sqrt{2}(I + A_d)^{-1}B_d : U &\rightarrow D(A_c^*)^{(\prime)}, \\ u &\longmapsto \sqrt{2}(I + A_d)^{-1}B_d(u)[x] \\ &:= \sqrt{2} < B_d(u), (I + A_d^*)^{-1}(x) >_X . \end{aligned}$$

3. *The operator* C_c *is given by*

$$\begin{aligned} C_c : D(C_c) &\rightarrow Y, \\ x &\longmapsto \lim_{\substack{\lambda\to 1 \\ \lambda>1}} \sqrt{2}C_d(\lambda I + A_d)^{-1}x, \end{aligned}$$

where $D(C_c) = D(A_c) + (I - A_c)^{-1}B_cU$. *On* $D(A_c)$ *we have*

$$C_{c|D(A_c)} = \sqrt{2}C_d(I + A_d)^{-1}.$$

4. $D_c := D_d - \lim_{\substack{\lambda\to 1 \\ \lambda>1}} C_d(\lambda I + A_d)^{-1}B_d$.

Moreover, let the admissible discrete-time system (A_d, B_d, C_d, D_d) *be a realization of the transfer function*

$$G_d : \mathbb{D}_e \rightarrow L(U, Y) ,$$

i.e., $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$ for $z \in \mathbb{D}_e$. *Then*

$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d))$$

is an admissible continuous-time realization of the transfer function

$$G_c(s) := G_d\left(\frac{1+s}{1-s}\right) : RHP \rightarrow L(U, Y). \qquad \square$$

The inverse map is considered in the next theorem [23].

THEOREM 3.2. *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$; *then* $T^{-1}((A_c, B_c, C_c, D_c)) :=(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, *where the operators* A_d, B_d, C_d, *and* D_d *are defined as follows:*

1. $A_d := (I+A_c)(I-A_c)^{-1}$, *and for* $x \in D(A_c)$ *we have* $A_dx = (I-A_c)^{-1}(I+A_c)x$.
2. $B_d := \sqrt{2}(I - A_c)^{-1}B_c$.
3. $C_d := \sqrt{2}C_c(I - A_c)^{-1}$.
4. $D_d := C_c(I - A_c)^{-1}B_c + D_c$.

Moreover, let the admissible continuous time system (A_c, B_c, C_c, D_c) *be a realization of the transfer function*

$$G_c : RHP \rightarrow L(U, Y) ,$$

i.e., $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$ *for* $s \in RHP$. *Then*

$$(A_d, B_d, C_d, D_d) = T^{-1}((A_c, B_c, C_c, D_c))$$

is an admissible discrete-time realization of the transfer function

$$G_d(z) := G_c\left(\frac{z-1}{z+1}\right) : \mathbb{D}_e \rightarrow L(U, Y). \qquad \square$$

We recall that two discrete-time systems $(A_{di}, B_{di}, C_{di}, D_{di}) \in D_{X_i}^{U,Y}$ $(i = 1, 2)$ are equivalent (unitarily equivalent) if there is a bounded operator (a unitary operator) V from X_1 onto X_2 such that

$$(A_{d1}, B_{d1}, C_{d1}, D_{d1}) = (VA_{d2}V^{-1}, VB_{d2}, C_{d2}V^{-1}, D_{d2}).$$

In [23] it was shown that T preserves (unitary) equivalence of systems and respects duality of systems.

Note that in the previous two theorems the state spaces for the continuous- and discrete-time realizations are the same. As will be seen in later sections for continuous-time systems it is more natural to work on a different yet unitarily equivalent state space that is a subspace of $H_Y^2(RHP)$. Here we point out the equivalence of the Hilbert spaces $H_Y^2(\mathbb{D})$ and $H_Y^2(RHP)$, where Y is a separable Hilbert space (see [25, Thm. 4.6]).

PROPOSITION 3.3. *The spaces $H_Y^2(\mathbb{D})$ and $H_Y^2(RHP)$ are unitarily equivalent by the map*

$$\begin{aligned} V_Y : \quad H_Y^2(\mathbb{D}) &\to H_Y^2(RHP), \\ f_d &\mapsto (V_Y f_d)(\bullet) := f_c(\bullet) := \frac{1}{\sqrt{\pi}(1+\bullet)} f_d\left(\frac{1-\bullet}{1+\bullet}\right). \end{aligned}$$

The inverse of V is given by

$$\begin{aligned} V_Y^{-1} : \quad H_Y^2(RHP) &\to H_Y^2(\mathbb{D}) \\ f_c &\mapsto (V_Y^{-1} f_c)(\bullet) := f_d(\bullet) := \frac{2\sqrt{\pi}}{(1+\bullet)} f_c\left(\frac{1-\bullet}{1+\bullet}\right). \qquad \square \end{aligned}$$

The next result shows that observability and reachability properties as well as the Gramians are preserved under T. This implies that the transformation preserves parbalancing of systems. This result is the translation of a result in [23] to the frequency domain.

THEOREM 3.4. *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ and $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ be such that*

$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d)).$$

Then

1. *(A_c, B_c, C_c, D_c) is observable (reachable) if and only if (A_d, B_d, C_d, D_d) is observable (reachable). In fact, if $\mathcal{O}_c$ ($\mathcal{R}_c$) and $\mathcal{O}_d$ ($\mathcal{R}_d$) are the observability (reachability) operators of (A_c, B_c, C_c, D_c) and (A_d, B_d, C_d, D_d), respectively, and if either (A_c, B_c, C_c, D_c) or (A_d, B_d, C_d, D_d) has a bounded observability (reachability) operator, then the following relations hold:*

$$V_1\mathcal{O}_d x = \mathcal{L}\mathcal{O}_c x, \quad x \in X \quad (\mathcal{R}_d V_2^{-1} u = \mathcal{R}_c \mathcal{L}^{-1} u, \quad u \in H_U^2(RHP))$$

where $V_1 : H_Y^2(\mathbb{D}) \to H_Y^2(RHP)$ and $V_2 : H_U^2(\mathbb{D}) \to H_U^2(RHP)$ are unitary transformations as defined in Proposition 3.3:

$$V_i f_i = \frac{1}{\sqrt{\pi}(1+s)} f_i\left(\frac{1-s}{1+s}\right), \quad f_1 \in H_Y^2(\mathbb{D}), \text{ and } f_2 \in H_U^2(\mathbb{D}), \quad i = 1, 2,$$

and $\mathcal{L}$ is the Laplace transform.

2. *If the reachability Gramians $\mathcal{W}_c$ and $\mathcal{W}_d$ (observability Gramians $\mathcal{M}_c$ and $\mathcal{M}_d$) of (A_d, B_d, C_d, D_d) and (A_c, B_c, C_c, D_c) are defined, then*

$$\mathcal{W}_c = \mathcal{W}_d \qquad (\mathcal{M}_c = \mathcal{M}_d).$$

Proof. In [23] a "time domain" version of this result was proven. The present result follows from the result in [23] by applying the z-transform (respectively, Laplace transform) and using the unitary transformation of Proposition 3.3. □

Therefore if $G_d(z) = G_c(\frac{z-1}{z+1})$, then $G_d \in TLD^{U,Y}$, i.e., G_d has a reachable and observable discrete-time admissible realization, if and only if $G_c \in TLC^{U,Y}$, i.e., if and only if G_c has a reachable and observable continuous-time admissible realization.

The combination of Theorems 3.1, 3.2, and 3.4 gives us an effective machinery to transform discrete-time results to the continuous-time case. Before doing this, we need to study Hankel operators which will be important in the analysis of parbalanced, input-normal, or output-normal realizations treated in the sequel.

4. Linear systems and Hankel operators. In the study of discrete-time systems Hankel operators on $H^2(\mathbb{D})$ play an important role [11]. Given a discrete-time transfer function, a Hankel operator can be associated with it in a natural way. The so-called *restricted shift realization* of the transfer function is constructed by using the range of the Hankel operator as its state space (see [11], [30], [24], and §5 below). When the Hankel operator is compact, a balanced realization can be obtained whose Gramians have diagonal representations with diagonal entries equal to the singular values of the Hankel operator [30]. In the continuous-time situation Hankel operators on $H^2(RHP)$ will be of equal importance. We therefore examine the relationship between discrete-time Hankel operators and their continuous-time counterparts.

4.1. Hankel operators and realizability. Let G_d be analytic on $\mathbb{D}_e$ and at infinity so that $G_d^{\perp}(z) = z^{-1}[G_d(z^{-1}) - G_d(\infty)]$ is analytic on $\mathbb{D}$. We define the operator $H_{G_d^{\perp},\mathbb{D}}$: $D(H_{G_d^{\perp},\mathbb{D}}) \to H_Y^2(\mathbb{D})$ by

$$(H_{G_d^{\perp},\mathbb{D}} f)(z) = P_+ G_d^{\perp} J f \quad (f \in D(H_{G_d^{\perp}})),$$

where $D(H_{G_d^{\perp},\mathbb{D}}) = \{f \in H_U^2(\mathbb{D}) : f \text{ polynomial}, \ G_d^{\perp} J f$ has nontangential limit in $\mathbb{D}$ almost everywhere (a.e.) at $\partial\mathbb{D}$ with limit in $L_Y^2(\partial\mathbb{D})\}$ and $(Jf)(z) = f(1/z)$. The operator $H_{G_d^{\perp}}$ is called the Hankel operator with symbol $G_d^{\perp}$. If $D(H_{G_d^{\perp},\mathbb{D}})$ is dense in $H_U^2(\mathbb{D})$ and $H_{G_d^{\perp},\mathbb{D}}$ extends to a bounded operator on $H_U^2(\mathbb{D})$, this extension is also called the Hankel operator with symbol $G_d^{\perp}$ and is denoted by $H_{G_d^{\perp},\mathbb{D}}$.

The following lemma [24] relates the existence of a reachable and observable realization of a discrete-time transfer function G_d to the boundedness of the Hankel operator $H_{G_d^{\perp},\mathbb{D}}$.

LEMMA 4.1. *The transfer function G_d is in $TLD^{U,Y}$; i.e., G_d has an admissible reachable and observable realization if and only if* (i) *G_d is analytic on $\mathbb{D}_e$ and at infinity,* (ii) *the limit* $\lim_{\substack{r\in\mathbb{R} \\ r<-1,\ r\to-1}} G_d(r)$ *exists in the norm topology, and* (iii) *the Hankel operator $H_{G_d^{\perp},\mathbb{D}}$ is bounded.* □

We analogously define Hankel operators for continuous-time transfer functions.

DEFINITION 4.2. *If G_c is an $L(U,Y)$-valued function analytic on RHP, then the operator*

$$\begin{aligned} H_{G_c,RHP} : \quad D(H_{G_c,RHP}) &\to H_Y^2(RHP), \\ f &\mapsto P_+ M_{G_c} R f, \end{aligned}$$

where

$$Rf(s) = f(-s),$$

M_{G_c} *is the multiplication operator by* G_c,

P_+ *is the projection on* $H_Y^2(RHP)$, *with* $D(H_{G_c,RHP}) = \{f \in H^2_U(RHP) : f$ *rational,* G_cRf *has a nontangential limit in* RHP *on a.e. on* $i\mathbb{R}$ *that is in* $L^2_Y(i\mathbb{R})\}$, *is called the* Hankel operator $H_{G_c,RHP}$ with symbol G_c. □

If $D(H_{G_c,RHP})$ is dense in $H^2_U(RHP)$ and $H_{G_c,RHP}$ extends to a bounded operator on $H^2_U(RHP)$, this extension is also called the Hankel operator with symbol G_c and is denoted by $H_{G_c,RHP}$.

If it is clear from the context that the Hankel operator is defined on RHP, we will drop the subscript RHP and write H_G instead of $H_{G,RHP}$.

It is important in our context that Hankel operators defined on the disk are unitarily equivalent to Hankel operators in the right half plane in the following way (see, e.g., [25, Thm. 4.6]).

PROPOSITION 4.3. *Let* V_U *and* V_Y *be the unitary operators defined in Proposition* 3.3.

1. *Let* $G_d \in TLD^{U,Y}$ *and* $G_c \in TLC^{U,Y}$. *If*

$$G_d(z) = G_c\left(\frac{z-1}{z+1}\right) \quad \textit{for } z \in \mathbb{D}_e,$$

or equivalently

$$G_c(s) = G_d\left(\frac{1+s}{1-s}\right) \quad \textit{for } s \in RHP,$$

then the Hankel operators $H_{G_d^\perp,\mathbb{D}}$ *and* $H_{G_c,RHP}$ *are unitarily equivalent, i.e.,*

$$H_{G_c,RHP} = V_Y H_{G_d^\perp,\mathbb{D}} V_U^{-1},$$

where $G_d^\perp(z) = z^{-1}[G_d(z^{-1}) - G_d(\infty)]$ $(z \in \mathbb{D})$.

2. *Let* $K_d \in H^\infty_{L(U,Y)}(\mathbb{D})$ *and* $K_c \in H^\infty_{L(U,Y)}(RHP)$ *be such that*

$$K_d(z) = K_c\left(\frac{1-z}{1+z}\right), \qquad z \in \mathbb{D},$$

or equivalently

$$K_c(s) = K_d\left(\frac{1-s}{1+s}\right), \qquad s \in RHP.$$

Then

$$V_Y(K_d H^2_U(\mathbb{D})) = K_c H^2_U(RHP), \;\; V_Y\left((K_d H^2_U(\mathbb{D}))^\perp\right) = (K_c H^2_U(RHP))^\perp$$

and

$$V_Y^{-1}(K_c H^2_U(RHP)) = K_d H^2_U(\mathbb{D}), \;\; V_Y^{-1}\left((K_c H^2_U(RHP))^\perp\right) = (K_d H^2_U(\mathbb{D}))^\perp.$$

Proof. The proposition follows from direct verification. □

Using this proposition we can give a characterization for a continuous-time transfer function G_c to be in $TLC^{U,Y}$, i.e., to have an observable, reachable, and admissible continuous-time realization.

COROLLARY 4.4. *The following two statements are equivalent.*

1. $G_c \in TLC^{U,Y}$, *that is,* G_c *has a reachable, observable, and admissible realization on some Hilbert space.*

2. $G_c(s)$ *is analytic on* RHP, *the limit* $\lim_{r\in\mathbb{R},r\to+\infty} G_c(r)$ *exists in the norm topology, and the Hankel operator* $H_{G_c} : H^2_U(RHP) \to H^2_Y(RHP)$ *is bounded.*

Proof. This follows from Theorem 3.4, Proposition 4.3, and Lemma 4.1. □

4.2. Range spaces of Hankel operators and factorizations of transfer functions. It is known that the orthogonal complement,

$$(\mathrm{range} H_{G_c})^{\perp} = H_Y^2(RHP) \ominus \overline{\mathrm{range}} H_{G_c},$$

of the range of the Hankel operator H_{G_c} is invariant under any multiplication operator with symbol in H_Y^{∞}. Hence by Beurling's theorem, the subspace $(\mathrm{range} H_{G_c})^{\perp}$ is either $\{0\}$ or $QH_Y^2(RHP)$, where $Q \in H_Y^{\infty}(RHP)$ is a rigid function. A rigid function is a function $Q \neq 0$ such that $Q(iy)$ is for a.e. $y \in \mathbb{R}$ a partial isometry with a fixed initial space (see, e.g., [11, p. 186], and [15]). In particular, inner functions are rigid functions.

Using the above-defined notions, we introduce the concept of cyclicity of continuous-time transfer functions, which relates Hankel operators with their symbols. The discrete-time case was studied in, e.g., Fuhrmann [11]. A general study of strictly noncyclic transfer functions can also be found in Dewilde [7].

DEFINITION 4.5. *Let $G_c \in H^{\infty}_{L(U,Y)}(RHP)$. Then G_c is called*

1. cyclic *if* $(\mathrm{range} H_{G_c,RHP})^{\perp} = \{0\}$;
2. noncyclic *if* $(\mathrm{range} H_{G_c,RHP})^{\perp} = QH_Y^2(RHP)$, *where* $Q \in H_Y^{\infty}(RHP)$ *is a rigid function*;
3. strictly noncyclic *if* $(\mathrm{range} H_{G_c,RHP})^{\perp} = QH_Y^2(RHP)$, *where* $Q \in H_Y^{\infty}(RHP)$ *is an inner function.* □

Evidently in the scalar case G_c is strictly noncyclic if and only if it is noncyclic.

In the sequel it will be seen that the cyclicity of the transfer functions has much to do with the stability and other properties of their realizations. Here we present more information on cyclicity of H^{∞} transfer functions.

DEFINITION 4.6. *Let G be in $H^{\infty}_{L(U,Y)}(RHP)$. Then the $L(U,Y)$-valued function $\hat{G}$ defined on LHP is called a* meromorphic pseudocontinuation of bounded type *of G if*

1. *$\hat{G}$ is of bounded type, i.e.,*

$$\hat{G} = \frac{F}{h},$$

where F is a $L(U,Y)$-valued function and h is a scalar-valued function and both functions are bounded and analytic in LHP.

2. *G and $\hat{G}$ have the same strong radial limits on $i\mathbb{R}$, i.e., for a.e. $y \in \mathbb{R}$*

$$\lim_{x<0, x\to 0} \hat{G}(x+iy) = \lim_{x>0, x\to 0} G(x+iy). \qquad \square$$

The following proposition summarizes the connection between discrete- and continuous-time transfer functions in terms of cyclicity, meromorphic pseudocontinuation of bounded type, and factorizations. We refer to [11] for a discussion of these concepts for discrete-time transfer functions, which are analogous to those that have been defined here for continuous-time transfer functions.

PROPOSITION 4.7. *Let $G_c \in TLC^{U,Y}$, $G_d \in TLD^{U,Y}$, and set $G_d^{\perp}(z) = z^{-1}[G_d(z^{-1}) - G_d(\infty)]$. Assume that*

$$G_d(z) = G_c\left(\frac{z-1}{z+1}\right) \quad (z \in \mathbb{D}_e),$$

or equivalently

$$G_c(s) = G_d\left(\frac{1+s}{1-s}\right) \quad (s \in RHP).$$

Then

1. *$G_d^\perp$ is strictly noncyclic (cyclic, noncyclic) if and only if G_c is strictly noncyclic (cyclic, noncyclic).*

2. *Let $Q_{d,1} \in H^\infty_{L(Y)}(\mathbb{D})$, $Q_{d,2} \in H^\infty_{L(U)}(\mathbb{D})$, $Q_{c,1} \in H^\infty_{L(Y)}(RHP)$, and $Q_{c,2} \in H^\infty_{L(U)}(RHP)$ be inner functions. Let $F_{d,1} \in H^\infty_{L(Y,U)}(\mathbb{D})$, $F_{d,2} \in H^\infty_{L(U,Y)}(\mathbb{D})$, $F_{c,1} \in H^\infty_{L(Y,Y)}(RHP)$, and $F_{c,2} \in H^\infty_{L(U,Y)}(RHP)$. Assume*

$$F_{d,1}(z) = F_{c,1}\left(\frac{1-z}{1+z}\right) - G_c(1)^* Q_{c,1}\left(\frac{1-z}{1+z}\right), \quad z \in \mathbb{D},$$

$$F_{d,2}(z) = F_{c,2}\left(\frac{1-z}{1+z}\right) - Q_{c,2}\left(\frac{1-z}{1+z}\right) G_c(1)^*, \quad z \in \mathbb{D},$$

$$Q_{d,i}(z) = Q_{c,i}\left(\frac{1-z}{1+z}\right), \quad z \in \mathbb{D}, \quad i = 1, 2,$$

or equivalently

$$F_{c,1}(s) = F_{d,1}\left(\frac{1-s}{1+s}\right) + G_d(\infty)^* Q_{d,1}\left(\frac{1-s}{1+s}\right), \quad s \in RHP,$$

$$F_{c,2}(s) = F_{d,2}\left(\frac{1-s}{1+s}\right) + Q_{d,2}\left(\frac{1-s}{1+s}\right) G_d(\infty)^*, \quad s \in RHP,$$

$$Q_{c,i}(s) = Q_{d,i}\left(\frac{1-s}{1+s}\right), \quad s \in RHP, \quad i = 1, 2.$$

Then G_c can be factored on $i\mathbb{R}$ as

$$G_c = Q_{c,1} F_{c,1}^* = F_{c,2}^* Q_{c,2}$$

if and only if $G_d^\perp$ can be factored on $\partial\mathbb{D}$ as

$$G_d^\perp(z) = Q_{d,1}(z)(zF_{d,1}(z))^* = (zF_{d,2}(z))^* Q_{d,2}(z) \quad (z \in \partial\mathbb{D}).$$

3. *Assume that $G_c \in H^\infty_{L(U,Y)}(RHP)$ and $G_d^\perp \in H^\infty_{L(U,Y)}(\mathbb{D})$. Let F_d (F_c) be a $L(U,Y)$-valued analytic function in $\mathbb{D}_e$ (LHP) and h_d (h_c) be a scalar-valued analytic function in $\mathbb{D}_e$ (LHP), both bounded, such that*

$$F_d(z) = \frac{1}{z}\left[F_c\left(\frac{1-z}{1+z}\right) - G_c(1) h_c\left(\frac{1-z}{1+z}\right)\right], \quad z \in \mathbb{D}_e,$$

$$h_d(z) = h_c\left(\frac{1-z}{1+z}\right), \quad z \in \mathbb{D}_e,$$

or equivalently

$$F_c(s) = \frac{1-s}{1+s} F_d\left(\frac{1-s}{1+s}\right) + G_d(\infty) h_d\left(\frac{1-s}{1+s}\right), \quad s \in LHP,$$

$$h_c(s) = h_d\left(\frac{1-s}{1+s}\right), \quad s \in LHP.$$

Then $G_d^\perp$ has a meromorphic pseudocontinuation $\hat{G}_d^\perp$ of bounded type in $\mathbb{D}_e$, which is given by

$$\hat{G_d^\perp} = \frac{F_d}{h_d}$$

if and only if G_c has a meromorphic pseudocontinuation $\hat{G}_c$ of bounded type in LHP, which is given by

$$\hat{G}_c = \frac{F_c}{h_c}.$$

Proof. The results can be directly verified. □

The next theorem provides some convenient ways to determine whether a transfer function is cyclic, noncyclic, or strictly noncyclic. Note that $Q \in H^\infty_{L(Z,Y)}(RHP)$ and $F \in H^\infty_{L(U,Y)}(RHP)$ are said to be *weakly left coprime* if $QH^2_Z(RHP) \vee FH^2_U(RHP) = H^2_Y(RHP)$, where $\vee$ denotes the closed linear span. In this case we write $(Q, F)_L = I_Y$. If two functions $Q_1 \in H^\infty_{L(U,Y)}(RHP)$ and $F_1 \in H^\infty_{L(U,Z)}(RHP)$ are such that $\tilde{Q}_1$ and $\tilde{F}_1$ are weakly left coprime, where $\tilde{Q}_1(s) = (Q_1(\bar{s}))^*$ and $\tilde{F}_1(s) = (F_1(\bar{s}))^*$ $(s \in RHP)$, they are said to be weakly right coprime, and we denote this by $(Q_1, F_1)_R = I_U$ (see Fuhrmann [11]).

