nger

lberg
ork
ona
est
Kong
n

Clara
ore

Identification, Adaptation, Learning

The Science of Learning Models from Data

Edited by

Sergio Bittanti

Politecnico di Milano
Piazza Leonardo da Vinci 32
I-20133 Milano, Italy

Giorgio Picci

Università di Padova
Via Gradenigo 6/A
I-35131 Padova, Italy

Springer

Published in cooperation with NATO Scientific Affairs Division

Proceedings of the NATO Advanced Study Institute "From Identification to Learning", held in Como, Italy, August 22 – September 2, 1994

Library of Congress Cataloging-in-Publication Data

Identification, adaptation, learning : the science of learning models from data / edited by Sergio Bittanti, Giorgio Picci.
p. cm. -- (NATO ASI series. Series F, Computer and systems sciences ; vol. 153.)
"Published in cooperation with NATO Scientific Affairs Division.
"Proceedings of the NATO Advanced Study Institute 'From Identification to Learning', held in Como, Italy, August 22-September 2, 1994"--T.p. verso.
Includes bibliographical references.
ISBN 3-540-61080-4 (hardcover : alk. paper)
1. System identification--Congresses. 2. Linear models (Statistics)--Congresses. 3. Learning models (Stochastic processes)--Congresses. I. Bittanti, Sergio. II. Picci, Giorgio, 1942- . III. North Atlantic Treaty Organization. Scientific Affairs Division. IV. NATO Advanced Study Institute 'From Identification to Learning' (1994 : Como, Italy) V. Series: NATO ASI series. Series F, Computer and systems sciences ; no. 153.
QA402.I327 1996
003'.74'0111--dc20 96-14783
CIP

CR Subject Classification (1991): I.5–6, G.1, G.3, I.2, J.2

ISBN 3-540-61080-4 Springer-Verlag Berlin Heidelberg New York

Printed in Germany

Typesetting: Camera-ready by authors
Printed on acid-free paper
SPIN: 10486185 45/3142 – 5 4 3 2 1 0

Dedicated to the men
E.J. Hannan (1921

Photograph by Pete
taken at the farewell to Geof V
Princeton Universit

Balanced Canonical Forms*

Raimund J. Ober

Center for Engineering Mathematics
University of Texas at Dallas
Richardson, TX75083, USA
E-mail: ober@utdallas.edu

1. Introduction

Canonical forms for linear systems are of importance since they provide a unique state space representation of linear systems. They therefore play a major role in system identification where a unique parametrization of the systems in the model set is essential to avoid identifiability problems. Various types of canonical forms for linear systems have been introduced and studied (see e.g. [31], [8], [13], [9]). Most of these canonical forms for multi-variable systems are generalizations of the observer or controller canonical form for single-input single-output systems. The purpose of this paper is to review canonical forms that are based on balanced realizations.

The usefulness of a canonical form depends on its properties. One of the standard canonical forms, the controller canonical form, is of particular significance since the parameters of the canonical form have an immediate interpretation as the coefficients of the transfer function. Moreover, this canonical form permits a straightforward proof of the pole-shifting theorem. There are, however, drawbacks of the controller canonical form especially concerning the resulting parametrization of linear systems. The set of parameters in the controller canonical form that lead to a minimal system is very complicated. This makes it difficult to use this canonical form in cases where it is important to have a geometrically well-behaved parameter space, e.g. in some optimization tasks. One of the main advantages of the balanced canonical forms that are discussed here, is that the parameter space has some desirable geometric properties. This is at the expense that even for single input single output systems discrete structural parameters have to be introduced.

The results presented in this paper extend the previous results ([21], [28], [26]) also to the case of systems with coefficients in the complex field C. In the multivariable case we use a modification due to Hanzon ([10]) of the previously published canonical forms for systems with identical singular values. The approach to the proofs of the canonical form for minimal systems, bounded real systems and positive real systems is also new and based on the bijections that were introduced in ([27]). The objective of this paper is to present these results in a comprehensive form and to derive them in a unified

* Dedicated to the memory of Ted Hannan.

way. Because of this tutorial nature of the presentation proofs are given of all essential results even if the proofs have been published already elsewhere.

For each of the classes of transfer functions which we consider we will now define a specific type of balanced realization. To simplify the presentation we introduce the following notation. The set of all p-dimensional output and m-dimensional input minimal continuous-time systems of McMillan degree n is denoted by $L_n^{p,m}$. We call a continuous-time system (A,B,C,D) *stable* if all the eigenvalues of A are in the open left half plane. The subset of $L_n^{p,m}$ of all stable systems is denoted by $S_n^{p,m}$. The subsets of $S_n^{p,m}$ of inner systems and bounded real systems are denoted by $I_n^{p,m}$ and $B_n^{p,m}$. A system (A,B,C,D) in $S_n^{p,m}$ with transfer function G is called *inner* if $(G(s))^*G(s)=I$ for all $s \in C$. It is called *bounded-real* if $I-(G(i\omega))^*+G(i\omega)>0$ for all $\omega \in \Re \cup \{\pm\infty\}$. If $p=m$, then P_n^m stands for the subset of $S_n^{m,m}$ of positive real systems. A system (A,B,C,D) in $S_n^{m,m}$ with transfer function G is called *positive-real* if $(G(i\omega))^*+G(i\omega)>0$ for all $\omega \in \Re\cup\{\pm\infty\}$. In this paper we will study systems with coefficients in the real field $\Re$ and in the complex field C. If a statement is valid for both situations we will use the symbol $\mathcal{K}$ to denote either $\Re$ or C. One of our aims is to study *canonical forms* for these classes of systems in terms of balanced realizations. Two systems $(A_i,B_i,C_i,D_i) \in L_n^{p,m}$, $i=1,2$, are called equivalent if there exists a nonsingular matrix $T \in \mathcal{K}^{n\times n}$ such that $(A_1,B_1,C_1,D_1)=(TA_2T^{-1},TB_2,C_2T,D_2)$.