THEOREM 4.8. *Let $G_c \in H^\infty_{L(U,Y)}(RHP)$ with finite-dimensional U and Y. Then the following statements are equivalent:*

1. *G_c is strictly noncyclic.*
2. *G_c has a meromorphic pseudocontinuation of bounded type on LHP.*
3. *On $i\mathbb{R}$ the function G_c can be factored as*

$$G_c = Q_1 F_1^* = F_2^* Q_2,$$

where Q_1 and Q_2 are inner functions in $H^\infty_{L(Y)}(RHP)$ and $H^\infty_{L(U)}(RHP)$, respectively. The functions F_1 and F_2 are in $H^\infty_{L(Y,U)}(RHP)$ and $H^\infty_{L(U,Y)}(RHP)$, respectively, and the coprimeness conditions

$$(Q_1, F_1)_R = I_Y, \qquad (Q_2, F_2)_L = I_U$$

hold. If part 3 holds, then $Q_1 H^2_U(RHP) = (\mathrm{range} H_{G_c})^\perp$ and $\tilde{Q}_2 H^2_U(RHP) = (\mathrm{range} H_{\tilde{G}_c})^\perp$, where $\tilde{Q}_2(s) = (Q_2(\bar{s}))^*$ and $\tilde{G}_c(s) = (G_c(\bar{s}))^*$.

Proof. Analogous results are shown in [11] for discrete-time transfer functions. Thus the theorem follows from Proposition 4.7. □

The factorization in the theorem is Fuhrmann's generalization of the Douglas, Shapiro, and Shields factorization [8] to matrix-valued functions. For a given function, part 2 of the theorem may be easy to check. For example, the function $e^{-\alpha s}R(s)$ is strictly noncyclic, where $\alpha > 0$ and $R(s)$ is any rational function in $H^\infty_{L(U,Y)}(RHP)$. This is because $e^{-\alpha s}R(s)$ has a meromorphic pseudocontinuation of bounded type on LHP of the form $F(s)/e^{\alpha s}h(s)$, where if $a_1, \ldots, a_n$ denote the poles of $R(s)$, then,

$$h(s) = \frac{(s-a_1)\cdots(s-a_n)}{(s+a_1)\cdots(s+a_n)},$$

and $F(s) = h(s)R(s)$. Part 2 of the theorem also gives the following corollary.

COROLLARY 4.9. *Under the assumption of the theorem, $G \in H^\infty_{L(U,Y)}(RHP)$ is strictly noncyclic if and only if $\tilde{G} \in H^\infty_{L(Y,U)}(RHP)$ is strictly noncyclic.* □

4.3. Hankel operators with closed range. Similarly to Theorem 4.8, the following theorem (see [11]) gives necessary and sufficient conditions for the Hankel operator to have closed range.

THEOREM 4.10. *Let $G_c \in H_{L(U,Y)^\infty}(RHP)$ with U and Y finite dimensional. Then the Hankel operator H_{G_c} has closed range if and only if on $i\mathbb{R}$ the function G_c has the factorization*

$$G_c(s) = Q(s)F(s)^*,$$

where $Q \in H^\infty_{L(Y)}(RHP)$, $F \in H^\infty_{L(Y,U)}(RHP)$, *and the equality*

$$WQ + VF = I_Y$$

holds for some $W \in H^\infty_{L(Y)}(RHP)$ *and* $V \in H^\infty_{L(U,Y)}(RHP)$; *that is,* Q *and* F *are* strongly right coprime. *In this case*

$$H_{G_c}\left(H^2_{L(U)}(RHP)\right) = H^2_{L(Y)}(RHP) \ominus QH^2_{L(Y)}(RHP). \qquad \square$$

This section essentially established that the unitary equivalence of the spaces $H^2(\mathbb{D})$ and $H^2(RHP)$ implies the unitary equivalence of the Hankel operators $H_{G_d^\perp}$ and H_{G_c}, where $G_d(z) = G_c(\frac{z-1}{z+1})$, $z \in \mathbb{D}$. Therefore the spaces $\overline{\text{range}} H_{G_d^\perp}$ and $\overline{\text{range}} H_{G_c}$ are unitarily equivalent. As a consequence, the discrete-time transfer function G_d and the continuous-time transfer function G_c have the same cyclicity properties.

These results will be repeatedly used in the next section when we obtain the restricted shift realization of a continuous-time system by applying the bilinear map in §3 to the corresponding discrete-time system.

5. Continuous-time shift realizations via a bilinear transformation. As a direct application of the bilinear transformation T given in §3 the continuous-time restricted and *-restricted shift realizations can be obtained from the corresponding discrete realizations. These realizations can be further analyzed via the connection between continuous and discrete-time transfer functions shown in §§3 and 4.

Restricted and *-restricted shift realizations are central to the development here since they serve as prototypes of output-normal (respectively, input-normal) realizations. It will be shown in Proposition 6.2 that each output-(input-)normal realization of an admissible transfer function G is unitarily equivalent to the restricted (*-restricted) shift realization. The concrete representations of the continuous-time shift realizations obtained in this section will allow us to analyze input- and output-normal realizations in some detail in later sections.

Another important result of this section is Proposition 5.11, in which the state spaces of the restricted shift realizations for strictly noncyclic transfer functions are characterized through the inner factors in the Douglas–Shapiro–Shields factorizations of the transfer function.

5.1. Discrete-time shift realizations. We first recall the discrete-time restricted and *-restricted shift realizations of a discrete-time transfer function (see [11], [30], and [24]).

THEOREM 5.1. *Let* $G_d \in TLD^{U,Y}$. *Then* G_d *has two state-space realizations:* (A_d, B_d, C_d, D_d) *with state space* X_d *and* $(A_{d,*}, B_{d,*}, C_{d,*}, D_{d,*})$ *with state space* $X_{d,*}$, *i.e., for* $z \in \mathbb{D}_e$

$$G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d = C_{d,*}(zI - A_{d,*})^{-1}B_{d,*} + D_{d,*}.$$

They are given in the following way:

1. *The state space* X_d *is given by* $X_d = \overline{\text{range}} H_{G^\perp} \subseteq H^2_Y(\mathbb{D})$, *where*

$$G_d^\perp(z) = \frac{1}{z}\left[G_d\left(\frac{1}{z}\right) - G_d(\infty)\right]$$

and $H_{G_d^\perp}$ *is the Hankel operator with symbol* $G_d^\perp$. *The operators* A_d, B_d, C_d, *and* D_d *are given as follows:*

$$(A_d f)(z) := (S^* f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in X, \quad z \in \mathbb{D},$$

$$(B_d u)(z) := G_d^{\perp}(z)u, \quad u \in U, \quad z \in \mathbb{D},$$

$$C_d f := f(0), \quad f \in X,$$

$$D_d u := G_d(+\infty)u, \quad u \in U,$$

where S is the (forward) shift operator $(Sf)(z) = zf(z)$, $f \in H_Y^2(\mathbb{D})$, $z \in \mathbb{D}$.

The realization (A_d, B_d, C_d, D_d) is called the restricted shift realization *of the transfer function G_d. It is admissible, observable, and reachable, and the observability and reachability operators $\mathcal{R}_d$ and $\mathcal{O}_d$ are, respectively,*

$$\mathcal{O}_d = I_{X_d}, \quad \mathcal{R} = H_{G_d^{\perp}}.$$

2. *The realization $(A_{d,*}, B_{d,*}, C_{d,*}, D_{d,*})$ is given as follows: The state space $X_{d,*}$ is given by $X_{d,*} = \overline{\text{range}} H_{\tilde{G}_d^{\perp}}$ with*

$$\tilde{G_d}(z) = (G_d(\bar{z}))^* \ \text{and} \ \tilde{G}_d^{\perp}(z) = \frac{1}{z}\left(\tilde{G_d}\left(\frac{1}{z}\right) - \tilde{G_d}(\infty)\right), \qquad z \in \mathbb{D}.$$

The operators $A_{d,*}$, $B_{d,*}$, $C_{d,*}$, and $C_{d,*}$ are defined as

$$A_{d,*} = P_{X_{d,*}} S_{|X_{d,*}},$$

$$B_{d,*} : U \to X_{d,*}; u \mapsto P_{X_{d,*}} u,$$

$$C_{d,*} : X_{d,*} \to Y; x \mapsto \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (z\tilde{G}_d^{\perp}(z))^* x(z)dz = P_Y H_{G_d^{\perp}} x = (H_{G_d^{\perp}} x)(0),$$

$$D_{d,*} = G_d(+\infty),$$

where Y is considered a subspace embedded in $H_Y^2(\mathbb{D})$: $Y = \{y_0 + 0z + 0z^2 + \cdots \mid y_0 \in Y\} \subseteq H_Y^2(\mathbb{D})$, and $P_{X_{d,*}}$ and P_Y are orthogonal projections from $H_Y^2(\mathbb{D})$ onto $X_{d,*}$ and Y, respectively.

The realization $(A_{d,*}, B_{d,*}, C_{d,*}, D_{d,*})$ is called the *-restricted shift realization *of the transfer function G_d. It is admissible, observable, and reachable, and the observability and reachability operators $\mathcal{O}_{d,*}$ and $\mathcal{R}_{d,*}$ are, respectively,*

$$\mathcal{R}_{d,*} = P_{X_{d,*}} : H_U^2(\mathbb{D}) \to X_* \ \text{and} \ \mathcal{O}_{d,*} = H^*_{\tilde{G}_d^{\perp}}|_{X_{d,*}} = H_{G_d^{\perp}}|_{X_{d,*}}. \qquad \square$$

5.2. Continuous-time restricted shift realization. Now we apply T to the realizations given in the theorem to get the continuous-time realizations. We need some simple lemmas.

LEMMA 5.2. *For any $x \in H_Y^2(\mathbb{D})$, $\lim_{r\in\mathbb{R},\, r>-1,\, r\to-1}(1+r)x(r) = 0$ in the norm of Y. For any $f \in H_Y^2(RHP)$, $\lim_{r\in\mathbb{R},\, r\to+\infty} f(r) = 0$ in the norm of Y.*

Proof. For $x \in H_Y^2(\mathbb{D})$ and $z \in \mathbb{D}$ we have $x(z) = \sum_{n\geq 0} z^n \hat{x}_n$, where $\hat{x}_n \in Y$ and $\sum_{n\geq 0} \|\hat{x}_n\|^2 = \|x\|^2_{H_Y^2(\mathbb{D})}$. Thus

$$\|x(z)\|_Y \leq \sum_{n\geq 0} |z^n|\, \|\hat{x}_n\| \leq \left(\sum_{n\geq 0} |z^n|^2\right)^{1/2} \left(\sum_{n\geq 0} \|\hat{x}_n\|^2\right)^{1/2}$$

$$= (1-|z|)^{-1/2}(1+|z|)^{-1/2}\|x\|.$$

Hence $\lim_{r>-1,\, r\to-1}(1+r)x(r) = 0$.

Now for $f \in H^2_Y(RHP)$ and any $s \in RHP$, it is shown in [22, p. 254] that

$$\|f(s)\| \le \delta(Re(s))^{-1/2},$$

where δ is a constant depending on f. Thus the lemma is proven. □

LEMMA 5.3. *With the notation of Theorem 5.1 we have*

1. $$\text{range}(I + A_d) = \left\{x \mid x(z) = \frac{(1+z)h(z) - h(0)}{z}, \quad (z \in \mathbb{D}), \quad h \in X_d\right\},$$

and for $x \in \text{range}(I + A_d)$ *the limit* $\lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} x(r)$ *exists;*

2. $$[(\lambda I + A_d)^{-1}x](z) = \frac{z}{1+\lambda z}x(z) + \frac{1}{\lambda(1+\lambda z)}x\left(-\frac{1}{\lambda}\right),$$

where $x \in X_d$, $\lambda \in \mathbb{D}_e$, $z \in \mathbb{D}$;

3. $$C_d(\lambda I + A_d)^{-1}x = \frac{1}{\lambda}x\left(-\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{D}_e, \quad x \in X_d;$$

4. $$[(I + A_d)^{-1}x](z) = \frac{z}{1+z}\left[x(z) + \frac{1}{z}x(-1)\right], \quad x \in \text{range}(I + A_d),$$

where $x(-1) = \lim_{r\in\mathbb{R},\, r>-1, r\to-1} x(r)$.

Proof. 1. Since $\text{range}(I + A_d) = \{x + A_d x \mid x \in X_d\} = \{\frac{x(z)-x(0)}{z} + x(z) \mid x \in X_d\}$, we have the equality in 1. If $x \in \text{range}(I + A_d)$, then $x(z) = \frac{(1+z)h(z)-h(0)}{z}$ for some $h \in X_d$. By Lemma 5.2,

$$\lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} x(r) = \lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} \frac{(1+r)h(r) - h(0)}{r} = h(0).$$

2. First we show that for $x \in X_d$ and $\lambda \in \mathbb{D}_e$, the element

$$\frac{z}{1+\lambda z}x(z) + \frac{1}{\lambda(1+\lambda z)}x\left(-\frac{1}{\lambda}\right) = P_+\left(\frac{z}{1+\lambda z}x(z)\right)$$

is in X_d. Take any y in the invariant space $H^2_Y(\mathbb{D}) \ominus X_d$. Since $\frac{1}{z+\bar\lambda} \in H^\infty$ for $\lambda \in \mathbb{D}_e$, we have $\frac{1}{z+\bar\lambda}y \in H^2_Y(\mathbb{D}) \ominus X_d$. Therefore,

$$\left\langle P_+\left(\frac{z}{1+\lambda z}x(z)\right),\ y\right\rangle_{H^2_Y(\mathbb{D})} = \left\langle x,\ \frac{1}{z+\bar\lambda}y\right\rangle = 0.$$

This shows that $P_+\left(\frac{z}{1+\lambda z}x(z)\right) \in X_d$. Since A_d is a contraction, $(\lambda I + A_d)^{-1}$ is a bounded operator on X_d for $\lambda \in \mathbb{D}_e$. Then the equality in 2. follows from the equality

$$(\lambda I + A_d)\left[\frac{z}{1+\lambda z}x(z) + \frac{1}{\lambda(1+\lambda z)}x\left(-\frac{1}{\lambda}\right)\right] = x(z).$$

3. Using 2. and the definition of C_d we get 3.

4. If $x \in \text{range}(I + A_d)$, then by 1. and its proof there exists $h \in X_d$ such that $x(z) = \frac{(1+z)h(z)-h(0)}{z}$ and $\lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} x(r) = h(0)$. Set

$$x(-1) = \lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} x(r) = h(0).$$

We have

$$\frac{z}{1+z}\left[x(z) + \frac{1}{z}x(-1)\right] = h(z),$$

which is in X_d. Note that $-1 \notin \sigma_p(A_d)$. Thus $(I + A_d)^{-1}$ is defined on range$(I + A_d)$. The equality in 4. then follows from

$$(I + A_d)\left(\frac{z}{1+z}\left[x(z) + \frac{x(-1)}{z}\right]\right) = [(I + A_d)h](z) = \frac{(1+z)h(z) - h(0)}{z} = x(z). \qquad \square$$

LEMMA 5.4. *Let $f \in L^2_Y(i\mathbb{R})$. Then in $L^2_Y(i\mathbb{R})$ norm,*

$$\lim_{n\to\infty} \frac{s}{n+s} f = \lim_{n\to\infty} \frac{s}{n-s} f = 0.$$

Proof. Since $\|\frac{s}{n+s} f(s)\|^2_Y \leq \|f(s)\|^2_Y$ for any $s \in i\mathbb{R}$ and $n > 0$ and

$$\lim_{n\to\infty} \left\| \frac{s}{n+s} f(s) \right\|^2_Y = 0$$

for a.e. $s \in i\mathbb{R}$, the lemma follows from the Lebesgue dominated convergence theorem. $\square$

LEMMA 5.5. *Let $G_c \in TLC^{U,Y}$. Set $X = \overline{\text{range}} H_{G_c}$ and $\mathcal{D} = P_X\{\frac{u}{1+s} : u \in U\}$. Then the map*

$$\begin{aligned} M_1 : \mathcal{D} &\rightarrow Y, \\ P_X \tfrac{u}{1+s} &\mapsto [\tilde{G}_c(1) - \tilde{G}(\infty)]u \end{aligned}$$

is well defined and the map

$$\begin{aligned} M_2 : X &\rightarrow X, \\ f &\mapsto P_X \tfrac{f}{1+s} \end{aligned}$$

is injective.

Proof. Assume $P_X \frac{u_1}{1+s} = P_X \frac{u_2}{1+s}$. Then $P_X \frac{u_1 - u_2}{1+s} = 0$. This shows that $\frac{u_1-u_2}{1+s} \in H^2_Y(RHP) \ominus X$. Therefore, for any $f \in H^2_Y(RHP)$,

$$\begin{aligned} 0 = \left\langle \frac{u_1 - u_2}{1+s}, H_{G_c} f \right\rangle_{H^2_U(RHP)} &= \left\langle \frac{u_1 - u_2}{1+s}, P_+ G_c f(-s) \right\rangle_{H^2_U(RHP)} \\ &= \left\langle \frac{u_1 - u_2}{1+s}, P_+[G_c - G_c(+\infty)] f(-s) \right\rangle \\ &= \left\langle \frac{u_1 - u_2}{1+s}, [G_c(s) - G_c(+\infty)] f(-s) \right\rangle \\ &= \left\langle [G_c(s) - G_c(+\infty)]^* \frac{u_1 - u_2}{1+s}, f(-s) \right\rangle \\ &= \left\langle [\tilde{G}_c(s) - \tilde{G}_c(\infty)] \frac{u_1 - u_2}{1-s}, f(s) \right\rangle. \end{aligned}$$

Hence $[\tilde{G}(s) - \tilde{G}(1)]\frac{u_1-u_2}{1-s} = 0$. So we have $[\tilde{G}(s) - \tilde{G}(1)](u_1 - u_2) = 0$. Taking the limit on the real line, we get

$$[\tilde{G}_c(1) - \tilde{G}_c(+\infty)](u_1 - u_2) = 0.$$

This shows that indeed M_1 is well defined.

To show that M_2 is injective, assume $P_X \frac{h_1(s)}{1+s} = P_X \frac{h_2(s)}{1+s}$, $h_1, h_2 \in X$. Then $P_X \frac{h_1(s)-h_2(s)}{1+s} = 0$. Hence

$$\frac{h_1(s) - h_2(s)}{1+s} \in H_Y^2 \ominus X.$$

By Lemma 5.4, we have

$$\lim_{n\to\infty} \left\| \frac{1+s}{1+s/n} \frac{h_1(s) - h_2(s)}{1+s} - (h_1 - h_2) \right\|_{H_Y^2(RHP)} = \lim_{n\to\infty} \left\| \frac{-s}{n+s}(h_1 - h_2) \right\| = 0.$$

Hence $\lim_{n\to+\infty} \frac{1+s}{1+s/n} \frac{h_1(s)-h_2(s)}{1+s} = h_1 - h_2$ in H_Y^2. Note that $H_Y^2 \ominus X$ is an invariant space and $\frac{1+s}{1+s/n} \in H^\infty$ for $n > 0$. So $\frac{1+s}{1+s/n} \frac{h_1(s)-h_2(s)}{1+s} \in H_Y^2 \ominus X$, and hence $h_1 - h_2 \in H_Y^2 \ominus X$. Since $h_1 - h_2 \in X$, we therefore have $h_1 - h_2 = 0$. This shows that M_2 is injective. □

We will need the following result on the reproducing kernel in $H^2(RHP)$ (see, e.g., [10]).

LEMMA 5.6. *For $f \in H_U^2(RHP)$, $u \in U$, and $\alpha \in RHP$ the following hold:*

$$\left\langle f, \frac{u}{s+\bar{\alpha}} \right\rangle_{H_U^2(RHP)} = 2\pi \langle f(\alpha), u \rangle_U,$$

$$\left\langle \frac{u}{s+\bar{\alpha}}, f \right\rangle_{H_U^2(RHP)} = 2\pi \langle u, f(\alpha) \rangle_U.$$

We are ready to present the continuous-time restricted shift realization using the bilinear transform T. For a continuous-time transfer function G_c we first realize the discrete-time transfer function G_d defined by

$$G_d(z) = G_c\left(\frac{z-1}{z+1}\right)$$

in terms of the restricted shift realization. Applying T to this discrete-time realization we obtain a realization of G_c with the same state space. Then we use a unitary transformation to get the continuous-time restricted shift realization with state space $\overline{\text{range}} H_{G_c}$.

THEOREM 5.7. *Let $G_c \in TLC^{U,Y}$. Then G_c has a state-space realization $(A_c, B_c, D_c, C_c) \in C_X^{U,Y}$, which is given in the following way:*

1. *The state space is given by*

$$X = \overline{\text{range}} H_{G_c,RHP} \subseteq H_Y^2(RHP).$$

2. *The semigroup $(e^{tA_c})_{t\geq 0}$ corresponding to the realization is given by*

$$\begin{aligned} e^{tA_c} :\quad & X \to \quad X, \\ & f \mapsto \quad (e^{tA_c} f)(s) = P_+ e^{ts} f(s). \end{aligned}$$

The infinitesimal generator $(A_c, D(A_c))$ of the semigroup $(e^{tA_c})_{t\geq 0}$ is given by

$$\begin{aligned} A_c :\quad & D(A_c) \to \quad X, \\ & f \mapsto \quad (A_c f)(s) = s f(s) - \lim_{\substack{r \in \mathbb{R} \\ r\to\infty}} r f(r). \end{aligned}$$

The domain $D(A_c)$ is dense in X, and we have

$$D(A_c) = \left\{ f \mid f(s) = \frac{1}{1-s}[h(s) - h(1)] :\ (s \in RHP),\ h \in X \right\}.$$

The domain of the adjoint A_c^* of the operator A_c is

$$D(A_c^*) = \left\{ f \mid f(s) = P_X \frac{h(s)}{1+s} : (s \in RHP),\ h \in X \right\}$$

and

$$A_c^* f = f - h \text{ for } f(s) = P_X \frac{h(s)}{1+s} \in D(A_c^*).$$

On $\mathcal{L}^{-1}(X) \subseteq L_U^2([0,+\infty))$ the semigroup is given by

$$\begin{aligned} e^{tA_c} :\ & \mathcal{L}^{-1}(X) \to \mathcal{L}^{-1}(X), \\ & f \mapsto (e^{tA_c} f)(\tau) = P_{L_U^2([0,+\infty))}(f(\tau + t))_{\tau \geq 0}. \end{aligned}$$

3. *The input operator is given by*

$$\begin{aligned} B_c :\ & U \to D(A_c^*)^{(\prime)}, \\ & u \mapsto B_c(u), \end{aligned}$$

where for $u \in U$ and $x(s) = P_X \frac{h(s)}{1+s} \in D(A_c^*)$,

$$\begin{aligned} [B_c(u)](x) &= \frac{1}{\sqrt{2\pi}} \left\langle \frac{1}{1-s}[G_c(s) - G_c(1)]u,\ (1 - A_c^*)x \right\rangle \\ &= \frac{1}{\sqrt{2\pi}} \left\langle H_{G_c} \frac{u}{1+s},\ (1 - A_c^*)x \right\rangle \\ &= \frac{1}{\sqrt{2\pi}} \left\langle H_{G_c} \frac{u}{1+s},\ h \right\rangle \\ &= \sqrt{2\pi} \left\langle u, (H_{\tilde{G}_c} h)(1) \right\rangle_U . \end{aligned}$$

4. *The output operator is given by*

$$\begin{aligned} C_c : D(C) = D(A_c) + (I - A_c)^{-1} B_c U \quad & \to \quad Y, \\ x \mapsto \quad & \sqrt{2\pi} \lim_{\substack{r \in \mathbb{R} \\ r \to \infty}} r x(r). \end{aligned}$$

5. *The feedthrough operator is given by*

$$\begin{aligned} D_c :\ & U \to Y, \\ & u \mapsto G_c(+\infty)u := \lim_{\substack{r \in \mathbb{R} \\ r \to +\infty}} G_c(r)u. \end{aligned}$$

The realization (A_c, B_c, C_c, D_c) of G_c is called the restricted shift realization.

Proof. These results are obtained by applying the map T of Theorem 3.1 to the restricted shift realization (A_d, B_d, C_d, D_d) of $G_d(z) = G_c(\frac{z-1}{z+1})$, $(z \in \mathbb{D}_e)$, with the state space then transformed by the unitary operator $V = V_Y$ defined in Proposition 3.3.

1. Let $(A_{c1}, B_{c1}, C_{c1}, D_{c1}) = T((A_d, B_d, C_d, D_d))$ and

$$(A_c, B_c, C_c, D_c) = (V A_{c1} V^{-1}, V B_{c1}, C V^{-1}, D_{c1}).$$

We use the following notation: $G_d^{\perp}(z) = \frac{1}{z}[G_d(\frac{1}{z}) - G_d(\infty)] = \frac{1}{z}[G_c(\frac{1-z}{1+z}) - G_c(1)]$ $(z \in \mathbb{D})$, $X_d = \overline{\text{range}} H_{G_d^{\perp}}$, and $\phi_t(z) = e^{t\frac{z-1}{z+1}}$, $t \geq 0$. Then by Proposition 4.3 $X = V X_d$, and $\phi_t(A_d)$ is the semigroup of contractions on X_d with infinitesimal generator $(A_d - I)(A_d + I)^{-1} = A_{c1}$ (see [28, p. 141]). Specifically, for $x \in X_d$ we have

$$\phi_t(A_d)x = P_+\phi_t\left(\frac{1}{z}\right)x = P_+e^{t\frac{\frac{1}{z}-1}{\frac{1}{z}+1}}x = P_+e^{t\frac{1-z}{1+z}}x.$$

Then it is easy to see that $A_c = VA_{c1}V^{-1}$ generates the semigroup of contractions $V\phi_t(A_d)V^{-1}$ on X. If we extend the unitary transformation $V : H^2(\mathbb{D}) \to H^2(RHP)$ naturally to

$$\begin{aligned} V : L^2(\partial\mathbb{D}) &\to L^2(i\mathbb{R}), \\ x_d &\mapsto (Vx_d)(\bullet) = \frac{1}{\sqrt{\pi}(1+\bullet)}x_d\left(\frac{1-\bullet}{1+\bullet}\right), \end{aligned}$$

we still have a unitary transformation. Moreover, by considering $z^n y$ for $n \in \mathbb{Z}$ and $y \in Y$ we can show that

$$VP_+^{\mathbb{D}} = P_+^{RHP}V,$$

where $P_+^{\mathbb{D}} : L^2(\partial\mathbb{D}) \to H^2(\mathbb{D})$ and $P_+^{RHP} : L^2(i\mathbb{R}) \to H^2(RHP)$ are the orthogonal projections. From this it follows that for $f \in X$,

$$e^{tA_c}f = V\phi_t(A_d)V^{-1}f = VP_+^{\mathbb{D}}e^{t\frac{1-z}{1+z}}V^{-1}f = P_+^{RHP}Ve^{t\frac{1-z}{1+z}}V^{-1}f = P_+e^{ts}f.$$

Clearly, $D(A_{c1}) = \text{range}(A_d+I)$, and by Lemma 5.3 $\text{range}(A_d+I) = \{\frac{(1+z)x(z)-x(0)}{z} \mid x \in X_d\}$. Since $D(A_c) = VD(A_{c1})$ and $x(0) = 2\sqrt{\pi}(Vx)(1)$ for $x \in X_d$, we have

$$D(A_c) = V\text{range}(A_d + I) = V\left\{\frac{(1+z)x(z) - x(0)}{z} \mid x \in X_d\right\}$$

$$= \left\{V\left(\frac{(1+z)x(z) - x(0)}{z}\right) \mid x \in X_d\right\} = \left\{\frac{(1+\frac{1-s}{1+s})f(s) - \frac{2f(1)}{1+s}}{\frac{1-s}{1+s}} \mid f \in X\right\}$$

$$= \left\{\frac{f(s) - f(1)}{1-s} \mid f \in X\right\}.$$

For $x \in D(A_{c1}) = \text{range}(A_d + I)$ the limit $\lim_{r\in\mathbb{R},\, r>-1,\, r\to-1} x(r)$ exists by Lemma 5.3. Denoting it by $x(-1)$ and using Lemma 5.3, part 4, we have

$$(A_{c1}x)(z) = [(A_d - I)(A_d + I)^{-1}x](z)$$

$$= (A_d - I)\left(\frac{z}{1+z}\left[x(z) + \frac{1}{z}x(-1)\right]\right) = \frac{(1-z)x(z) - 2x(-1)}{1+z}.$$

From this we obtain, for $f \in D(A_c) = VD(A_{c1})$,

$$\begin{aligned} (A_cf)(s) &= (VA_{c1}V^{-1}f)(s) = V\left(\frac{(1-z)(V^{-1}f)(z) - 2(V^{-1}f)(-1)}{1+z}\right) \\ &= sf(s) - \lim_{r\in\mathbb{R},\, r\to+\infty} rf(r), \end{aligned}$$

where we have used the fact that for $f \in D(A_c)$

$$(V^{-1}f)(-1) = \sqrt{\pi}\lim_{\substack{r\in\mathbb{R}\\ r\to+\infty}}(1+r)f(r) = \sqrt{\pi}\lim_{\substack{r\in\mathbb{R}\\ r\to+\infty}} rf(r).$$

Now we show the form of A_c^*. Recall that A_c^* is the generator of the strongly continuous semigroup $(e^{tA_c})^*$. Let

$$D(\hat{A}) = \left\{ f \mid f(s) = P_X \frac{h(s)}{1+s} \text{ for some } h \in X \right\}$$

and

$$\begin{aligned} \hat{A} : D(\hat{A}) &\rightarrow X, \\ \hat{A} f &= f - h \quad \left(f = P_X \frac{h(s)}{1+s} \in D(\hat{A}) \right). \end{aligned}$$

By Lemma 5.5, the operator $\hat{A}$ is well defined.

For $f \in D(\hat{A})$ and $g \in D(A_c)$ there are v and w in X such that $f = P_X \frac{v}{1+s}$ and $g = \frac{w - w(1)}{1-s}$. By the definition of A_c and $\hat{A}$, we have $A_c g = w$ and $\hat{A} f = v$. It then follows that

$$\langle A_c g,\ f \rangle = \left\langle w,\ P_X \frac{v}{1+s} \right\rangle = \left\langle \frac{w}{1-s},\ v \right\rangle = \left\langle \frac{w - w(1)}{1-s}, v \right\rangle = \langle g,\ \hat{A} f \rangle.$$

This shows that $D(\hat{A}) \subseteq D(A_c^*)$ and $\hat{A} = A_c^*|_{D(\hat{A})}$. On the other hand, we clearly have $(I - \hat{A}) D(\hat{A}) = X$ and hence

$$(I - A_c^*) D(\hat{A}) = X.$$

Let $x \in D(A_c^*)$. Then there exists $x_1 \in D(\hat{A})$ such that

$$(I - A_c^*) x_1 = (I - A_c^*) x.$$

Since A_c^* is the infinitesimal generator of a semigroup of contractions, the number 1 is not in the spectrum of A_c^*. Thus we must have $x_1 = x$. This shows that

$$D(A_c^*) \subseteq D(\hat{A}).$$

Therefore $D(A_c^*) = D(\hat{A})$ and hence $A_c^* = \hat{A}$.

2. For the operator B_c we first compute B_{c1}, following the definition of T:

$$\begin{aligned} B_{c1} := \sqrt{2}(I + A_d)^{-1} B_d : U &\rightarrow D(A_c^*)^{(\prime)}, \\ u &\longmapsto \sqrt{2}(I + A_d)^{-1} B_d(u)[\cdot], \\ &:= \sqrt{2} < B_d(u), (I + A_d^*)^{-1}(\cdot) >_{X_d} . \end{aligned}$$

Note that $V^* = V^{-1}$, $(I + A_d^*)^{-1} = \frac{1}{2}(I - A_{c1}^*)$, and

$$(V B_d u)(s) = \frac{1}{\sqrt{\pi}} \frac{G_c(s) - G_c(1)}{1-s} u \quad (s \in RHP).$$

Thus for $x = P_X \frac{h(s)}{1+s} \in D(A_c^*) \subseteq X$, we have $(I - A_c^*) x = h$ and

$$(B_c u)(x) = (V B_{c1})(x) = (B_{c1} u)(V^* x) = \sqrt{2} \langle B_d u,\ (I + A_d^*)^{-1} V^{-1} x \rangle_{X_d}$$

$$= \sqrt{2} \langle V B_d u,\ V(I + A_d^*)^{-1} V^{-1} x \rangle_X = \sqrt{2} \left\langle V B_d u,\ \frac{1}{2} V(I - A_{c1}^*) V^{-1} x \right\rangle_X$$

$$= \frac{1}{\sqrt{2}} \langle V B_d u,\ (I - A_c^*) x \rangle_X$$

$$= \frac{1}{\sqrt{2\pi}} \left\langle \frac{G_c(s) - G_c(1)}{1-s} u,\ (I - A_c^*)x \right\rangle_X$$

$$= \frac{1}{\sqrt{2\pi}} \left\langle \frac{G_c(s) - G_c(1)}{1-s} u,\ h \right\rangle_X$$

$$= \frac{1}{\sqrt{2\pi}} \left\langle H_{G_c} \frac{u}{1+s},\ h \right\rangle_X .$$

Since $(H_{G_c})^* = H_{\tilde{G}_c}$, we have

$$(B_c u)(x) = \frac{1}{\sqrt{2\pi}} \left\langle \frac{u}{1+s},\ H_{\tilde{G}_c} h \right\rangle.$$

By Lemma 5.6 the right-hand side is $\sqrt{2\pi}\langle u,\ (H_{\tilde{G}_c} h)(1)\rangle_U$.

3. To compute C_c we use Lemma 5.3, part 3, to get

$$C_d(\lambda I + A_d)^{-1} x = \frac{1}{\lambda} x\left(-\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{D}_e.$$

So for $x \in D(C_{c1}) = D(A_{c1}) + (I - A_{c1})^{-1} B_{c1} U$ we have

$$C_{c1} x = \sqrt{2} \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1} x$$

$$= \sqrt{2} \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} \frac{1}{\lambda} x(-\frac{1}{\lambda}) = \sqrt{2} \lim_{\substack{\lambda > -1 \\ \lambda \to -1}} x(\lambda).$$

The existence of the limit for $x \in D(A_{c1})$ follows from Lemma 5.3, part 1, because $D(A_{c1}) = \text{range}(A_d + I)$. For $x \in (I - A_{c1})^{-1} B_{c1} U$ we have that the limit

$$\sqrt{2} \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1} x$$

also exists by the admissibility of G_d, since $(I - A_{c1}) B_{c1} u = \frac{1}{\sqrt{2}} B_d u = \frac{1}{\sqrt{2}} G_d^{\perp} u$ (see [23]). Now it can be verified that $V D(C_{c1}) = D(A_c) + (I - A_c)^{-1} B_c U$, i.e., $V D(C_{c1}) = D(C_c)$. Hence we get, for $f \in D(C_c)$,

$$C_c f = C_{c1} V^{-1} f = \sqrt{2} \lim_{\substack{\lambda > -1 \\ \lambda \to -1}} \frac{2\sqrt{\pi}}{1+\lambda} f\left(\frac{1-\lambda}{1+\lambda}\right)$$

$$= \sqrt{2\pi} \lim_{\substack{r \in \mathbf{R} \\ r \to +\infty}} (1+r) f(r) = \sqrt{2\pi} \lim_{\substack{r \in \mathbf{R} \\ r \to +\infty}} r f(r).$$

Finally, the obvious expression $D_c = G_c(\infty)$ can also be verified as follows:

$$D_c u = D_{c1} u = D_d u - \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1} B_d u$$

$$= G_d(\infty) u - \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1} G_d^{\perp} u = G_c(1) u - \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} \frac{1}{\lambda} G_d^{\perp} \left(-\frac{1}{\lambda}\right) u$$

$$= G_c(1) u - \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} (-1) \left[G_c\left(\frac{\lambda+1}{\lambda-1}\right) - G_c(1) \right] u = \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} G_c\left(\frac{\lambda+1}{\lambda-1}\right)$$

$$= G_c(+\infty). \qquad \square$$

Regarding the expressions for the operator B_c in the theorem we have the following corollary.

COROLLARY 5.8. *$B_c u \in X$, $(u \in U)$ if and only if $[G_c - G_c(+\infty)]u \in X$ $(u \in U)$. In this case*

$$(B_c u)(s) = \frac{1}{\sqrt{2\pi}}[G_c(s) - G_c(+\infty)]u \quad (u \in U).$$

In particular, if G_c satisfies

$$\sup_{x>0} \int_{-\infty}^{+\infty} \|G_c(x+iy) - G_c(+\infty)\|^2 dy < \infty,$$

where for $s \in RHP$ the expression $\|G_c(s) - G_c(+\infty)\|$ denotes the operator norm of the operator $G_c(s) - G_c(+\infty) \in L(U, Y)$, then $B_c u \in X$ and

$$(B_c u)(s) = \frac{1}{\sqrt{2\pi}}[G_c(s) - G_c(+\infty)]u$$

for any $u \in U$.

Proof. First we assume that $[G_c - G_c(+\infty)]u \in X$ $(u \in U)$. Define $F(s) = G_c(s) - G_c(+\infty)$. Then $Fu \in X$. It follows from the formula for $D(A_c)$ that

$$\frac{1}{1-s}[G_c(s) - G_c(1)]u = \frac{1}{1-s}[F(s)u - F(1)u] \in D(A_c)$$

and hence for $x \in D(A_c^*)$,

$$[B_c(u)](x) = \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1-s}[G_c(s) - G_c(1)]u,\ (I - A_c^*)x \right\rangle_{H_Y^2(RHP)}$$

$$= \frac{1}{\sqrt{2\pi}}\left\langle (I - A_c)\frac{1}{1-s}[G_c(s) - G_c(1)]u,\ x \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\left\langle \left(\frac{1}{1-s} - \frac{s}{1-s}\right)[G_c(s) - G_c(1)]u + \lim_{\substack{r\in\mathbb{R} \\ r\to+\infty}} \frac{r}{1-r}[G_c(r) - G_c(1)]u,\ x \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\langle [G_c(s) - G_c(+\infty)]u,\ x\rangle = \frac{1}{\sqrt{2\pi}}\langle Fu,\ x\rangle.$$

Here we have used the definition of A_c and the fact that the limit $\lim_{\substack{r\in\mathbb{R} \\ r\to+\infty}} G_c(r)u$ exists, which follows from the admissibility of G_c. Thus we have shown that $B_c(u) \in X$ and $B_c(u) = \frac{1}{\sqrt{2\pi}}Fu$ for any $u \in U$.

On the other hand, if $B_c(u) \in X$, $(u \in U)$, then there is $f_u \in X$ such that $[B_c(u)](x) = \langle f_u, x\rangle$ for any $x \in X$. Therefore

$$\frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1-s}[G_c(s) - G_c(1)]u,\ (I - A_c^*)x \right\rangle = \langle f_u, x\rangle \ (x \in X).$$

This shows that $\frac{1}{1-s}[G_c(s) - G_c(1)]u \in D((I - A_c^*)^*) = D(I - A_c) = D(A_c)$. So there is $h \in X$ such that

$$\frac{1}{1-s}[G_c(s) - G_c(1)]u = \frac{h(s) - h(1)}{1-s}.$$

Hence $G_c(s) - G_c(1) = h(s) - h(1)$. Since $\lim_{\substack{s\in\mathbb{R}\\ s\to\infty}} h(s) = 0$ (see Lemma 5.2), we have $G_c - G_c(+\infty) = h \in X$.

To complete the proof of the corollary, it suffices to show that the condition that G_c is analytic for $Re(s) > 0$ and satisfies

$$\sup_{x>0} \int_{-\infty}^{+\infty} \|G_c(x+iy) - G_c(+\infty)\|^2 dy < \infty$$

implies that $[G_c - G_c(+\infty)]u \in X$ for any $u \in U$.

Again let $F(s) = G_c(s) - G_c(+\infty)$. We have the equality of Hankel operators:

$$H_{G_c} = H_F.$$

The assumption on G_c implies that $Fu \in L^2_Y(i\mathbb{R})$ for any $u \in U$. Now we show that in $L^2_Y(i\mathbb{R})$ norm

$$Fu = \lim_{n\to\infty} H_F \frac{n}{n+s} u$$

and hence $Fu \in X = \overline{\text{range}} H_F$. The proof will then be complete.

Consider

$$\left\| Fu - H_F \frac{n}{n+s} u \right\|_{L^2_Y(i\mathbb{R})} = \left\| P_+ F \frac{-s}{n-s} u \right\| = \left\| P_+ \frac{-s}{n-s} Fu \right\|.$$

By Lemma 5.4, we have $\lim_{n\to\infty} \|\frac{-s}{n-s} Fu\|_{L^2_Y(i\mathbb{R})} = 0$. Therefore

$$\lim_{n\to\infty} \|P_+ \frac{-s}{n-s} Fu\| = 0.$$

So we indeed have $Fu = \lim_{n\to\infty} H_F \frac{n}{n+s} u$, converging in $L^2_Y(i\mathbb{R})$ norm. □

5.3. Continuous-time $*$-restricted shift realization. If we apply the map T in Theorem 3.1 to the $*$-restricted shift realization of $G_d(z) = G_c(\frac{z-1}{z+1})$ and then transform the state space by the unitary operator of Proposition 3.3, we obtain the $*$-restricted shift realization of G_c. Alternatively, we can find the restricted shift realization of the transfer function $\tilde{G}_c \in TLC^{Y,U}$ first, and then the dual system of this restricted shift realization will be the $*$-restricted shift realization of G_c.

THEOREM 5.9. *Let $G_c \in TLC^{U,Y}$. Then G_c has a state-space realization $(A_{c,*}, B_{c,*}, C_{c,*}, D_{C,*}) \in C_X^{U,Y}$, which is given in the following way:*

1. *The state space is given by*

$$X_* = \overline{\text{range}} H_{\tilde{G}_c, RHP} \subseteq H^2_U(RHP),$$

where $\tilde{G}_c(s) = (G(\bar{s}))^*$ for $s \in RHP$.

2. *The semigroup $(e^{tA_{c,*}})_{t\geq 0}$ corresponding to the realization is given by*

$$\begin{aligned} e^{tA_{c,*}}: \quad X_* &\to \quad X_*, \\ f &\mapsto \quad (e^{tA_{c,*}} f)(s) = P_{X_*} e^{-ts} f(s), \end{aligned}$$

where the operator $A_{c,*}$ has domain

$$D(A_{c,*}) = P_{X_*} \left\{ \frac{1}{1+s} h(s) : h \in X_* \right\}$$

and for $f(s) = P_{X_*}\frac{1}{1+s}h(s) \in D(A_{c,*})$,

$$A_{c,*}f = f - h.$$

On $\mathcal{L}^{-1}(X_*) \subseteq L^2_U([0,\infty))$ *the semigroup is given by*

$$\begin{aligned} e^{tA_{c,*}}: \quad \mathcal{L}^{-1}(X_*) &\to \quad \mathcal{L}^{-1}(X_*), \\ f &\mapsto \quad (e^{tA_{c,*}}f)(s) = P_{\mathcal{L}^{-1}(X)_*}f(s-t). \end{aligned}$$

3. *The input operator* $B_{c,*}: U \to D(A^*_{c,*})^{(')}$ *is given by*

$$u \mapsto B_c(u)$$

with

$$\begin{aligned} \left[B_{c,*}(u)\right](x) &= \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1+s}u,\ (1-A^*_{c,*})x\right\rangle \\ &= \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1+s}u,\ h\right\rangle \\ &= \sqrt{2\pi}\langle u, h(1)\rangle_U,\quad x = \frac{h(s)-h(1)}{1-s} \in D(A^*_{c,*}),\ h \in X^*. \end{aligned}$$

4. *The output operator has the following form:*

$$\begin{aligned} D(C_{c,*}) &= D(A_{*,c}) + (I - A_{*,c})^{-1}B_{c,*}U \\ &= P_{X_*}\left\{\frac{h(s)}{1+s} \mid h \in X_*\right\} + P_{X_*}\left\{\frac{u}{1+s} \mid u \in U\right\}. \end{aligned}$$

If $x = P_{X_*}\frac{h(s)}{1+s}$, *then*

$$C_{c,*}x = \sqrt{2\pi}(H_{G_c}h)(1),$$

and if $x = P_{X_*}\frac{u}{1+s}$, *then*

$$C_{c,*}x = \sqrt{2\pi}[G_c(1) - G(+\infty)]u.$$

5. *The feedthrough operator is given by*

$$\begin{aligned} D_{c,*}: \quad U &\to \quad Y, \\ u &\mapsto \quad G_c(+\infty)u := \lim_{\substack{r\in\mathbb{R} \\ r\to\infty}} G_c(r)u. \end{aligned}$$

The realization $(A_{c,*}, B_{c,*}, C_{c,*}, D_{c,*})$ *of* G_c *is called the* *- restricted shift realization.

Proof. Let (A, B, C, D) be the restricted shift realization of the transfer function $\tilde{G}_c(s) = (G(\bar{s}))^*$. Take $(A_{c,*}, B_{c,*}, C_{c,*}, D_{c,*})$ to be the dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of (A, B, C, D). Then $(A_{c,*}, B_{c,*}, C_{c,*}, D_{c,*})$ is a realization of G (see Definition 2.2). We show that $(A_{c,*}, B_{c,*}, C_{c,*}, D_{c,*})$ obtained this way has the expressions as given in the theorem. Notice that $\tilde{A} = A^*$, i.e., $A_{c,*} = A^*$.

1. By Theorem 5.1 the state space of the realization (A, B, C, D) is $\overline{\text{range}}H_{\tilde{G}_c}$. Thus the dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ has the same state space. That is,

$$X_* = \overline{\text{range}}H_{\tilde{G}_c}.$$

2. The semigroup generated by A is defined as

$$e^{tA} f = P_+ e^{ts} f \quad (f \in X_*).$$

It is easy to verify

$$e^{tA^*} f = (e^{tA})^* f = P_{X_*} e^{-ts} f \quad (f \in X_*).$$

That is,

$$e^{tA_{c,*}} f = P_{X_*} e^{-ts} f \quad (f \in X_*).$$

By Theorem 5.1,

$$D(A_{c,*}) = D(A^*) = \left\{ P_{X_*} \frac{1}{1+s} h(s) \mid h \in X_* \right\},$$

$$A_{c,*} f = f - h \text{ for } f(s) = P_{X_*} \frac{1}{1+s} h(s) \in D(A_{c,*}),$$

and $A_{c,*}$ is well defined.

3. By the definition of the dual system (Definition 2.2), we have

$$\tilde{B} : U \to D(A)^{(\prime)}; \qquad u \longmapsto \tilde{B}(u)[\cdot] := \langle u, \ C(\cdot) \rangle.$$

For $x(s) = \frac{1}{1-s}[h(s) - h(1)] \in D(A)$ $(h \in X_*)$, we have

$$\tilde{B}(u)[x] = \langle u, \ Cx \rangle = \left\langle u, \sqrt{2\pi} \lim_{\substack{r \in \mathbf{R} \\ r \to \infty}} r x(r) \right\rangle = \sqrt{2\pi} \langle u, h(1) \rangle.$$

By Lemma 5.6, $\sqrt{2\pi} \langle u, h(1) \rangle_U = \frac{1}{\sqrt{2\pi}} \langle \frac{1}{1+s} u, \ h \rangle_{H^2_U(RHP)}$.

4. Now we compute $C_{c,*} = \tilde{C}$. Again use Definition 2.2:

$$D(\tilde{C}) = D(\tilde{A}) + (I - \tilde{A})^{-1} \tilde{B} U = D(A_{c,*}) + (I - A_{c,*})^{-1} B_{c,*} U,$$

and $\tilde{C} x_0$ is defined by

$$\begin{cases} \langle y, \tilde{C} x_0 \rangle = B(y)[x_0], & x_0 \in D(A_{c,*}) \\ \langle \tilde{C} x_0, y \rangle = \langle u_0, C(I - A)^{-1} B y \rangle, & x_0 = (I - A_{c,*})^{-1} B_{c,*} u_0, \ u_0 \in U, \ y \in Y. \end{cases}$$

Since by Theorem 5.7

$$B(y)[x] = \sqrt{2\pi} \langle y, \ (H_{G_c} h)(1) \rangle_Y \quad \left(x = P_X \frac{h(s)}{1+s} \in D(A_{c,*}) \right),$$

we have

$$\tilde{C} x = \sqrt{2\pi} (H_{G_c} h)(1) \ \text{ for } \ x = P_X \frac{h(s)}{1+s} \in D(A_{c,*}).$$

From Lemma 5.5 it follows that $\tilde{C}$ is well defined for $x \in D(A_{c,*})$.

Note that $C(I - A)^{-1} B y = [\tilde{G}_c(1) - \tilde{G}_c(+\infty)] y$. Thus

$$\tilde{C} x_0 = [G_c(1) - G_c(+\infty)] u_0 \ \text{ for } \ x_0 = (I - A_{c,*})^{-1} B_{c,*} u_0.$$

Now we show that $(I - A_{c,*})^{-1}B_{c,*}u_0 = \frac{1}{\sqrt{2\pi}}P_{X_*}\frac{u_0}{1+s}$. Let $x \in D(A^*_{c,*})$. Since by Theorem 5.7 $(I - A^*_{c,*})^{-1}x = \frac{x - x(1)}{1-s}$, we have

$$\begin{aligned}
\langle (I - A_{c,*})^{-1}B_{c,*}u_0, x\rangle &= [(I - A_{c,*})^{-1}B_{c,*}u_0](x) \\
&= [B_{c,*}u_0]((I - A^*_{c,*})^{-1}x) \\
&= [B_{c,*}u_0]\left(\frac{x - x(1)}{1-s}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left\langle \frac{u_0}{1+s}, x\right\rangle = \frac{1}{\sqrt{2\pi}}\left\langle P_{X_*}\frac{u_0}{1+s}, x\right\rangle.
\end{aligned}$$

This shows that $(I - A_{c,*})^{-1}B_{c,*}u_0 = \frac{1}{\sqrt{2\pi}}P_{X_*}\frac{u_0}{1+s}$. Hence, to sum up, the operator $C_{c,*} = \tilde{C}$ is defined in the following way:

$$D(C_{c,*}) = P_{X_*}\left\{\frac{h(s)}{1+s} \mid h \in X_*\right\} + P_{X_*}\left\{\frac{u}{1+s} \mid u \in U\right\}.$$

If $x = P_{X_*}\frac{h(s)}{1+s}$, then

$$C_{c,*}x = \sqrt{2\pi}(H_{G_c}h)(1),$$

and if $x = P_{X_*}\frac{u}{1+s}$, then

$$C_{c,*}x = \sqrt{2\pi}[G_c(1) - G_c(+\infty)]u.$$

Note that by Lemma 5.5 $C_{c,*}x$ is also well defined for $x \in P_{X_*}\{\frac{u}{1+s} \mid u \in U\}$.

4. It is straightforward to get

$$D_{c,*} = \tilde{D}^* = ((G_c(+\infty))^*)^* = G_c(+\infty). \qquad \square$$

Note that the restricted and *-restricted shift realizations of admissible transfer functions in H^∞ are well posed in the sense of Curtain and Weiss [5] and Salamon [27]. Indeed we have the following corollary concerning the reachability and observability of the restricted and *-restricted shift realizations.

COROLLARY 5.10. 1. *The reachability operator of the restricted shift realization is given by*

$$\mathcal{R}_c : L^2_U[0, +\infty) \to X, \quad f \mapsto H_{G_c,RHP}\mathcal{L}f.$$

The observability operator of the restricted shift realization is given by

$$\mathcal{O}_c|_X : X \to L^2_Y[0, +\infty), \quad x \mapsto \mathcal{L}^{-1}x.$$

2. *The reachability operator of the *-restricted shift realization is given by*

$$\mathcal{R}_{c,*} : L^2_U[0, +\infty) \to X_*, \quad f \mapsto P_{X_*}\mathcal{L}f.$$

*The observability operator of the *-restricted shift realization is given by*

$$\mathcal{O}_{c,*} : X_* \to L^2_Y[0, +\infty), \quad x \mapsto \mathcal{L}^{-1}H_{G_c,RHP}x.$$

Here $\mathcal{L}$ denotes the Laplace transform.

Proof. This follows from Theorem 3.4 and Theorem 5.7 □

We categorize the state spaces of the restricted and *-restricted shift realizations here for later use.

PROPOSITION 5.11. *Let X and X_* be, respectively, the state spaces of restricted and *-restricted shift realizations of $G_c \in TLC^{U,Y}$. Then*

1. *if G_c is cyclic, then $X = H^2_Y(RHP)$ and $X_* = H^2_U(RHP)$;*
2. *if G_c is noncyclic, then $X = H^2_Y(RHP) \ominus Q_1 H^2_Y(RHP)$ and $X_* = H^2_U(RHP) \ominus Q_2 H^2_U(RHP)$, where $Q_1 \in H^\infty_{L(Y)}(RHP)$ and $Q_2 \in H^\infty_{L(U)}(RHP)$ are rigid functions;*
3. *if G_c is in $H^\infty_{L(U,Y)}(RHP)$, is strictly noncyclic, and has factorization $G_c = Q_1 F_1^* = \tilde{F}_2^* \tilde{Q}_2$, where $Q_1 \in H^\infty_{L(Y)}(RHP)$ and $Q_2 \in H^\infty_{L(U)}(RHP)$ are inner, Q_1 and F_1 are left coprime, and Q_2 and F_2 are also left coprime, then $X = H^2_Y(RHP) \ominus Q_1 H^2_Y(RHP)$ and $X_* = H^2_U(RHP) \ominus Q_2 H^2_U(RHP)$.*

Proof. This follows from Definition 4.5 and Theorems 4.8, 5.7, and 5.9.

6. Continuous-time input-normal, output-normal, parbalanced realizations and their asymptotic stability. Recall that a reachable and observable admissible system (A_c, B_c, C_c, D_c) is said to be input-normal if $\mathcal{W}_c = I$. It is output-normal if $\mathcal{M}_c = I$. The reachable and observable admissible systems are said to be parbalanced if

$$\mathcal{W}_c = \mathcal{M}_c.$$

Here $\mathcal{W}_c$ and $\mathcal{M}_c$ are, respectively, the reachability and observability Gramians of the system. Given a transfer function $G_c \in TLC^{U,Y}$, by Corollary 5.10 the restricted and *-restricted shift realizations are examples of, respectively, output-normal and input-normal realizations of G_c. Proposition 6.2 shows that up to unitary equivalence all observable input-normal and reachable output-normal realizations of an admissible transfer function G are up to unitary equivalence *-restricted and restricted shift realizations, respectively.

In this section we establish the existence of a parbalanced realization for any $G_c \in TLC^{U,Y}$ and study the stability properties of input-normal, output-normal, and parbalanced realizations.

A parbalanced realization of a continuous-time transfer function $G_c \in TLC^{U,Y}$ can be obtained from the map T in Theorem 3.1 applied to a discrete-time parbalanced realization of the corresponding discrete-time transfer function G_d. The existence of parbalanced realizations was shown by Young [30]. In [23] Young's results are cast into the continuous-time situation and the following theorem is proven.

THEOREM 6.1. 1. *For $G_c \in TLC^{U,Y}$, there exists a parbalanced realization $(A_c, B_c, C_c, D_c) \in C^{U,Y}_X$ of G_c. The state space of this realization is given by the closure of the range of the Hankel operator with symbol G_c, i.e., $X = \overline{\text{range}} H_{G_c}$. If $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is another parbalanced realization of G_c with state space $\tilde{X}$, then (A_c, B_c, C_c, D_c) and $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ are unitarily equivalent.*

2. *If in addition $G_c(s)$ is continuous on the extended $i\mathbb{R}$ (i.e. on $i\mathbb{R} \cup \{i\infty\}$) and is a compact operator for each $s \in i\mathbb{R}$, then there is a basis of $X = \overline{\text{range}} H_{G_c}$ on which the Gramians of the above realization have a diagonal matrix representation with its diagonal consisting of the Hankel singular values of G_c. We will call this realization a balanced realization of G_c.* □

6.1. Characterization of the realizations. Concerning the equivalence of different realizations, we have the following proposition.

PROPOSITION 6.2. 1. *Any two input-normal (output-normal) realizations of $G_c \in TLC^{U,Y}$ are unitarily equivalent. Hence every input-normal (output-normal) realization of G_c is unitarily equivalent to the *-restricted (restricted) shift realization of G_c.*

2. *An input-normal realization and an output-normal realization of* $G_c \in TLC^{U,Y}$ *are equivalent if and only if the Hankel operator* H_{G_c} *has closed range.*

3. *All reachable and observable admissible realizations of* G_c *are equivalent if and only if the Hankel operator* H_{G_c} *has closed range.*

Proof. Analogous results in the discrete-time case are shown in [24] (see Theorem 3.1, Corollary 3.1, and Proposition 4.1 therein). Applying Theorem 3.1, Theorem 3.4, and Proposition 4.3 to these results we have the proposition. □

A consequence of the proposition is that the study of input-normal (output-normal) realizations reduces to the study of the *-restricted (restricted) shift realizations. This point will be used repeatedly.

Part 2 of the proposition shows when the state-space isomorphism theorem holds. Note that the Hankel operator H_{G_c} to have closed range is a very strong condition. This condition can be stated in terms of the Douglas–Shapiro–Shields factorization of the transfer function G_c (see Theorem 4.10 and [11]).

6.2. Asymptotic stability. Now we turn to the study of stability properties of continuous-time systems and use the classes $C_{i,j}$ to describe different asymptotic stability properties of systems [28].

DEFINITION 6.3. *Let* $(e^{tA_c})_{t\geq 0}$ *be a semigroup of contractions on the Hilbert space* H. *Then*

1. $(e^{tA_c})_{t\geq 0} \in C_{0\cdot}$ *if* $\lim_{t\to\infty} e^{tA_c}h = 0$ *for all* $h \in H$,
2. $(e^{tA_c})_{t\geq 0} \in C_{\cdot 0}$ *if* $\lim_{t\to\infty} e^{tA_c^*}h = 0$ *for all* $h \in H$,
3. $(e^{tA_c})_{t\geq 0} \in C_{1\cdot}$ *if* $\lim_{t\to\infty} e^{tA_c}h \neq 0$ *for all* $h \in H$,
4. $(e^{tA_c})_{t\geq 0} \in C_{\cdot 1}$ *if* $\lim_{t\to\infty} e^{tA_c^*}h \neq 0$ *for all* $h \in H$.

We further set

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j}, \qquad i, j = 0, 1. \qquad \square$$

The notions of stability that we consider are the following.

DEFINITION 6.4. *A continuous-time system* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ *is*

1. asymptotically stable *if for all* $x \in X$,

$$e^{tA_c}x \to 0$$

as $t \to \infty$, *i.e.,* $(e^{tA_c})_{t\geq 0} \in C_{0\cdot}$;

2. exponentially stable *if*

$$\omega := \inf\{\alpha \in \mathbb{R} \mid \text{there exists } M_\alpha \geq 0 \text{ such that } \|e^{tA_c}\| \leq M_\alpha e^{\alpha t}\ (t \geq 0)\} < 0.$$

The number ω *is called the* growth bound *of the semigroup.* □

We comment that the asymptotic and exponential stability of a system is preserved by system equivalence. Moreover, if two systems are unitarily equivalent, they will have the same growth bound.

An important result in [28, Prop. 9.1, p. 148] implies that a continuous-time system is asymptotically stable if and only if the corresponding discrete-time system is asymptotically stable.

PROPOSITION 6.5. *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ *and* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ *such that*

$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d)).$$

Then for all $x \in X$,

$$\lim_{n\to\infty} \|A_d^n x\| = \lim_{t\to\infty} \|e^{tA_c}x\|$$

and

$$\lim_{n\to\infty} \|(A_d^*)^n x\| = \lim_{t\to\infty} \|e^{tA_c^*} x\|. \qquad \square$$

Therefore, the study of asymptotic stability of a continuous-time system reduces to the study of the asymptotic stability of the corresponding discrete-time system.

Now we state the main result of this section, which asserts that any admissible parbalanced realization of an admissible continuous-time transfer function is asymptotically stable.

THEOREM 6.6. *Let $G_c \in TC^{U,Y}$. Let (A_b, B_b, C_b, D_b), (A_i, B_i, C_i, D_i), and (A_o, B_o, C_o, D_o) be, respectively, a parbalanced, an input-normal, and an output-normal observable and reachable realization of G_c. Then*

1. (a) $(e^{tA_i})_{t\geq 0} \in C_{\cdot 0}$,
   (b) $(e^{tA_i})_{t\geq 0} \in C_{00}$ *if $G_c^\perp$ is strictly noncyclic,*
   (c) $(e^{tA_i})_{t\geq 0} \in C_{10}$ *if $G_c^\perp$ is cyclic,*
2. (a) $(e^{tA_o})_{t\geq 0} \in C_{0\cdot}$, *i.e., asymptotically stable,*
   (b) $(e^{tA_o})_{t\geq 0} \in C_{00}$ *if G_c is strictly noncyclic,*
   (c) $(e^{tA_o})_{t\geq 0} \in C_{01}$ *if G_c is cyclic,*
3. $(e^{tA_b})_{t\geq 0} \in C_{00}$.

Proof. The corresponding asymptotic stability results for discrete-time systems were obtained in Theorem 3.2 and Theorem 4.2 of [24]. Hence, combining those theorems with Proposition 6.5 and part 1 of Proposition 4.7, we have the theorem. $\square$

Since by Proposition 6.2 all reachable and observable realizations of G_c are equivalent when the Hankel operator H_{G_c} has closed range, and equivalent realizations have the same asymptotic stability properties, the theorem has the following corollary.

COROLLARY 6.7. *If the Hankel operator H_{G_c} has closed range, then all reachable, observable, and admissible realizations of G_c are asymptotically stable.* $\square$

7. Spectral minimality and exponential stability of input-normal, output-normal, and parbalanced realizations. This section aims to examine the exponential stability of continuous-time input-normal, output-normal, and parbalanced realizations of certain classes of transfer functions. The results are mainly based upon a detailed spectral analysis of input-normal and output-normal realizations. While the asymptotic stability properties of continuous-time systems can be obtained directly from the discrete-time case as we did in the previous section, exponential stability properties of continuous-time systems do not follow in the same way. However, we can relate the spectrum of the discrete-time system to that of the continuous-time system and thus establish the exponential stability results.

Recall that a continuous-time system (A_c, B_c, C_c, D_c) is exponentially stable if

$$\inf\{\alpha \in \mathbb{R} | \text{ there exists } M_\alpha \geq 0 \text{ such that } \|e^{tA_c}\| \leq M_\alpha e^{\alpha t} \text{ for } t \geq 0\} < 0.$$

The following proposition gives an interpretation of the growth bound of a semigroup in terms of the spectral radius of the semigroup (see, e.g., [21, p. 60]).

PROPOSITION 7.1. *Let ω be the growth bound of the semigroup $(e^{tA_c})_{t\geq 0}$ and $r(e^{tA_c})$ the spectral radius of e^{tA_c}; then*

$$r(e^{tA_c}) = e^{\omega t}$$

for $t \geq 0$. $\square$

Note that it follows from this proposition that equivalent systems have the same growth bound.

7.1. Spectral analysis. Thus we have to investigate the spectral properties of a continuous-time linear system (A_c, B_c, C_c, D_c) in order to study its exponential stability.

The way we do this is to relate the spectral properties of (A_c, B_c, C_c, D_c) to those of the corresponding discrete-time system (A_d, B_d, C_d, D_d). First we have the following relation between $\sigma(A_d)$ and $\sigma(A_c)$.

PROPOSITION 7.2. *Let A_c be the infinitesimal generator of a semigroup of contractions and A_d the co-generator such that $A_c = (A_d - I)(A_d + I)^{-1}$. Then*

$$\sigma_p(A_c) = \left\{\frac{z-1}{z+1} : \ z \in \sigma_p(A_d)\right\} \ \textit{and} \ \sigma_p(A_d) = \left\{\frac{1+s}{1-s} : \ s \in \sigma_p(A_c)\right\},$$

$$\sigma(A_c) = \left\{\frac{z-1}{z+1} : \ z \in \sigma(A_d), \ z \neq -1\right\} \ \textit{and} \ \sigma(A_d) \setminus \{-1\} = \left\{\frac{1+s}{1-s} : \ s \in \sigma(A_c)\right\}.$$

Proof. First note that $1 \notin \sigma(A_c)$ since e^{tA_c} is a semigroup of contractions and that by Theorem 3.1,

$$A_c x = (A_d - I)(A_d + I)^{-1} x = (A_d + I)^{-1}(A_d - I)x \ \text{for} \ x \in D(A_c),$$

where $D(A_c) = \text{range}(A_d + I)$. Hence the following relations hold:

$$\begin{aligned} (7.1) \qquad (sI - A_c)(A_d + I)x &= [sI - (A_d - I)(A_d + I)^{-1}](A_d + I)x \\ &= [s(A_d + I) - (A_d - I)]x \\ &= (1-s)\left(\frac{1+s}{1-s}I - A_d\right)x, \qquad x \in X_d, \ s \neq 1; \end{aligned}$$

$$\begin{aligned} (7.2) \qquad (A_d + I)(sI - A_c)x &= (A_d + I)[sI - (A_d - I)(A_d + I)^{-1}]x \\ &= (1-s)\left(\frac{1+s}{1-s}I - A_d\right)x, \qquad x \in D(A_c), \ s \neq 1. \end{aligned}$$

The equations (7.1) and (7.2) show that

$$\sigma_p(A_d) = \left\{\frac{1+s}{1-s} : \ s \in \sigma_p(A_c)\right\}.$$

Now if $\frac{1+s}{1-s} \notin \sigma(A_d)$, i.e., if $(\frac{1+s}{1-s}I - A_d)^{-1}$ exists and is bounded, then

$$(A_d + I)\left(\frac{1+s}{1-s}I - A_d\right)^{-1} = \left(\frac{1+s}{1-s}I - A_d\right)^{-1}(A_d + I).$$

Thus by (7.1) and (7.2)

$$\begin{aligned} (7.3) \qquad (sI - A_c)^{-1}x &= (1-s)^{-1}(A_d + I)\left(\frac{1+s}{1-s}I - A_d\right)^{-1} x \\ &= (1-s)^{-1}\left(\frac{1+s}{1-s}I - A_d\right)^{-1}(A_d + I)x, \ x \in D(A_c). \end{aligned}$$

So $(sI - A_c)^{-1}$ is bounded and densely defined, i.e., $s \notin \sigma(A_c)$. Hence

$$(7.4) \qquad \left\{\frac{1+s}{1-s} : \ s \in \sigma(A_c)\right\} \subseteq \sigma(A_d).$$

On the other hand, if $s \neq 1$ and $s \notin \sigma(A_c)$, then $(sI - A_c)^{-1}$ is a bounded operator. It is easy to verify in this case that

$$\left(\frac{1+s}{1-s}I - A_d\right)^{-1} x = \frac{(1-s)^2}{2}(sI - A_c)^{-1} + \frac{1-s}{2}x, \qquad x \in D(A_c).$$

In fact from (7.3) we have

$$\left(\frac{1+s}{1-s}I - A_d\right)\left[\frac{(1-s)^2}{2}(sI - A_c)^{-1} + \frac{1-s}{2}I\right]x$$

$$= \frac{(1-s)^2}{2}\left(\frac{1+s}{1-s}I - A_d\right)(sI - A_c)^{-1}x + \frac{1-s}{2}\left(\frac{1+s}{1-s}I - A_d\right)x$$

$$= \frac{(1-s)^2}{2}(1-s)^{-1}(A_d + I)x + \frac{1-s}{2}\left(\frac{1+s}{1-s}I - A_d\right)x$$

$$= x, \quad x \in D(A_c).$$

Similarly,

$$\left[\frac{(1-s)^2}{2}(sI - A_c)^{-1} + \frac{1-s}{2}x\right]\left(\frac{1+s}{1-s}I - A_d\right)x = x, \qquad x \in D(A_c).$$

Thus $\frac{1+s}{1-s} \notin \sigma(A_d)$. So we have

$$\left\{s : \frac{1+s}{1-s} \in \sigma(A_d)\right\} \subseteq \sigma(A_c). \tag{7.5}$$

Combining (7.4) and (7.5) we have that

$$\sigma(A_c) = \left\{s : \frac{1+s}{1-s} \in \sigma(A_d)\right\} = \left\{\frac{z-1}{z+1} : z \in \sigma(A_d), z \neq -1\right\},$$

which implies

$$\sigma(A_d) \setminus \{-1\} = \left\{\frac{1+s}{1-s} : s \in \sigma(A_c)\right\}. \qquad \square$$

In our application of the proposition, A_c is the state propagation operator of a continuous-time system $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ and A_d is the state propagation operator of the corresponding discrete-time system $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, which is related to (A_c, B_c, C_c, D_c) by

$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d)),$$

where T is the bilinear mapping in Theorem 3.1.

A powerful tool in spectral analysis is the spectral mapping theorem for C_0 operators (see, e.g., [22, p. 74]). A contraction $W \in L(M)$, where M is a separable Hilbert space, is called a C_0 operator, denoted $W \in C_0$, if there exists no subspace $V \in M$ such that $W|_V : V \to V$ is unitary and if there exists an inner function $m \in H^\infty(\mathbb{D})$ such that $m(W) = 0$. The least common divisor of all such inner functions is called the minimal function of W, denoted

m_W, which is still an inner function such that $m_W(W) = 0$. Note that if W is a C_0 operator, W is unitarily equivalent to a left shift S^* restricted to a left invariant space of the form $H_Y^2(\mathbb{D}) \ominus QH_Y^2(\mathbb{D})$, where Q is inner (see, [22, p. 72]). It can be seen that minimal functions are the generalizations of minimal polynomials of matrices. As in the matrix case, the spectrum of a C_0 operator is given by the "zeros" of its minimal function in the following sense (see [22, p. 72]).

LEMMA 7.3. *If $W \in C_0$, then $\sigma(W) = \sigma(m_W)$ and $\sigma_p(W) = \sigma(m_W) \cap \mathbb{D}$, where for an inner function $Q \in H^\infty_{L(Y)}(\mathbb{D})$, Y is a Hilbert space, and the spectrum of Q is defined as*

$$\sigma(Q) = \left\{ \lambda \in \overline{\mathbb{D}} \mid \lim_{\delta \to 0} \inf_{\substack{|\xi-\lambda|<\delta \\ \xi\in\mathbb{D}}} \inf_{\substack{\|y\|=1 \\ y\in Y}} \|Q(\xi)y\| = 0 \right\}. \qquad \square$$

Given a C_0 operator W and a function $\phi \in H^\infty(\mathbb{D})$, the operator

$$\phi(W) := \lim_{r<1,\ r\to 1} \phi(rW)$$

is well-defined. The following theorem relates the spectra of these two operators (see [22, p. 74]).

THEOREM 7.4 (the spectral mapping theorem). *Let $\phi \in H^\infty(\mathbb{D})$ and $W \in C_0$. Then*

$$\sigma(\phi(W)) \subseteq \left\{ \xi \in \mathbb{C} \mid \inf_{z\in\mathbb{D}} (|\phi(z) - \xi| + |m_W(z)|) = 0 \right\},$$

where m_W is the minimal function of W. $\square$

7.2. Spectral minimality. We are going to use these results to transpose the spectral properties of the discrete-time input- and output-normal realizations to those of the continuous-time case. First we recall the discrete-time results. Assume that the input and output spaces are of finite dimension. If the transfer function G_d is such that $G_d^\perp$ is strictly noncyclic, then G_d has a pseudomorphic continuation of bounded type to the unit disk $\mathbb{D}$ (see [11]). Take this continuation as the definition of G_d on $\mathbb{D}$, wherever defined. Consider the analytic continuation of the extended G_d. Let $\sigma_s(G_d)$ be the set of points at which G_d has no analytic continuation. We are interested in the relationship between $\sigma_s(G_d)$ and $\sigma(A_d)$. The following theorem shows $\sigma_s(G_d) = \sigma(A_d)$ for input-normal or output-normal realizations. If G_d is not strictly noncyclic, the spectrum of A_d can also be characterized (see [24] and [11]).

THEOREM 7.5. *Let $G_d \in TLD^{U,Y}$ with U and Y finite dimensional and let $(A_{d,o}, B_{d,o}, C_{d,o}, D_{d,o})$, $(A_{d,i}, B_{d,i}, C_{d,i}, D_{d,i})$, and $(A_{d,b}, B_{d,b}, C_{d,b}, D_{d,b})$ be, respectively, an output-normal, an input-normal, and a parbalanced realization of G_d.*

1. *If $G_d^\perp$ is in $H^\infty_{L(Y)}(\mathbb{D})$ and strictly noncyclic, then $(A_{d,o}, B_{d,o}, C_{d,o}, D_{d,o})$, $(A_{d,i}, B_{d,i}, C_{d,i}, D_{d,i})$, and $(A_{d,b}, B_{d,b}, C_{d,b}, D_{d,b})$ are* spectrally minimal, *i.e.,*

$$\sigma(A_{d,o}) = \sigma(A_{d,i}) = \sigma(A_{d,b}) = \sigma_s(G_d).$$

In this case $A_{d,o}$, $A_{d,i}$, and $A_{d,b}$ are all C_0 operators and have the same minimal function—say, m. Moreover, if $G_c(e^{it}) = Q_1(e^{it})(e^{it}F_1(e^{it}))^*$ is the Douglas–Shapiro–Shields factorization of G_d (see Theorem 4.8) and $\tilde{G}_d(e^{it}) = Q_2(e^{it})(e^{it}F_2(e^{it}))^*$ is the factorization of $\tilde{G}_d$, then the following equalities hold:

$$\sigma(m) = \sigma_s(G_d) = (\sigma(Q_1))^* = \sigma(Q_2),$$

$$\sigma_p(A_{d,o}) = \{\overline{\lambda} \in \mathbb{D} \mid \operatorname{Ker} Q_1(\lambda)^* \neq \{0\}\},$$

and

$$\sigma_p(A_{d,i}) = \{\lambda \in \mathbb{D} |\ \mathrm{Ker} Q_2(\lambda) \neq \{0\}\},$$

where $(\sigma(Q_1))^* = \{\bar{\lambda} |\ \lambda \in \sigma(Q_1)\}$ *and*

$$\sigma(Q_i) = \{\lambda \in \overline{\mathbb{D}} \mid \liminf_{\substack{\xi \to \lambda \\ \xi \in \mathbb{D}}} \inf_{\substack{\|y\|=1 \\ y \in Y}} \|Q_i(\xi)y\| = 0\} \ \ (i = 1, 2).$$

2. *If* $G_d^{\perp}$ *is noncyclic but not strictly noncyclic, then*

$$\sigma(A_o) = \sigma(A_i) = \overline{\mathbb{D}}.$$

3. *If* $G_d^{\perp}$ *is cyclic, then*

$$\sigma_p(A_o) = \mathbb{D}, \quad \sigma_p(A_i) = \emptyset, \quad \sigma(A_o) = \sigma(A_i) = \overline{\mathbb{D}}\ . \qquad \square$$

Corresponding to this theorem we have the following continuous-time analogue. For a strictly noncyclic continuous-time transfer function G_c, we define $\sigma_s(G_c)$ similarly as in the discrete-time situation. Specifically, G_c has a pseudomorphic continuation of bounded type to LHP (see Theorem 4.8), which is taken to be the definition of G_c on LHP. We consider the analytic continuation of the redefined G_c and denote by $\sigma_s(G_c)$ the set of points in the complex plane at which G_c has no analytic continuation.

We note that results in part 1 of the following theorem can be found in the thesis by Gearheart [12] and a paper by Moeller [19].

THEOREM 7.6. *Let* $G_c \in TLC^{U,Y}$ *and let* $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o}) \in C_X^{U,Y}$ *be an output-normal realization with U and Y finite dimensional. Then*

1. *if* G_c *is in* $H^\infty_{L(U,Y)}(RHP)$ *and is strictly noncyclic with factorization* $G_c = Q_1 F_1^*$, *where* $Q_1 \in H^\infty_{L(Y)}(RHP)$ *is inner and* Q_1 *and* $F_1 \in H^\infty_{L(Y,U)}(RHP)$ *are weakly coprime, then* $\lambda \notin \sigma(e^{A_{c,o}})$, $|\lambda| < 1$, *if and only if for*

$$w_n = -\log\bar{\lambda} + 2\pi n i, \qquad n \in \mathbb{Z},$$

$Q_1(w_n)$ *is invertible for all* $n \in \mathbb{Z}$ *and*

$$\sup_{-\infty < n < \infty} \|Q_1(w_n)^{-1}\| < \infty.$$

For $|\lambda| = 1$, $\lambda \notin \sigma(e^{A_{c,o}})$ *if and only if there exists a* $\delta > 0$ *and* $M > 0$ *such that* $Q_1(w_n)^{-1}$ *exists for all* $n \in \mathbb{Z}$ *and* $Q_1(s)^{-1}$ *is bounded by M in the* δ *neighborhood of each point* w_n.

For the point spectrum, we have

$$\sigma_p(e^{A_{c,o}}) \setminus \{0\} = \{e^{-\bar{s}} : s \in RHP \text{ and } \mathrm{Ker} Q_1(s)^* \neq \{0\}\}.$$

2. *under the same assumptions on* G_c *as in* 1., *we have*

$$\sigma(A_{c,o}) = \{-\bar{s} : s \in \sigma(Q_1)\} = \sigma_s(G_c)$$
$$\sigma_p(A_{c,o}) = \{-\bar{s} : s \in RHP \text{ and } \mathrm{Ker} Q_1(s)^* \neq \{0\}\}.$$

3. *if* G_c *is noncyclic but not strictly noncyclic, then*

$$\sigma(A_{c,o}) = \text{the closed left half plane.}$$

4. *if G_c is cyclic, then*

$$\sigma(e^{A_{c,o}}) = \bar{\mathbb{D}}, \qquad \sigma_p(e^{A_{c,o}}) \setminus \{0\} = \mathbb{D} \setminus \{0\},$$
$$\sigma(A_{c,o}) = \text{the closed left half plane}, \quad \sigma_p(A_{c,o}) = LHP.$$

Proof. Without loss of generality we may assume that $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$ is the restricted shift realization. We write (A_c, B_c, C_c, D_c) for $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$. Let $G_d(z) = G_c(\frac{z-1}{z+1})$ for $z \in \mathbb{D}_e$ and let

$$G_d^{\perp}(z) = z^{-1}[G_d(z^{-1}) - G_d(\infty)] = z^{-1}\left[G_c\left(\frac{1-z}{1+z}\right) - G_c(1)\right], \quad z \in \mathbb{D}.$$

Suppose (A_d, B_d, C_d, D_d) is the restricted shift realization of the discrete-time transfer function G_d. We use the mapping T defined in Theorem 3.1.

1. The formula for $\sigma(e^{A_c})$ can be found in [19] in the case $0 < |\lambda| < 1$. If $|\lambda| = 1$ or $\lambda = 0$, see [12].

For the formula of $\sigma_p(e^{A_c})$ see the proof of 2. below.

2. Note that by Proposition 4.7 $G_d^{\perp}$ is also strictly noncyclic and has a factorization $G_d = Q_{d,1}F_{d,1}^*$, where $Q_{d,1}(z) = Q_1(\frac{1-z}{1+z})$ and $F_{d,1}(z) = F_1(\frac{1-z}{1+z})$. The spectra of $Q_{d,1}$ and Q_1 are related as

$$\sigma(Q_{d,1}) = \left\{\frac{1-s}{1+s} | s \in \sigma(Q_1)\right\},$$

and the sets $\sigma_s(G_d)$ and $\sigma_s(G_c)$ are related as

$$\sigma_s(G_c) = \left\{s : \frac{1+s}{1-s} \in \sigma_s(G_d)\right\}.$$

Then the equalities about $\sigma(A_c)$ and $\sigma_s(G_c)$ follow from Proposition 7.2 and Theorem 7.5. Similarly the expression for $\sigma_p(A_c)$ also follows from Proposition 7.2 and Theorem 7.5.

The point spectrum $\sigma_p(e^{A_c})$ can be obtained by the general formula (see [26, Thm. 2.4, p. 46])

$$\sigma_p(e^{tA_c}) \setminus \{0\} = e^{\sigma_p(tA_c)}.$$

3. This also follows from Proposition 7.2 and Theorem 7.5.

4. We offer a direct proof here, although the result again follows from Proposition 7.2 and Theorem 7.5.

If G_c is cyclic, then the state space is $X_c = H_Y^2(RHP)$. It is easy to see that for any $\mu \in LHP, t \geq 0$, and $y \in Y$ we have $\frac{y}{s-\mu} \in X_c = H_Y^2(RHP)$ and

$$\frac{e^{ts} - e^{t\mu}}{s-\mu} y \in H_Y^2(LHP) = (H_Y^2(RHP))^{\perp},$$

where the orthogonal complement is taken in $L_Y^2(i\mathbb{R})$. Therefore,

$$e^{tA_c}\frac{1}{s-\mu}y = P_+\frac{e^{ts}}{s-\mu}y = P_+\left[\frac{e^{t\mu}}{s-\mu}y + \frac{e^{ts}-e^{t\mu}}{s-\mu}y\right] = \frac{e^{t\mu}}{s-\mu}y.$$

Hence $e^{t\mu} \in \sigma_p(e^{tA_c})$. This shows that $\sigma_p(e^{tA_c}) \setminus \{0\} = \mathbb{D} \setminus \{0\}$ and hence $\sigma(e^{tA_c}) = \overline{\mathbb{D}}$.

Also for any $\mu \in LHP$ and $y \in Y$ we have $h = \frac{1-\mu}{s-\mu} y \in H^2_Y(RHP)$ and

$$\frac{y}{s-\mu} = \frac{h(s) - h(1)}{1-s}.$$

Hence $\frac{y}{s-\mu} \in D(A_c)$. Using the definition of A_c we have

$$A_c \frac{y}{s-\mu} = \frac{sy}{s-\mu} - \lim_{\substack{r\in\mathbb{R} \\ r\to\infty}} \frac{ry}{r-\mu} = \mu \frac{y}{s-\mu}.$$

Therefore $\mu \in \sigma_p(A_c)$. This shows that $\sigma_p(A_c) = LHP$ and hence $\sigma(A_c) = \overline{LHP}$. □

For input-normal realizations we have results analogous to the results above. The proof is similar to the proof of the previous results.

THEOREM 7.7. *Let $G_c \in TLC^{U,Y}$ and let $(A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i}) \in C_X^{U,Y}$ be an observable input-normal realization with U and Y finite dimensional. Then*

1. *if G_c is in $H^\infty_{L(U,Y)}(RHP)$ and is strictly noncyclic with $\tilde{G}_c = Q_2 F_2^*$, where $Q_2 \in H^\infty_{L(U)}(RHP)$ is inner and Q_2 and $F_2 \in H^\infty_{L(U,Y)}(RHP)$ are weakly coprime, then $\lambda \notin \sigma(e^{A_{c,i}})$, $|\lambda| < 1$, if and only if for*

$$w_n = -\log\lambda + 2\pi n i, \qquad n \in \mathbb{Z},$$

$Q_2(w_n)$ is invertible for all $n \in \mathbb{Z}$ and

$$\sup_{-\infty<n<\infty} \|Q_2(w_n)^{-1}\| < \infty.$$

For $|\lambda| = 1$, $\lambda \notin \sigma(e^{A_{c,i}})$ if and only if there exists a $\delta > 0$ and $M > 0$ such that $Q_2(w_n)^{-1}$ exists for all $n \in \mathbb{Z}$ and $Q_2(s)^{-1}$ is bounded by M in a δ neighborhood of each point w_n. As to the point spectrum, we have

$$\sigma_p(e^{A_{c,i}}) \setminus \{0\} = \{e^{-s} : s \in RHP, \ \mathrm{Ker} Q_2(s) \neq \{0\}\}.$$

2. *Under the same assumption as in 1., for the generator $A_{c,i}$ we have*

$$\sigma(A_{c,i}) = \{-\lambda : \lambda \in \sigma(Q_2)\} = \sigma_s(G_c),$$
$$\sigma_p(A_{c,i}) = \{-s : s \in RHP, \ \mathrm{Ker} Q_2(s) \neq \{0\}\}.$$

3. *If $\tilde{G}_c$ is noncyclic and $\overline{\mathrm{range}}(H_{\tilde{G}_c}) = (Q_2 H^2_U(RHP))^\perp$, where Q_2 is a non-inner rigid function, then*

$$\sigma(A_{c,i}) = \text{the closed left half plane.}$$

4. *If G_c is cyclic, then*

$$\sigma(e^{A_{c,i}}) = \bar{\mathbb{D}}, \qquad \sigma_p(e^{A_{c,i}}) \setminus \{0\} = \emptyset,$$
$$\sigma(A_{c,i}) = \text{the closed left half plane.} \quad □$$

The following proposition gives the spectral properties of parbalanced realizations in the case of strictly noncyclic transfer functions.

PROPOSITION 7.8. *If $G_c \in H^\infty_{L(U,Y)}(RHP)$ is strictly noncyclic with finite dimensional U and Y, then*

$$\sigma(A_{c,o}) = \sigma(A_{c,i}) = \sigma(A_{c,b}) = \sigma_s(G_c),$$

where $(A_{c,b}, B_{c,b}, C_{c,b}, D_{c,b})$ is a parbalanced realization of G_c.

Proof. The analogous results in the discrete-time case are proven in [24, Cor. 4.3]. Since $\sigma_s(G_c) = \{\frac{z-1}{z+1} | \ z \in \sigma_s(G_d), \ z \neq -1\}$, where $G_d(z) = G_c(\frac{z-1}{z+1})$, $(z \in \mathbb{D}_e)$, the statement follows from Propositions 4.7 and 7.2 and Theorem 7.5. □

7.3. Exponential stability. Before we can give a criterion for the exponential stability of input- and output-normal realizations, we need some results concerning the relation between the spectrum of a semigroup and the spectrum of its generator. The following lemma can be deduced from [9, p. 622] (see also [21, p. 84]).

LEMMA 7.9. *Let e^{tA} be a strongly continuous semigroup of operators on a Hilbert space X with infinitesimal generator A. If $\sigma(e^{tA}) \subseteq \{\lambda : |\lambda| \le e^{\alpha t}\}$ $(t > 0)$, then $\sigma(A) \subseteq \{s : Re(s) \le \alpha\}$.* □

Note that in particular if $\|e^{tA}\| \le Me^{\alpha t}$ for some $M > 0$, then $r(e^{tA}) \le e^{\alpha t}$ and hence $\sigma(A) \subseteq \{s : Re(s) \le \alpha\}$, where $r(e^{tA})$ is the spectral radius.

It is well known that in general the converse of the lemma is not true (see [21, Chap. A-III]). However, the converse can be proven in some particular cases.

PROPOSITION 7.10. *Let e^{tA} be a strongly continuous semigroup of contractions on a Hilbert space X with infinitesimal generator A. Let A_d be its co-generator; that is, A_d is a contraction with $-1 \notin \sigma(A_d)$ and*

$$Ax = (A_d - I)(I + A_d)^{-1}x \quad (x \in D(A) = \text{range}(I + A_d)).$$

Assume that A_d is a C_0 operator with minimal function m. Then $\sigma(e^{tA}) \subseteq \{\lambda : |\lambda| \le e^{\alpha t}\}$ $(t > 0)$, if and only if $\sigma(A) \subseteq \{s : Re(s) \le \alpha\}$. Here α is a real number.

Proof. The necessity part follows from Lemma 7.9.

Now assume $\sigma(A) \subseteq \{s : Re(s) \le \alpha\}$. Since $\sigma(e^{tA}) \subseteq \{\lambda : |\lambda| \le 1\}$, we may assume $\alpha < 0$.

By Lemma 7.3, we have $\sigma(A_d) = \sigma(m)$. On the other hand Proposition 7.2 shows that

$$\sigma(A_d) \setminus \{-1\} = \left\{\frac{1+s}{1-s} : s \in \sigma(A)\right\}.$$

Since $\sigma(A) \subseteq \{s : Re(s) \le \alpha\}$, we have

$$\sigma(A_d) \setminus \{-1\} \subseteq \left\{\frac{1+s}{1-s} : Re(s) \le \alpha\right\}.$$

Thus

$$\sigma(m) \setminus \{-1\} \subseteq \left\{\frac{1+s}{1-s} : Re(s) \le \alpha\right\}.$$

Let $\xi = \frac{1+s}{1-s}$. Then $Re(s) \le \alpha$ if and only if $|\xi - \frac{\alpha}{2-\alpha}| \le 1 + \frac{\alpha}{2-\alpha}$. This shows that

$$\sigma(m) \subseteq \left\{\xi : |\xi - \frac{\alpha}{2-\alpha}| \le 1 + \frac{\alpha}{2-\alpha}\right\}.$$

Therefore if $\xi \in \bar{\mathbb{D}}$ and $|\xi - \frac{\alpha}{2-\alpha}| > 1 + \frac{\alpha}{2-\alpha}$, then $\xi \notin \sigma(m)$. Hence there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|m(z)| \ge \delta_1 \text{ for any } z \in \mathbb{D} \text{ satisfying } |z - \xi| \le \delta_2.$$

Now for each $t > 0$, let $u(z) = e^{t\frac{z-1}{z+1}}$. Then $u \in H^\infty(\mathbb{D})$ and $e^{tA} = u(A_d)$. Using the spectral mapping theorem (Theorem 7.4) we have (note again that if $\xi = \frac{1+s}{1-s}$, then $Re(s) \le \alpha$ if and only if $|\xi - \frac{\alpha}{2-\alpha}| \le 1 + \frac{\alpha}{2-\alpha}$)

$$\sigma(e^{tA}) = \sigma(u(A_d))$$
$$\subseteq \left\{\lambda : \inf_{\xi \in \mathbb{D}} \{|u(\xi) - \lambda| + |m(\xi)|\} = 0\right\}$$
$$= \left\{\lambda : \inf_{|\xi - \frac{\alpha}{2-\alpha}| \leq 1 + \frac{\alpha}{2-\alpha}} \{|u(\xi) - \lambda| + |m(\xi)|\} = 0\right\}$$
$$\subseteq \left\{\lambda : \inf_{|\xi - \frac{\alpha}{2-\alpha}| \leq 1 + \frac{\alpha}{2-\alpha}} \{|u(\xi) - \lambda|\} = 0\right\}$$
$$= \left\{\lambda : \inf_{|\xi - \frac{\alpha}{2-\alpha}| \leq 1 + \frac{\alpha}{2-\alpha}} \{|e^{t\frac{\xi-1}{\xi+1}} - \lambda|\} = 0\right\}$$
$$= \text{closure}\left\{e^{t\frac{\xi-1}{\xi+1}} : \ |\xi - \frac{\alpha}{2-\alpha}| \leq 1 + \frac{\alpha}{2-\alpha}\right\}$$
$$= \text{closure}\{e^{st} : \ Re(s) \leq \alpha\}$$
$$= \{\lambda : \ |\lambda| \leq e^{\alpha t}\}, \ \ (t > 0).$$

This completes the proof. □

We are ready to show when an input- or output-normal realization is exponentially stable. For exponentially stable realizations we also characterize the growth bound in terms of the analyticity of the transfer function. The results remarkably resemble the related results for finite dimensional systems.

THEOREM 7.11. *Let G_c be in $H^\infty_{L(U,Y)}(RHP)$ with finite dimensional U and Y. Then an input-normal or output-normal realization of G_c is exponentially stable if and only if G_c is strictly noncyclic and there is $\alpha < 0$ such that G_c has analytic continuation on $Re(s) > \alpha$.*

In this case the growth bound is given by

$$\omega = \inf\{\alpha : \ G_c \textit{ has analytic continuation on } Re(s) > \alpha\}.$$

Proof. We prove the theorem for output-normal realizations. The proof in the input-normal case is exactly the same. For output-normal realizations, it suffices to prove the result for the restricted shift realization.

Thus we assume that the restricted shift realization (A, B, C, D) of G_c is exponentially stable. Hence there are $\alpha < 0$ and $M > 0$ such that

$$\|e^{tA}\| \leq Me^{\alpha t} \ \text{ for } t \geq 0.$$

Then by the remark after Lemma 7.9 $\sigma(A) \subseteq \{s| \ Re(s) \leq \alpha\}$. As $\alpha < 0$, from Theorem 7.6 it follows that G_c has to be strictly noncyclic since otherwise $\sigma(A) = \overline{LHP}$. Now applying Proposition 7.8 we have

$$\sigma_s(G_c) = \sigma(A) \subseteq \{s : \ Re(s) \leq \alpha\}.$$

Hence G_c has analytic continuation on $Re(s) > \alpha$. This also shows that

$$\inf\{\alpha' : \ G_c \text{ has analytic continuation on } Re(s) > \alpha'\}$$

is not greater than the growth bound of (A, B, C, D).

Conversely, assume that G_c is strictly noncyclic and there is $\alpha < 0$ such that G_c has an analytic continuation on $Re(s) > \alpha$. Let (A, B, C, D) be the restricted shift realization of G_c and (A_d, B_d, C_d, D_d) be the discrete-time restricted shift realization of $G_d(z) = G_c(\frac{z-1}{z+1})$. Note that $(A, B, C, D) = T((A_d, B_d, C_d, D_d))$.

Again by Proposition 7.8 we have

$$\sigma(A) = \sigma_s(G_c).$$

Therefore $\sigma(A) \subseteq \{s : Re(s) \leq \alpha\}$. Note that G_d is also strictly noncyclic. It follows from Theorem 7.5 that A_d is a C_0 operator. Now we can apply Proposition 7.10 to get

$$\sigma(e^{tA}) \subseteq \{\lambda : \ |\lambda| \leq e^{\alpha t}\}.$$

This shows that $r(e^{tA}) \leq e^{\alpha t}$. Thus by Proposition 7.1, (A, B, C, D) is exponentially stable. This also implies that the growth bound of (A, B, C, D) is not greater than

$$\inf\{\alpha' : \ G_c \text{ has analytic continuation on } Re(s) > \alpha'\}.$$

The proof is now complete. □

The following proposition shows that for strictly noncyclic transfer functions parbalanced realizations have the same exponential stability properties as input- and output-normal realizations.

PROPOSITION 7.12. *Let $G_c \in H^\infty_{L(U,Y)}(RHP)$ be strictly noncyclic with U and Y finite dimensional and let $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$, $(A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i})$, and $(A_{c,b}, B_{c,b}, C_{c,b}, D_{c,b})$ be, respectively, an output-normal, an input-normal, and a parbalanced realization of G_c. Then the following are equivalent:*

1. *$(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$ is exponentially stable with growth bound α;*
2. *$(A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i})$ is exponentially stable with growth bound α;*
3. *$(A_{c,b}, B_{c,b}, C_{c,b}, D_{c,b})$ is exponentially stable with growth bound α.*

Proof. By Theorem 7.11, 1. and 2. are equivalent. Hence we need only to prove the equivalence of 1. and 3. Assume that 1. is true. Then there exist $M > 0$ and $\alpha < 0$ such that

$$\|e^{tA_{c,o}}\| \leq Me^{t\alpha} \quad (t > 0).$$

From the remark after Lemma 7.9 it follows that $\sigma(A_{c,o}) \subseteq \{s| \ Re(s) \leq \alpha\}$. Since now by Proposition 7.8 $\sigma(A_{c,b}) = \sigma(A_{c,o})$ we have

$$\sigma(A_{c,b}) \subseteq \{s \mid \ Re(s) \leq \alpha\}.$$

Let $A_{d,b} = (I + A_{c,b})(I - A_{c,b})^{-1}$ be the propagation operator of the corresponding discrete-time parbalanced realization of $G_d(z) = G_c(\frac{z-1}{z+1}) \ \ (z \in \mathbb{D}_e)$. That is, $A_{d,b}$ is the co-generator of the semigroup $e^{tA_{c,b}}$. Note that $G_d^{\perp}$ is strictly noncyclic. By Theorem 7.5 $A_{d,b}$ is a C_0 operator. Therefore it follows from Proposition 7.10 that

$$\sigma(e^{tA_{c,b}}) \subseteq \{\lambda : \ |\lambda| \leq e^{\alpha t}\}.$$

This, by Proposition 7.1, shows that $(A_{c,b}, B_{c,b}, C_{c,b}, D_{c,b})$ is exponentially stable with growth bound no greater than α and hence no greater than the growth bound of $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$.

If we assume 3., a similar argument will lead to 1. □

If the Hankel operator H_{G_c} has closed range, then by Proposition 6.2 all reachable, observable, and admissible realizations of G_c are equivalent. Hence we have the following corollary.

COROLLARY 7.13. *Assume that the spaces U and Y are finite dimensional and the Hankel operator H_{G_c} has closed range. Then a reachable, observable, and admissible realization of G_c is exponentially stable if and only if there is a number $\alpha < 0$ such that G_c is strictly noncyclic and can be analytically continued to the half plane $\{s|\ Re(s) > \alpha\}$. The growth bound of these systems is* $\inf \alpha$. □

8. Boundedness of the system operators. We have seen that for an admissible continuous-time transfer function $G_c(s)$ there are always output-normal, input-normal, and parbalanced realizations with well-defined bounded observability and reachability operators. In this sense those realization are well posed. As expected for all infinite-dimensional continuous-time realizations, the propagation, input, and output operators of those realizations are in general unbounded. The input operators are defined in such a way that the range may not be contained in the state space. In this section we are going to investigate when those operators are bounded. We will use the specific form of the restricted and *-restricted shift realizations obtained in §5.

8.1. Boundedness of A_c. First we have the following lemma which shows that the input and output operators are bounded when the propagation operators are.

LEMMA 8.1. *Let (A_c, B_c, C_c, D_c) be an admissible system in $C_X^{U,Y}$. If $A_c : X \to X$ is bounded, then $C_c \in L(X, Y)$ and B_c can be considered as an operator in $L(U, X)$.*

Proof. By definition $C_c|_{D(A_c)} : (D(A_c), \|\cdot\||_{A_c}) \to Y$ is bounded. Now that A_c is bounded, $D(A_c) = X$. Hence for any $x \in X = D(A_c)$,

$$\begin{aligned}\|C_c x\| &\le \|C_c|_{D(A_c)}\|\ (\|x\|^2 + \|A_c x\|^2)^{1/2}\\ &\le \|C_c|_{D(A_c)}\|\ (\|x\|^2 + \|A_c\|^2\ \|x\|^2)^{1/2}\\ &= \|C_c|_{D(A_c)}\|(1 + \|A_c\|^2)^{1/2}\|x\|.\end{aligned}$$

So $C_c \in L(X, Y)$.

For B we know that $B_c u \in D(A_c^*)^{(\prime)}$ and by definition $\|B_c u\|^{(\prime)} \le b\|u\|$ for any $u \in U$ and some fixed number $b > 0$. By the Riesz representation theorem, there exists $x_u \in D(A_c^*) = X$ such that

$$\|B_c u\|^{(\prime)} = \|x_u\|_{A_c^*}$$

and for $x \in D(A_c^*) = X$,

$$(B_c u)(x) = \langle x_u, x\rangle_{A_c^*} = \langle x_u, x\rangle + \langle A_c^* x_u, A_c^* x\rangle = \langle (1 + A_c A_c^*) x_u, x\rangle.$$

Therefore $B_c u = (1 + A_c A_c^*) x_u \in X$ and

$$\begin{aligned}\|B_c u\| &= \|(1 + A_c A_c^*) x_u\|\\ &\le \|1 + A_c A_c^*\|\ \|x_u\| \le \|1 + A_c A_c^*\|\ \|x_u\|_{A_c^*}\\ &= \|1 + A_c A_c^*\|\ \|B_c u\|^{(\prime)} \le \|1 + A_c A_c^*\|\ b\ \|u\|.\end{aligned}$$

Hence $B_c \in L(U, X)$. □

Now we give a necessary and sufficient condition for the propagation operators in the input-normal and output-normal realizations to be bounded.

THEOREM 8.2. *Let G_c be in $H^\infty_{L(U,Y)}(RHP)$ with U and Y finite dimensional and let (A_c, B_c, C_c, D_c) be an input-normal (or output-normal) realization of G_c. Then A_c is bounded* *if and only if $G_c(s)$ is strictly noncyclic and analytic at infinity. Here analyticity at infinity means that $G_c(\frac{1}{s})$ is analytic at the origin.*

Proof. Since all output-normal (input-normal) realizations are unitarily equivalent to the restricted (*-restricted) shift realizations, we prove the theorem for the restricted shift realization and *-restricted shift realization. Let $G_d(z) = G_c(\frac{z-1}{z+1})$ $(z \in \mathbb{D}_e)$ and (A_d, B_d, C_d, D_d) be the restricted realization of G_d on $X_d = \overline{\text{range}}(H_{G_d^\perp})$. Then $A_c = V(A_d - I)(A_d + I)^{-1}V^{-1}$, where V is the unitary operator defined in Proposition 3.3.

If $G_c(s)$ is strictly noncyclic and analytic at infinity, then $G_d(z)$ is strictly noncyclic and analytic at -1. Hence by the spectral minimality of the discrete-time restricted shift realization (see [11]) $-1 \notin \sigma(A_d)$; i.e., $(A_d + I)^{-1}$ is bounded and so $A_c = V(A_d - I)(A_d + I)^{-1}V^{-1}$ is bounded.

Conversely, if A_c is bounded, then $(A_d+I)^{-1} = \frac{1}{2}(I - V^{-1}A_cV)$ is also bounded and thus $-1 \notin \sigma(A_d)$. By Theorem 7.5 G_d has to be strictly noncyclic since otherwise $\sigma(A_d) = \bar{\mathbb{D}}$. Also $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$ is analytic at -1. Therefore $G_c(s)$ is strictly noncyclic and analytic at infinity.

Exactly the same argument can also be applied to the *-restricted case. □

Regarding the boundedness of a parbalanced realization, we have the following.

COROLLARY 8.3. *Let $G_c \in H^\infty_{L(U,Y)}(RHP)$ be strictly noncyclic with finite dimensional U and Y and let $(A_{c,o}, B_{c,o}, C_{c,o}, D_{c,o})$, $(A_{c,i}, B_{c,i}, C_{c,i}, D_{c,i})$, and $(A_{c,b}, B_{c,b}, C_{c,b}, D_{c,b})$ be, respectively, an output-normal, an input-normal, and a parbalanced realization of G_c. Then the boundedness of one of $A_{c,o}$, $A_{c,i}$, and $A_{c,b}$ implies the boundedness of the other two.*

Proof. By Theorem 8.2, it suffices to prove that the boundedness of $A_{c,o}$ implies and is implied by that of $A_{c,b}$. We do this by connecting the continuous-time and discrete-time systems as in Theorem 3.1.

Assume that $A_{c,o}$ is bounded. Then, as in the proof of Theorem 8.2, $-1 \notin \sigma(A_{d,o})$. Since G_c and hence $G_d^\perp$ are strictly noncyclic, $\sigma(A_{d,o}) = \sigma(A_{d,b})$. Thus $-1 \notin \sigma(A_{d,b})$ and hence $A_{c,b} = (A_{d,b} - I)(A_{d,b} + I)^{-1}$ is bounded. The same argument can also go the other direction, and the result is proven. □

8.2. Boundedness of B_c in output-normal realizations. We now consider the boundedness of the input and output operators. First we recall that for the input operator B_c of the restricted shift realization with state space X, we have that $B_cu \in X$ $(u \in U)$ if and only if

$$[G_c(s) - G_c(+\infty)]u \in X \quad (u \in U),$$

and in this case

$$(B_cu)(s) = \frac{1}{\sqrt{2\pi}}[G_c(s) - G_c(+\infty)]u$$

(see Theorem 5.7 and Corollary 5.8).

PROPOSITION 8.4. *Let $G_c \in TLC^{U,Y}$.*

1. *The input operator of an output-normal realization of G_c is bounded if and only if there is $M > 0$ such that*

$$\sup_{x>0} \int_{-\infty}^{+\infty} \|[G_c(x + iy) - G_c(+\infty)]u\|^2 dy \le (M\|u\|)^2 \text{ for any } u \in U,$$

where $M > 0$ is a constant.

2. *The output operator of an input-normal realization of G_c is bounded if and only if there is $M > 0$ such that*

$$\sup_{x>0} \int_{-\infty}^{+\infty} \|[\tilde{G}_c(x + iy) - \tilde{G}_c(+\infty)]v\|^2 dy \le (M\|v\|)^2 \text{ for any } v \in Y.$$

3. *If*

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[G_c(x+iy)-G_c(+\infty)]\|^2dy<\infty,$$

then the input operator of any output-normal realization and the output operator of any input-normal realization are bounded. If in addition the Hankel operator H_{G_c} has closed range, then both the input and the output operators of any output-normal, input-normal, and parbalanced realizations are bounded.

Proof. 1. It suffices to prove the result for the restricted shift realization of G_c. Let B_c be the input operator of the restricted shift realization (see Theorem 5.7).

Assume

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[G_c(x+iy)-G_c(+\infty)]u\|^2dy\le(M\|u\|)^2\ \text{ for any }\ u\in U.$$

This condition implies that $[G_c-G_c(+\infty)]u\in X$ for any $u\in U$ because as in the proof of Corollary 5.8 we have in $L^2(i\mathbb{R})$ norm

$$[G_c-G_c(+\infty)]u=\lim_{n\to\infty}\frac{G_c(s)-G_c(n)}{1-s/n}u=\lim_{n\to\infty}H_{G_c}\frac{n}{n+s}u.$$

Hence $(B_cu)(s)=\frac{1}{\sqrt{2\pi}}[G_c(s)-G_c(+\infty)]u\ \ (u\in U)$ and $\|B_cu\|\le\frac{1}{\sqrt{2\pi}}M\|u\|$.

Conversely, if B_c is bounded, then there is $M>0$ such that $\|B_cu\|\le M\|u\|$. Also $B_cu\in X$ for any $u\in U$. By Corollary 5.8

$$(B_cu)(s)=\frac{1}{\sqrt{2\pi}}[G_c(s)-G_c(+\infty)]u\ \ (u\in U).$$

Thus we have

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[G_c(x+iy)-G_c(+\infty)]u\|^2dy=\frac{1}{2\pi}\|B_cu\|^2\le\frac{1}{2\pi}M^2\|u\|^2.$$

2. Similarly we only need to prove the result for the $*$-restricted shift realization. Since the $*$-restricted shift realization of G_c is the dual of the restricted shift realization of $\tilde{G}_c$, the result follows from 1.

3. First note that

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[G_c(x+iy)-G_c(+\infty)]\|^2dy<\infty$$

if and only if

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|\tilde{G}_c(x+iy)-\tilde{G}_c(+\infty)\|^2dy<\infty.$$

Clearly these conditions imply

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[G_c(x+iy)-G_c(+\infty)]u\|^2dy\le(M\|u\|)^2\ \text{ for any }\ u\in U$$

and

$$\sup_{x>0}\int_{-\infty}^{+\infty}\|[\tilde{G}_c(x+iy)-\tilde{G}c(+\infty)]u\|^2dy\le(\tilde{M}\|u\|)^2\ \text{ for any }\ u\in U$$

for some constants $M < \infty$ and $\tilde{M} < \infty$. Hence by 1. and 2. the input operator of any output-normal realization and the output operator of any input-normal realization are bounded. If in addition H_{G_c} has closed range, then by Proposition 6.2 all reachable and observable admissible realizations of G_c are equivalent. Thus all have bounded input and output operators. □

The following corollary gives a simple condition for the input operator of an output-normal realization and the output operator of an input-normal realization to be bounded.

COROLLARY 8.5. *Let $G_c \in H^\infty_{L(U,Y)}(RHP)$ be analytic at ∞. Then the input operator of the output-normal realizations and the output operator of the input-normal realizations are bounded.*

Proof. Let $F_d(z) = G_c(\frac{1-z}{1+z}) - G_c(\infty)$. The analyticity of G_c at ∞ means that F_d and $\frac{F_d(z)}{1+z}$ are both analytic at -1. Hence $\frac{F_d(z)}{1+z} \in H^\infty_{L(U,Y)}(\mathbb{D})$ and for any $u \in U$,

$$\left\| \frac{F_d(z)}{1+z} u \right\|_{H^2_Y(\mathbb{D})} \leq M \|u\|_U,$$

where $M = \sup_{z\in\mathbb{D}} \| \frac{F_d(z)}{1+z} \|_{L(U,Y)}$. Applying the unitary transformation V in Proposition 3.3, we have

$$\|(G_c - G_c(\infty))u\|_{H^2_Y(RHP)} = 2\sqrt{\pi} \| V \frac{F_d(z)}{1+z} u \|_{H^2_Y(\mathbb{D})} \leq 2\sqrt{\pi} M \|u\|_U \quad (u \in U).$$

Since the analyticity of G_c at ∞ implies the analyticity of $\tilde{G}_c$ at ∞, we have similarly

$$\|(\tilde{G}_c - \tilde{G}_c(\infty))y\|_{H^2_U(RHP)} \leq 2\sqrt{\pi} M \|y\|_Y \quad (y \in Y).$$

By Proposition 8.4 it follows that the input operator of the restricted shift realization and the output operator of the *-restricted shift realization are bounded. This proves the corollary. □

8.3. Boundedness of C_c in output-normal realization. Now we consider the boundedness of the input operator of the *-restricted shift realization and the output operator of the restricted shift realization. We present here results for noncyclic scalar transfer functions.

It is well known that a scalar inner function $q_d \in H^\infty(\mathbb{D})$ admits a factorization of the form $q_d(z) = \lambda \mathcal{B}_d(z) \mathcal{S}_d(z)$, where λ is a complex number, $|\lambda| = 1$;

$$\mathcal{B}_d(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$$

is a Blaschke product, and

$$\mathcal{S}_d(z) = \exp\left[-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_d(\theta) \right]$$

is a singular inner function with μ_d a finite singular positive measure on the unit circle $\partial\mathbb{D}$ (see [17]). Here we take $\frac{\bar{\alpha}_n}{|\alpha_n|}$ to be 1 when $\alpha_n = 0$. Ahern and Clark [1] have proved the following theorem.

THEOREM 8.6. *Set $X = H^2(\mathbb{D}) \ominus q_d H^2(\mathbb{D})$ and denote the compressed shift operator on X by $S(q_d) := P_{X_d} z|_{X_d}$. Then the following statements are equivalent.*

1. *For every $x \in X$ the nontangential limit of $x(z)$ exists at -1.*
2. $P_X 1 \in \text{range}(I + S(q_d))$.
3. *For the function q_d*

$$\sum_{n=1}^{\infty} \frac{1 - |\alpha_n|^2}{|1 + \alpha_n|^2} < \infty \quad \text{and} \quad \int_0^{2\pi} \frac{d\mu_d(\theta)}{|1 + e^{i\theta}|} < \infty.$$

Furthermore, if one of these conditions hold, then there exists a function $k \in X$ such that the nontangential limit of any $x \in X$ at -1 is

$$x(-1) := \lim_{\substack{z \to -1,\ z \in \mathbb{D} \\ \text{nontangential}}} x(z) = \langle x, k \rangle. \qquad \square$$

This theorem can be cast into left invariant spaces on the right half plane. Let q_c be an inner function in $H^\infty(RHP)$. Then q_c has the form $q_c(s) = \lambda \mathcal{B}_c(s)\mathcal{S}_c(s)$, where $\mathcal{B}_c$ is a Blaschke product on the right half plane,

$$\mathcal{B}_c(s) = \prod_{n=1}^{\infty} \frac{|1-\beta_n^2|}{1-\beta_n^2} \frac{s-\beta_n}{s+\bar{\beta}_n},$$

and

$$\mathcal{S}_c(s) = e^{-as} \exp\left[-\int_{-\infty}^{\infty} \frac{ys+i}{y+is} d\mu_c(y) \right]$$

is a singular inner function with μ_c a finite singular positive measure on $i\mathbb{R}$ and $a \geq 0$ (see [17]). Here $\frac{|1-\beta_n^2|}{1-\beta_n^2}$ is taken to be 1 if $\beta_n = 1$. Let V be the transformation defined in Proposition 3.3. Applying V to X_d in Theorem 8.6, we obtain the following theorem.

THEOREM 8.7. *Set $X = H^2(RHP) \ominus q_c H^2(RHP)$ and $\mathcal{D} = \{P_X \frac{h}{1+s} : h \in X\}$. Then the following statements are equivalent.*

1. *For every $f \in X$, the limit*

$$\lim_{\substack{s \to \infty,\ Re(s) > 0 \\ Re(s) > \epsilon |s|}} s f(s)$$

exists for any $\epsilon > 0$.

2. $P_X \frac{1}{1+s} \in \mathcal{D}$.
3. *For the inner function q_c,*

$$a = 0, \quad \sum_{n=1}^{\infty} Re\beta_n < \infty, \quad \text{and} \quad \int_{-\infty}^{\infty} \sqrt{1+y^2}\, d\mu_c(y) < \infty.$$

Moreover, if one of the statements holds, then there exists $k \in X$ such that

$$\lim_{\substack{s \to \infty, Re(s) > 0 \\ Re(s) > \epsilon |s|}} s f(s) = \langle f, k \rangle \quad (f \in X).$$

Proof. Let $q_d(z) = q_c(\frac{1-z}{1+z})$ $(z \in \mathbb{D})$. Then q_d admits factorization as in Theorem 8.6: $q_d(z) = \lambda \mathcal{B}_d(z)\mathcal{S}_d(z)$. It can be easily seen that the Blaschke products $\mathcal{B}_d(z)$ and $\mathcal{B}_c(z)$ can be related by $\beta_n = \frac{1-\alpha_n}{1+\alpha_n}$; that the functions $\mathcal{S}_c(s)$ and $\mathcal{S}_d(z)$ are related by

$$\mathcal{S}_c(s) = \mathcal{S}_d\left(\frac{1-s}{1+s}\right) \quad (s \in RHP),$$

with $a = \mu_d(\{-1\})$; and that the measure μ_c is the measure μ_d transformed by the bilinear transformation

$$s = \frac{1-z}{1+z} : \partial\mathbb{D} \backslash \{-1\} \to i\mathbb{R}.$$

Hence the condition $\sum_{n=1}^{\infty} \frac{1-|\alpha_n|^2}{|1+\alpha_n|^2} < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} Re\beta_n < \infty,$$

and the condition $\int_0^{2\pi} \frac{d\mu_d(\theta)}{|1+e^{i\theta}|} < \infty$ is equivalent to

$$a = 0, \quad \int_{-\infty}^{\infty} \sqrt{1+y^2} d\mu_c(y) < \infty.$$

This shows that condition 3 of Theorem 8.6 and condition 3 of Theorem 8.7 are equivalent.

Let V be the unitary transformation defined in Proposition 3.3. Then $V(H^2(\mathbb{D}) \ominus q_d H^2(\mathbb{D})) = H^2(RHP) \ominus q_c H^2(RHP)$, $V(\text{range}(I + S(q_d))) = \{P_X \frac{h}{1+s} : h \in H^2(RHP) \ominus q_c H^2(RHP)\} = \mathcal{D}$, and

$$P_{H^2(RHP)\ominus q_c H^2(RHP)} \frac{1}{1+s} = \sqrt{\pi} V P_{H^2(\mathbb{D})\ominus q_d H^2(\mathbb{D})} 1.$$

Therefore condition 2 of Theorem 8.6 and condition 2 of Theorem 8.7 are equivalent.

Finally, for $x \in H^2(\mathbb{D}) \ominus q_d H^2(\mathbb{D})$, we have $f := Vx \in H^2(RHP) \ominus q_c H^2(RHP)$ and

$$\lim_{\substack{z\in\mathbb{D}, z\to -1 \\ \text{nontangential}}} x(z) := \lim_{\substack{z\in\mathbb{D}, z\to -1 \\ (1-|z|)>\epsilon|z+1|}} x(z) = \sqrt{\pi} \lim_{\substack{s\to\infty,\ Re(s)>0 \\ Re(s)>\epsilon|s|}} sf(s)$$

for any $\epsilon > 0$. This completes the proof. □

If in the theorem we replace q_c by $\tilde{q}_c(s) = \overline{q_c(\bar{s})}$ and $H^2(RHP) \ominus q_c H^2(RHP)$ by $H^2(RHP) \ominus \tilde{q}_c H^2(RHP)$, then we have the results to be true for the space $H^2(RHP) \ominus \tilde{q}_c H^2(RHP)$ while condition 3 of Theorem 8.7 remains unchanged in terms of a, β_n's and the singular measure μ_c.

These results can be immediately used to show the boundedness of the output operators of output-normal realizations and the input operators of input-normal realizations.

COROLLARY 8.8. *Let $G_c \in H^\infty(RHP)$ be a scalar noncyclic transfer function admitting the factorization $G_c = q_c f^*$, where $q_c \in H^\infty(RHP)$ is inner and q_c and $f \in H^\infty(RHP)$ are weakly coprime. Assume q_c has decomposition as in Theorem 8.7, and set $X = H^2(RHP) \ominus q_c H^2(RHP)$. Then the following statements are equivalent:*

1. *The output operator C_c of the restricted shift realization of G_c is bounded.*
2. *The input operator $B_{c,*}$ of the *-restricted shift realization of G_c is bounded.*
3. *One of the statements in Theorem 8.7 is true.*

Hence the output operator of every output-normal realization and the input operator of every input-normal realization are bounded if and only if one of the statements in Theorem 8.7 is true.

If in addition the Hankel operator H_{G_c} has closed range, then both the input operator and the output operator of every reachable and observable admissible realization of G_c are bounded.

Proof. First assume one and hence all of the statements in Theorem 8.7 to be true. We prove 1. and 2.

By Theorem 5.7, the output operator of the shift realization of G_c is given by

$$\begin{aligned} C_c : D(C) = D(A_c) + (I - A)^{-1} B_c U \subseteq X_c &\to Y, \\ x &\mapsto \sqrt{2\pi} \lim_{\substack{r\in\mathbb{R} \\ r\to\infty}} rx(r), \end{aligned}$$

where $X_c = H^2(RHP) \ominus q_c H^2(RHP)$. Now by Theorem 8.7 there exists $k \in X_c$ such that

$$\lim_{\substack{r\in\mathbb{R} \\ r\to\infty}} rx(r) = \langle x, k \rangle \quad (x \in X_c).$$

Hence

$$C_c x = \sqrt{2\pi} \lim_{\substack{r\in\mathbb{R} \\ r\to\infty}} rx(r) = \sqrt{2\pi} \langle x, k \rangle$$

for any $x \in D(C_c)$, and it follows that C_c is bounded: $\|C_c\| \le \sqrt{2\pi} \|k\|$.

To show the boundedness of the input operator $B_{c,*}$ of the *-restricted shift realization we use the expression of $B_{c,*}$ as given in Theorem 5.9: The state space is $X_{c,*} = H^2(RHP) \ominus \tilde{q}_c H^2(RHP)$ and

$$\begin{array}{rrcl} B_{c,*}: & U & \to & D(A_{c,*}^*)^{(')}, \\ & u & \mapsto & B_c(u), \\ & [B_{c,*}(u)](x) & = & \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1+s}u,\ (1-A_{c,*}^*)x \right\rangle,\ \ x \in D(A_{c,*}^*), \end{array}$$

where the operator $A_{c,*}$ has domain $D(A_{c,*}) = \{P_{X_{c,*}}\frac{h}{1+s} : h \in X_{c,*}\}$. Here $U = \mathbb{C}$. Since by Theorem 8.7 we have $P_{X_{c,*}}\frac{1}{1+s} \in D(A_{c,*})$, it follows that

$$[B_{c,*}(u)](x) = \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1+s}u,\ (1-A_{c,*}^*)x \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\left\langle P_{X_{c,*}}\frac{1}{1+s}u,\ (1-A_{c,*}^*)x \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\left\langle (1-A_{c,*})P_{X_{c,*}}\frac{1}{1+s}u,\ x \right\rangle.$$

This shows that $B_{c,*}(u) \in X_c$ and $B_{c,*}(u) = \frac{1}{\sqrt{2\pi}}(I-A_{c,*})P_{X_{c,*}}\frac{1}{1+s}u$ for $u \in \mathbb{C}$. Hence B_c is bounded: $\|B_c\| \le \frac{1}{\sqrt{2\pi}}\|(I-A_{c,*})P_{X_{c,*}}\frac{1}{1+s}\|_{H^2(RHP)}$.

Now we assume 2. and prove that this implies 3. As in the above, $X_{c,*} = H^2(RHP) \ominus \tilde{q}_c H^2(RHP)$ and

$$[B_{c,*}(u)](x) = \frac{1}{\sqrt{2\pi}}\left\langle P_{X_{c,*}}\frac{1}{1+s}u,\ (1-A_{c,*}^*)x \right\rangle \quad (x \in D(A_{c,*}^*))$$

where $D(A_{c,*}) = \{P_{X_{c,*}}\frac{h}{1+s} : h \in X_{c,*}\}$. As $B_{c,*}$ is assumed to be bounded, for any $u \in \mathbb{C}$ there exists $k(u) \in X_{c,*}$ such that

$$[B_{c,*}(u)](x) = \langle k(u),\ x\rangle \ (x \in D(A_{c,*}^*)).$$

Hence $\langle k(u),\ x\rangle = \frac{1}{\sqrt{2\pi}}\langle P_{X_{c,*}}\frac{1}{1+s}u,\ (1-A_{c,*}^*)x\rangle$ for any $x \in D(A_{c,*}^*)$. This shows that

$$P_{X_{c,*}}\frac{u}{1+s} \in D((1-A_{c,*}^*)^*) = D(1-A_{c,*}) = D(A_{c,*}) = \left\{P_{X_{c,*}}\frac{h}{1+s} : h \in X_{c,*}\right\}.$$

Thus statement 2 in Theorem 8.7 is true if the space X is replaced by $X_{c,*}$ and the inner function q_c is replaced by $\tilde{q}_c$. By the remark following Theorem 8.7, we know that the statements in Theorem 8.7 are true for X and q_c.

Finally, we show that 1. implies 3. Let (A_c, B_c, C_c, D_c) be the restricted shift realization of $G_c = q_c f^*$ and assume that C_c is bounded. Denote by $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ the dual system of (A_c, B_c, C_c, D_c). Then $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is the *-restricted shift realization of $\tilde{G}_c = \tilde{q}_c \tilde{f}^*$ with state space $X = H^2(RHP) \ominus q_c H^2(RHP)$, and

$$\tilde{B}_c = C_c^*.$$

Hence $\tilde{B}_c$ is bounded. By the preceding proof, statement 2 in Theorem 8.7 is true for the space X and the inner function q_c. This completes the proof. □

8.4. Boundedness of B_c, C_c for parbalanced realizations. To conclude this section we show some results on the boundedness of the input and output operators of parbalanced realizations.

PROPOSITION 8.9. *If the output operator of an output-normal realization of $G_c \in TLC^{U,Y}$ is bounded, then the output operator of a parbalanced realization of G_c is bounded.*

Proof. Consider the discrete-time transfer function $G_d(z) = G_c(\frac{z-1}{z+1})$. Let $(A_{do}, B_{do}, C_{do}, D_{do})$ be the discrete-time restricted shift realization of G_d with state space X and $(A_{co}, B_{co}, C_{co}, D_{co}) = T((A_{do}, B_{do}, C_{do}, D_{do}))$ be the continuous-time restricted shift realization of G_c with the same state space X. Denote their observability operators by $\mathcal{O}_{do}$ and $\mathcal{O}_{co}$, respectively. By Theorem 3.4 we have

$$V\mathcal{O}_{do}x = \mathcal{L}\mathcal{O}_{co}x, \quad x \in X,$$

where $V : H_Y^2(\mathbb{D}) \to H_Y^2(RHP)$ is the unitary transformation as defined in Proposition 3.3, and $\mathcal{L}$ is the Laplace transform.

In [30] (see also [24]) it has been shown that there is a parbalanced realization $(A_{db}, B_{db}, C_{db}, D_{db})$ of G_d with state space X that satisfies the following:

$$\mathcal{W}^{1/4}A_{db} = A_{do}\mathcal{W}^{1/4}$$

and

$$\mathcal{O}_{db} = \mathcal{W}^{1/4},$$

where $\mathcal{W} = H_{G^\perp}H_{G^\perp}^*|_X$, and $\mathcal{O}_{db}$ is the observability operator of $(A_{db}, B_{db}, C_{db}, D_{db})$.

Let $(A_{cb}, B_{cb}, C_{cb}, D_{cb}) = T((A_{db}, B_{db}, C_{db}, D_{db}))$, where T is the transformation defined in Theorem 3.1. Since $D(A_{cb}) = \text{range}(A_{db} + I)$ and $D(A_{co}) = \text{range}(A_{do} + I)$, we have, by the equality $\mathcal{W}^{1/4}A_{db} = A_{do}\mathcal{W}^{1/4}$, that $\mathcal{W}^{1/4}D(A_{cb}) \subseteq D(A_{co})$. By Theorem 3.4, $(A_{cb}, B_{cb}, C_{cb}, D_{cb})$ is a parbalanced realization of G_c and for the observability operator $\mathcal{O}_{cb}$ of $(A_{cb}, B_{cb}, C_{cb}, D_{cb})$ we have

$$\mathcal{L}\mathcal{O}_{cb}x = V\mathcal{O}_{db}x = V\mathcal{W}^{1/4}x, \quad x \in X.$$

Notice that in fact by Theorem 5.1 we have $\mathcal{O}_{do} = I_X$. Thus

$$\mathcal{L}\mathcal{O}_{cb}x = V\mathcal{O}_{do}\mathcal{W}^{1/4}x = \mathcal{L}\mathcal{O}_{co}\mathcal{W}^{1/4}x, \quad x \in X.$$

Since $\mathcal{L}$ is unitary, this shows that $\mathcal{O}_{cb}x = \mathcal{O}_{co}\mathcal{W}^{1/4}x$ for $x \in X$. By the definition of $\mathcal{O}_{co}$ and $\mathcal{O}_{cb}$ we have

$$C_{cb}e^{tA_{cb}}x = C_{co}e^{tA_{co}}\mathcal{W}^{1/4}x, \quad x \in D(A_{cb}).$$

Note that C_{cb} is a bounded operator from $(D(A_{cb}), \|\cdot\|_{A_{cb}})$ to Y (see Definition 2.1) and C_{co} has the analogous property. For $x \in D(A_{cb})$ the function $e^{tA_{cb}}x$ is continuous in t in the graph norm $\|\cdot\|_{A_{cb}}$. Similarly, since $\mathcal{W}^{1/4}x \in D(A_{co})$ for $x \in D(A_{cb})$, $e^{tA_{co}}\mathcal{W}^{1/4}x$ is continuous in t in the graph norm $\|\cdot\|_{A_{co}}$. Therefore both $C_{cb}e^{tA_{cb}}x$ and $C_{co}e^{tA_{co}}\mathcal{W}^{1/4}x$ are continuous in t in the norm of Y. Taking $t = 0$, we have

$$C_{cb}x = C_{co}\mathcal{W}^{1/4}x, \quad x \in D(A_{cb}).$$

Since by assumption C_{co} and hence $C_{co}\mathcal{W}^{1/4}$ are bounded, the operator

$$C_{cb}|_{A_{cb}} : D(A_{cb}) \to Y$$

is bounded, where $D(A_{cb})$ is equipped with the norm of X. As $D(A_{cb})$ is dense in X, C_{cb} can be boundedly extended to X.

To complete the proof we just note that the restricted shift realization is unitarily equivalent to any output-normal realization of G_c and that all parbalanced realizations of G_c are equivalent. □

COROLLARY 8.10. *If the input operator of an input-normal realization of* $G_c \in TLC^{U,Y}$ *is bounded, then the input operator of a parbalanced realization of* G_c *is bounded.*

Proof. Let (A_*, B_*, C_*, D_*) be the *-restricted shift realization of G_c and let (A_o, B_o, C_o, D_o) be its dual realization. Then (A_o, B_o, C_o, D_o) is the restricted shift realization of $\tilde{G}_c$. By the assumption, the operator B_* is bounded. Hence so is the operator C_o. By Proposition 8.9 the output operator of a parbalanced realization of $\tilde{G}_c$ is bounded. Consider the dual system (A, B, C, D) of this parbalanced realization of $\tilde{G}_c$. We have B to be bounded. Notice that the dual system of a parbalanced realization of $\tilde{G}_c$ is a parbalanced realization of G_c. Therefore the input operator of any parbalanced realization of G_c is bounded. □

COROLLARY 8.11. *Let* G_c *be in* $TLC^{U,Y}$. *Assume that the Hankel operator* H_{G_c} *has closed range. Then the input (output) operator of a parbalanced realization of* G_c *is bounded if and only if there is a constant* $M > 0$ *such that*

$$\sup_{x>0} \int_{-\infty}^{+\infty} \|[G_c(x+iy) - G_c(+\infty)]u\|^2 dy \le (M\|u\|)^2 \ \textit{for any} \ u \in U,$$

$$\left(\sup_{x>0} \int_{-\infty}^{+\infty} \|[\tilde{G}_c(x+iy) - \tilde{G}c(+\infty)]v\|^2 dy \le (M\|v\|)^2 \ \textit{for any} \ v \in Y \right).$$

Proof. Since the Hankel operator H_{G_c} has closed range, by Proposition 6.2 all input-normal, output-normal, and parbalanced realizations of G_c are equivalent. The corollary then follows from Proposition 8.4. □

9. Examples.

Example 1: Rational transfer function. Let $g(s)$ be a scalar-valued rational proper transfer function in $H^\infty(RHP)$, i.e., $g(s)$ has all its poles in the open left half plane.

Note that $g(s)$ has, up to a unitary scalar, a unique factorization as

$$g(s) = q(s)f(-s),$$

where $q(s)$ is an inner function, i.e., a Blaschke product with poles in LHP, and $f(s)$ is a rational function in $H^\infty(RHP)$, i.e., a proper rational function with poles in LHP. The functions $q(s)$ and $f(s)$ are strongly coprime, which is for rational functions equivalent to both functions not having common zeros in the extended RHP, i.e., $\{s \in \mathbb{C} \mid Re(s) \ge 0\} \cup \{\infty\}$.

The Blaschke product q is determined by the poles of g. For example if

$$g(s) = \frac{(s-1)(s+2)}{(s+3)(s+4)(s+5)},$$

then the Blaschke product is given by

$$q(s) = \frac{(s-3)(s-4)(s-5)}{(s+3)(s+4)(s+5)}$$

and

$$f(s) = \frac{(s+1)(s-2)}{(s+3)(s+4)(s+5)}.$$

It follows from the results in §6 that the state space of the restricted and *-restricted shift realization of the transfer function g is given by

$$X = (qH^2(RHP))^\perp.$$

Note that by Kronecker's theorem (see, e.g., [22]) X is a finite-dimensional space with dimension equal to the number of zeros or poles (counted with multiplicities) of the Blaschke product. From the construction it is clear that the Blaschke product is completely determined by the poles of the transfer function. Hence we have recovered the well-known result that the dimension of a minimal state-space realization equals the number of poles of the transfer function.

Example 2: Delay system with strictly proper rational part. In this example we consider single-input single-output delay systems. We continue with the notation in the above example and let the transfer function have the form $g_1(s) = e^{-\alpha s}g(s)$ with $\alpha > 0$. Let $p(s) = e^{-\alpha s}q(s)$. Clearly p is in $H^\infty(RHP)$ and inner. Later we will show that in fact p and f are weakly coprime. For now assume that this is true. Thus by Theorem 4.8 g_1 is strictly noncyclic, and by Proposition 5.11 the state space X of the restricted shift realization (A_c, B_c, C_c, D_c) has the form

$$X = H^2(RHP) \ominus pH^2(RHP).$$

The domain of A_c is $D(A_c) = \{\frac{x(s)-x(1)}{1-s} \mid x \in X\}$. Hence for $h \in D(A_c)$ we will have $h(s) = \frac{x(s)-x(1)}{1-s}$ for some $x \in X$, $\lim_{r\in\mathbb{R}, r\to+\infty} rh(r) = x(1)$ and

$$(A_c h)(s) = sh(s) - \lim_{r\in\mathbb{R}, r\to+\infty} rh(r) = sh(s) - x(1).$$

Note that g_1 satisfies the condition in Proposition 8.4. So the operator B_c is defined as

$$(B_c u)(s) = \frac{1}{\sqrt{2\pi}}[g_1(s) - g_1(+\infty)]u = \frac{1}{\sqrt{2\pi}} g_1(s)u, \qquad u \in \mathbb{C},$$

and B_c is bounded. Hence $(I - A_c)^{-1}B\mathbb{C} \subseteq D(A_c)$ and

$$D(C_c) = D(A_c) + (I - A_c)^{-1}BU = D(A_c).$$

We have, for $h \in D(A_c)$,

$$C_c h = \sqrt{2\pi} \lim_{r\in\mathbb{R}, r\to+\infty} rh(r).$$

Note that because $\alpha \neq 0$, by Corollary 8.8 C_c is unbounded.

The operator D_c is $D_c = g_1(+\infty) = 0$.

We can directly verify that this is a realization of g_1. Let $\xi \in RHP$. An easy calculation will show that for $h \in D(A_c)$

$$((\xi I - A_c)^{-1}h)(s) = \frac{h(s) - h(\xi)}{\xi - s}.$$

(We remark here that this formula is true in general, not just for this particular example.) Then

$$((\xi I - A_c)^{-1}B_c u)(s) = \frac{1}{\sqrt{2\pi}}(\xi I - A_c)^{-1}g_1(s)u = \frac{1}{\sqrt{2\pi}}\frac{g_1(s) - g_1(\xi)}{\xi - s}.$$

Hence

$$C_c(\xi I - A_c)^{-1}B_c u = \lim_{r\in\mathbb{R},\, r\to+\infty} r\frac{g_1(r) - g_1(\xi)}{\xi - r} = g_1(\xi).$$

This realization is exponentially stable by Theorem 7.11 since g_1 is clearly analytic on $Re(s) > -3$. It also follows from Theorem 7.11 that the degree of stability is $-3 = \max\{s : s \text{ is a pole of } g\}$. Consequently the parbalanced realization will also be exponentially stable with the same degree of stability. Notice that g_1 is continuous in the extended $i\mathbb{R}$. Hence the Hankel operator H_{g_1} is compact. Therefore by Theorem 6.1 there exists a balanced realization.

To show that p and f are weakly coprime, consider the closed linear span $S := pH^2(RHP) \vee fH^2(RHP)$. We need to show that $S = H^2(RHP)$. The space S is obviously a (right) invariant subspace of $H^2(RHP)$. Hence by Beurling's theorem [22] there is an inner function $\Theta \in H^\infty(RHP)$ such that

$$S = \Theta H^2(RHP).$$

Hence $pH^2(RHP) \subseteq \Theta H^2(RHP)$ and $\overline{fH^2(RHP)} \subseteq \Theta H^2(RHP)$. Let $q_1(s) = \frac{s-2}{s+2}$ (which is the inner part of the inner-outer factorization of f; see [22, p. 11]). Then

$$q_1 H^2(RHP) = \overline{fH^2(RHP)}.$$

So by [22, Cor. 5, p. 13] we must have that p/Θ and q_1/Θ are both inner functions. Note that $\Theta(2) \neq 0$ since otherwise $h(2) = 0$ for any $h \in pH^2(RHP) \subseteq \Theta H^2(RHP)$, and this is certainly not true. Thus the inner function $q_1(s)/\Theta(s)$ has a zero at 2. Hence the function $\frac{s+2}{s-2}\frac{q_1(s)}{\Theta(s)}$ will still be in $H^\infty(RHP)$. That is, $1/\Theta \in H^\infty(RHP)$. Hence $H^2(RHP) = \Theta(1/\Theta)H^2(RHP) \subseteq \Theta H^2(RHP) = S$.

Note that exactly the same argument in this example will apply for any transfer function $g_1 = e^{-\alpha s}g(s)$, where g is a stable and strictly proper rational function and $\alpha > 0$. Also, in a similar manner we can obtain the *-restricted shift realization which will have bounded output operator and has the same stability properties as the restricted shift realization.

We summarize these as follows.

PROPOSITION 9.1. *If a scalar transfer function G has the form $G(s) = e^{-\alpha s}g(s)$, $\alpha > 0$, where g is a stable and strictly proper rational function, then*

1. *G has a balanced realization;*
2. *all reachable output-normal realizations of G have bounded input operator and unbounded output operator, whereas all observable input-normal realizations have bounded output operator and unbounded input operator;*
3. *all reachable and observable input- and output-normal realizations and all parbalanced realizations are exponentially stable with growth bound equal to* $\max\{Re(s) : s$ *is a pole of* $G\}$. □

Example 3: Delay system with not strictly proper rational part. When the rational transfer function g in the previous example is not strictly proper, the resulting realizations will be different: the input operator of the restricted shift realization is not necessarily bounded, and it is not clear whether there is a balanced realization of g_1 because the Hankel operator H_{g_1} is not compact. A parbalanced realization, however, exists by Theorem 6.1. We first consider the simplest case with $g(s) = 1$. This is a simple delay $g_1(s) = e^{-\alpha s}$ ($\alpha > 0$). The state space of the restricted shift realization is $X = H^2 \ominus e^{-\alpha s}H^2$, which is the image of the Laplace transform $\mathcal{L}$ on $L^2([0,\alpha])$. Let (A_c, B_c, C_c, D_c) be the restricted shift realization and let

$$(A, B, C, D) = (\mathcal{L}^{-1}A_c\mathcal{L}, \mathcal{L}^{-1}B_c, C_c\mathcal{L}, D_c).$$

We know that (see Theorem 5.7)

$$(e^{tA_c}f)(x) = f(x+t)|_{[0,\alpha]}, \qquad f \in L^2([0,\alpha]),\ x \in [0,\alpha], t \geq 0,$$

where $f(x+t)|_{[0,\alpha]} = f(t+x)$ if $t + x \in [0,\alpha]$ and 0 otherwise. Thus

$$Af = f', \qquad f \in D(A),$$

with $D(A) = \{x \in L^2([0,\alpha]) : x$ is absolutely continuous, $x' \in L^2([0,\alpha]),\ x(\alpha) = 0\}$. By Theorem 5.7, for $x \in D(A_c^*)$ and $u \in \mathbb{C}$,

$$\begin{aligned}
[B_c(u)](x) &= \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1-s}[G_c(s) - G_c(1)]u,\ (1 - A_c^*)x \right\rangle \\
&= \frac{1}{\sqrt{2\pi}}\left\langle \frac{1}{1-s}(e^{-\alpha s} - e^{-\alpha})u,\ (1 - A_c^*)x \right\rangle \\
&= \frac{1}{\sqrt{2\pi}}\left\langle \mathcal{L}^{-1}\frac{e^{-\alpha s} - e^{-\alpha}}{1-s}u,\ \mathcal{L}^{-1}(1 - A_c^*)\mathcal{L}\mathcal{L}^{-1}x \right\rangle \\
&= \langle e^{t-\alpha}u|_{[0,\alpha]},\ \mathcal{L}^{-1}(1 - A_c^*)\mathcal{L}\mathcal{L}^{-1}x\rangle_{L^2([0,\alpha])} \\
&= \langle e^{t-\alpha}u,\ (1 - A^*)\mathcal{L}^{-1}x\rangle_{L^2([0,\alpha])},
\end{aligned}$$

where $e^{t-\alpha}u|_{[0,\alpha]} = \left(\frac{1}{\sqrt{2\pi}}\mathcal{L}^{-1}\frac{e^{-\alpha s}-e^{-\alpha}}{1-s}u\right)(t)$ is $e^{t-\alpha}u$ for $t \in [0,\alpha]$ and 0 otherwise. This shows that for $x \in D(A^*) \subseteq L^2([0,\alpha])$,

$$[B(u)](x) = [\mathcal{L}^{-1}B_c u](x) = [\mathcal{L}^* B_c u](x) = [B_c u](\mathcal{L}x) = \langle e^{t-\alpha}u,\ (1 - A^*)x\rangle_{L^2([0,\alpha])}.$$

It can be shown that

$$D(A^*) = \{x \in L^2([0,\alpha]) : x \text{ is absolutely continuous}, x' \in L^2([0,\alpha]),\ x(0) = 0\},$$

and $A^*x = -x'$ for $x \in D(A^*)$. Hence

$$[B(u)](x) = \langle e^{t-\alpha}u,\ (1 - A^*)x\rangle_{L^2([0,\alpha])} = \langle e^{t-\alpha}u, x + x'\rangle_{L^2([0,\alpha])} = ux(\alpha).$$

Since for $x \in D(C_c)$,

$$C_c x = \sqrt{2\pi}\lim_{\substack{r\in\mathbb{R}\\ r\to\infty}} r x(r),$$

we have for $x \in D(C) \subseteq L^2([0,\alpha])$,

$$Cx = C_c\mathcal{L}x = \sqrt{2\pi}\lim_{\substack{r\in\mathbb{R}\\ r\to\infty}} r(\mathcal{L}x)(r) = \lim_{\substack{\lambda\to 0\\ \lambda>0}} x(\lambda) = x(0).$$

Finally, $D_c = g(+\infty) = 0$.

This realization is, by Theorem 7.11, exponentially stable. In fact, the spectrum of e^{tA} is $\{0\}$ ($t > 0$). The operators B and C are both unbounded.

Now consider the factorization $e^{-\alpha s} = qf^*$, where $q(s) = e^{-\alpha s}$ and $f(s) = 1$. Clearly this is a strongly coprime factorization. Therefore by Proposition 6.2 all reachable and observable realizations of $e^{-\alpha s}$ are equivalent. This shows that all reachable and observable realizations are exponentially stable and have unbounded input, output, and state propagation operators.

As in the previous example, we can generalize this result.

PROPOSITION 9.2. *If a scalar transfer function G has the form $G(s) = e^{-\alpha s}g(s)$, where g is a stable proper rational function and $g(\infty) \neq 0$, $\alpha > 0$, then*

1. *all reachable and observable admissible realizations of G are equivalent;*
2. *if (A, B, C, D) is a reachable and observable admissible realization of G, then the operators A, B, and C are all unbounded;*
3. *every reachable and observable admissible realization of G is exponentially stable with growth bound equal to* $\max\{Re(s) : s$ *is a pole of* $G\}$.

Proof. Since g is a stable proper rational function, g has a factorization $g = qf^*$ such that q and f are stable proper rational and strongly coprime (see Theorem 4.10 for the definition of strong coprimeness). Hence

$$\inf_{s \in RHP} [|q(s)| + |f(s)|] > 0.$$

Since $g(\infty) \neq 0$, we must have that $f(\infty) \neq 0$. Therefore

$$\inf_{s \in RHP} [|q(s)e^{-\alpha s}| + |f(s)|] > 0.$$

This, by the Corona theorem (see [22, p. 66]), shows that $qe^{-\alpha s}$ and f are strongly coprime. So by Theorem 4.10 the Hankel operator H_G has closed range and by Proposition 6.2 all reachable and observable realizations of G are equivalent. Thus 1. is proven.

Since G is not analytic at infinity, by Theorem 8.2 the state propagation operator of any reachable output-normal realization is unbounded. Note that in the factorization $G = (qe^{-\alpha s})f^*$ the inner function does not satisfy condition 3 in Theorem 8.6 because now $\alpha \neq 0$. Therefore by Corollary 8.8 the output operator of the restricted shift realization and the input operator of the *-restricted shift realization are unbounded. Thus 2. follows from 1.

Since G is strictly noncyclic and

$$\begin{aligned} \inf\{\alpha : \; & G(s) \text{ has analytic continuation on } Re(s) > \alpha\} \\ &= \max\{Re(s) : s \text{ is a pole of } g\} \\ &< 0, \end{aligned}$$

by Theorem 7.11 all reachable output-normal realizations of G are exponentially stable with growth bound $\max\{Re(s) : s \text{ is a pole of } g\}$. As equivalent systems have the same exponential stability property and growth bound, 3. also follows from 1. □

Example 4: Systems with infinite Blaschke product. In this example we consider transfer functions of the form $g(s) = R(s)B(s)$, where $R(s)$ is a proper rational function in $H^\infty(RHP)$ and $B(s)$ is an infinite Blaschke product also in $H^\infty(RHP)$. We assume that there is no pole-zero cancellation. That is, the zeros of $R(s)$ ($B(s)$) do not coincide with any of the poles of $B(s)$ (respectively, $R(s)$). We point out that B has the form

$$B(s) = \prod_{n=1}^{\infty} \frac{|1-\beta_n^2|}{1-\beta_n^2} \frac{s-\beta_n}{s+\bar{\beta}_n},$$

where $\frac{|1-\beta_n^2|}{1-\beta_n^2}$ is assumed to be 1 if $\beta_n = 1$. The zeros $\beta_n (n = 1, 2, \ldots)$ of B satisfy the condition (see [17])

$$\sum_{n=1}^{\infty} \frac{Re(\beta_n)}{1+|\beta_n|^2} < \infty.$$

Note that either infinity is an accumulation point of the zeros (and the poles) of B, or else, the zeros (and the poles) of B are bounded and have accumulation points which are on the imaginary line.

First we consider the case that $R(s)$ is not strictly proper and the zeros of $R(s)$ do not coincide with any of the accumulation points of the poles of $B(s)$.

Write $R(s) = n(s)/d(s)$, where $n(s)$ and $d(s)$ are coprime polynomials. Then we have

$$g(s) = \frac{d^*(s)}{d(s)} B(s) \frac{n(s)}{d^*(s)},$$

where $d^*(s) = \overline{d(-\bar{s})}$. Set $q(s) = \frac{d^*(s)}{d(s)}B(s)$ and $f(s) = \frac{n^*(s)}{d(s)}$. We have $g = qf^*$. The inner function $q(s)$ is again a Blaschke product and $f(s)$ is in $H^\infty(RHP)$ and rational. Furthermore, from the assumption on $R(s)$ and $B(s)$ it follows that the zeros of $f(s)$ do not coincide with any of the zeros or accumulation points of the zeros of $q(s)$. Thus we must have

$$\inf_{s\in RHP}|f(s)| + |q(s)| > 0.$$

This shows that g has a strongly coprime Douglas–Shapiro–Shields factorization. Hence the Hankel operator H_g has closed range. Thus by Proposition 6.2 all reachable and observable admissible realizations of g are equivalent. Therefore all these realizations are asymptotically stable. They are exponentially stable if and only if there exists $\alpha > 0$ such that g is analytic on $Re(s) > -\alpha$. Since $R(s)$ is rational and in $H^\infty(RHP)$, we know that g is analytic on $Re(s) > -\alpha$ for some $\alpha > 0$ if and only if there is $\lambda > 0$ such that $B(s)$ is analytic on $Re(s) > -\lambda$. Note that the last condition on $B(s)$ is equivalent to that there is $\lambda > 0$ such that $Re(\beta_n) > \lambda, n = 1, 2, \ldots$.

By Corollary 8.8 we know that the input and output operators of any reachable and observable admissible realization of g are bounded if and only if $\sum Re(\beta_n) < \infty$.

The second case is that $R(s)$ is strictly proper, no zero of $R(s)$ coincides with any accumulation point of the poles of $B(s)$, and infinity is not an accumulation point of the poles of $B(s)$. In this case B is analytic at infinity and the poles of B have accumulation points on the imaginary line. As in the first case, g has a strongly coprime factorization and hence H_g has closed range. Thus all reachable and observable admissible realizations of g are equivalent and asymptotically stable. However, no reachable and observable realization of g is exponentially stable, since the poles of B have accumulation points on the imaginary line and hence g is not analytic on $Re(s) > -\alpha$ for any $\alpha > 0$.

Since in this case we have $g \in H^2(RHP)$ by Proposition 8.4, the input and output operators of any reachable and observable realization of g are bounded.

The third case is that $R(s)$ is strictly proper, no zero of $R(s)$ coincides with any accumulation point of the poles of $B(s)$, and infinity is an accumulation point of the poles of $B(s)$. In this case we can show as was done in Example 2 that the factorization of g in the first case is a weakly coprime factorization. Hence g is strictly noncyclic. Thus all input-normal, output-normal, and parbalanced realizations of g are asymptotically stable. As in the first case, an input-normal, an output-normal, or a parbalanced realization of g is exponentially stable if and only if there exists $\lambda > 0$ such that $Re(\beta_n) > \lambda$, $(n = 1, 2, \ldots)$.

From Corollary 8.8 it follows that the input operator of an input-normal realization or the output operator of an output-normal realization is bounded if and only if $\sum Re(\beta_n) < \infty$. Thus by Proposition 8.9 and Corollary 8.10 the input operator and output operator of any parbalanced realization of g are bounded if $\sum Re(\beta_n) < \infty$.

Since clearly $g \in H^2(RHP)$, by Proposition 8.4 the input operator of an output-normal realization and the output operator of an input-normal realization of g are bounded. If in addition no accumulation point of the poles of $B(s)$ is on the imaginary line, then g is continuous in the extended imaginary line and therefore g has a balanced realization.

We observe that in this case an output-normal realization cannot have a bounded output operator and still be exponentially stable. An analogous fact holds for an input-normal realization and its input operator.

The fourth and final case is that at least one of the zeros of $R(s)$ coincides with an accumulation point of the poles of $B(s)$. Note that this accumulation point must be on the imaginary line.

As in the previous case, the factorization of g in the first case is a weakly coprime factorization. Hence g is strictly noncyclic. Thus all input-normal, output-normal, and parbalanced realizations of g are asymptotically stable. They are not exponentially stable because g is not analytic on $Re(s) > -\alpha$ for any $\alpha > 0$.

Again by Corollary 8.8 the input operator of an input-normal realization or the output operator of an output-normal realization is bounded if and only if $\sum Re(\beta_n) < \infty$. Thus by Proposition 8.9 and Corollary 8.10 the input operator and output operator of any parbalanced realization of g are bounded if $\sum Re(\beta_n) < \infty$.

If every accumulation point of the poles of B is a zero of R, then g is continuous on the extended imaginary line. Hence g has a balanced realization.

We now summarize the results as follows.

PROPOSITION 9.3. *Consider $g(s) = R(s)B(s)$, where $R(s)$ is a proper rational function and $B(s)$ is an infinite Blaschke product, both in $H^\infty(RHP)$, and B and R have no pole-zero cancellation.*

1. *If $R(s)$ is not strictly proper and no zero of $R(s)$ coincides with any accumulation point of the poles of $B(s)$, then*
   (a) *all reachable and observable admissible realizations of g are equivalent;*
   (b) *all reachable and observable admissible realizations of g are asymptotically stable;*
   (c) *all reachable and observable admissible realizations of g are exponentially stable if and only if there exists $\alpha > 0$ such that $Re(\beta_n) > \alpha$, $n = 1, 2, \ldots$, where β_n, $n = 1, 2, \ldots$, are the zeros of $B(s)$;*
   (d) *all reachable and observable admissible realizations of g have bounded input and output operators if and only if $\sum Re(\beta_n) < \infty$.*
2. *If $R(s)$ is strictly proper, no zero of $R(s)$ coincides with any accumulation point of the poles of $B(s)$, and infinity is not an accumulation point of the poles of $B(s)$, then*
   (a) *all reachable and observable admissible realizations of g are equivalent;*
   (b) *all reachable and observable admissible realizations of g are asymptotically stable;*
   (c) *no reachable and observable admissible realization of g is exponentially stable;*
   (d) *all reachable and observable admissible realizations of g have bounded input and output operators.*
3. *If $R(s)$ is strictly proper, no zero of $R(s)$ coincides with any accumulation point of the poles of $B(s)$, and infinity is an accumulation point of the poles of $B(s)$, then*
   (a) *all input-normal, output-normal, and parbalanced realizations of g are asymptotically stable;*
   (b) *all input-normal, output-normal, and parbalanced realizations of g are exponentially stable if and only if there exists $\alpha > 0$ such that $Re(\beta_n) > \alpha$ $(n = 1, 2, \ldots)$;*
   (c) *the input operator of an input-normal realization or the output operator of an output-normal realization of g is bounded if and only if $\sum Re(\beta_n) < \infty$. The input operator and output operator of any parbalanced realization of g are bounded if $\sum Re(\beta_n) < \infty$;*
   (d) *the input operator of an output-normal realization and the output operator an input-normal realization of g are bounded.*

   If, in addition, no accumulation point of the poles of B is on the imaginary line, then g has a balanced realization.
4. *If at least one of the zeros of R coincides with an accumulation point of the poles of B, then*
   (a) *all input-normal, output-normal, and parbalanced realizations of g are asymptotically stable;*
   (b) *no input-normal, output-normal, or parbalanced realization of g is exponentially stable;*
   (c) *the input operator of an input-normal realization or the output operator of an output-normal realization of g is bounded if and only if $\sum Re(\beta_n) < \infty$. The input operator and output operator of any parbalanced realization of g are bounded if $\sum Re(\beta_n) < \infty$.*

If every accumulation point of the poles of B is a zero of R, then g has a balanced realization.

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