Definition 1.1. *A* canonical form *for system equivalence on a subset $\mathcal{A} \subseteq L_n^{p,m}$ is a map*

$$\Gamma : \mathcal{A} \to \mathcal{A},$$

such that

1. *$\Gamma(a) \sim a$ for all $a \in \mathcal{A}$.*
2. *if $a,b \in \mathcal{A}$, and $a \sim b$ then $\Gamma(a)=\Gamma(b)$.*

We also refer to $\Gamma(a)$ as the canonical form of $a \in \mathcal{A}$.

We now introduce the different types of balancing. The principle behind the definition of the different types of balancing is that associated with each class of systems there is a natural pair of Riccati or Lyapunov equations. A system is then called balanced if specified solutions of each of the two equations are identical and diagonal.

Definition 1.2. *1.* (**LQG-balancing**) *The system $(A,B,C,D) \in L_n^{p,m}$ is called* LQG-balanced *if the stabilizing solutions Y and Z to the control and filter algebraic Riccati equations,*

$$0=A_L^*Y+YA_L-YBR^{-1}B^*Y+C^*S^{-1}C,$$

$$0=A_LZ+ZA_L^*-ZC^*S^{-1}CZ+BR^{-1}B^*,$$

where $A_L:=A-BR^{-1}D^*C$, $R=I+D^*D$, $S=I+DD^*$, are such that

$$Y = Z = diag(\sigma_1, \sigma_2, \dots, \sigma_n) := \Sigma_L,$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. *The matrix* Σ_L *is called the* LQG-grammian *of the system. The positive numbers* $\sigma_1, \sigma_2, \dots, \sigma_n$ *are called the* LQG-singular values *of the system.*

2. *(***Lyapunov-balancing***) The system* $(A, B, C, D) \in S_n^{p,m}$ *is called* Lyapunov-balanced *if the positive definite solutions* Y *and* Z *to the Lyapunov equations,*

$$0 = AZ + ZA^* + BB^*,$$
$$0 = A^*Y + YA + C^*C,$$

are such that

$$Y = Z = diag(\sigma_1, \sigma_2, \dots, \sigma_n) := \Sigma_S,$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. *The matrix* Σ_S *is called the* Lyapunov-grammian *of the system. The positive numbers* $\sigma_1, \sigma_2, \dots, \sigma_n$ *are called the* Lyapunov-singular values *of the system.*

3. *(***Bounded-real-balancing***) The system* $(A, B, C, D) \in B_n^{p,m}$ *is called* bounded-real-balanced *if the stabilizing solutions* Y *and* Z *to the control and filter bounded-real Riccati equations,*

$$0 = A_B^*Y + YA_B + YBR^{-1}B^*Y + C^*S^{-1}C,$$
$$0 = A_BZ + ZA_B^* + ZC^*S^{-1}CZ + BR^{-1}B^*,$$

where $A_B := A + BR^{-1}D^*C$, $R = I - D^*D$, $S = I - DD^*$, *are such that*

$$Y = Z = diag(\sigma_1, \sigma_2, \dots, \sigma_n) := \Sigma_B,$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. *The matrix* Σ_B *is called the* bounded-real-grammian *of the system. The positive numbers* $\sigma_1, \sigma_2, \dots, \sigma_n$ *are called the* bounded-real-singular values *of the system.*

4. *(***Positive-real-balancing***) The system* $(A, B, C, D) \in P_n^m$ *is called* positive-real-balanced *if the stabilizing solutions* Y *and* Z *to the control and filter positive-real Riccati equations,*

$$0 = A_P^*Y + YA_P + YBR^{-1}B^*Y + C^*R^{-1}C,$$
$$0 = A_PZ + ZA_P^* + ZC^*R^{-1}CZ + BR^{-1}B^*,$$

where $A_P := A - BR^{-1}C$, $R = D + D^*$, *are such that*

$$Y = Z = diag(\sigma_1, \sigma_2, \dots, \sigma_n) =: \Sigma_P,$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. *The matrix* Σ_P *is called the* positive-real-grammian *of the system. The positive numbers* $\sigma_1, \sigma_2, \dots, \sigma_n$ *are called the* positive-real-singular values *of the system.*

The definition of LQG balancing is due to Jonckheere and Silverman ([14]) and Verriest ([33]). The notion of Lyapunov balancing has been introduced by the author to distinguish this type of balancing from others that were defined later. Balancing for stable systems has been considered by many authors for different purposes. Roberts and Mullis ([20]) studied these types of realizations because of their good sensitivity properties for the implementation of filters. The good behavior of Lyapunov balanced realizations from the point of view of model reduction has apparently first been recognized by Moore ([19]). For the definition of positive-real balancing see the work by Desai and Pal (e.g [2]). Bounded-real balancing has been considered by Opdenacker and Jonckheere ([29]).

The existence of these various types of balanced realizations is established in the following theorem.

Theorem 1.1. *Let* G *be a proper rational (stable, antistable, bounded-real, positive-real) function. Then* G *has a LQG- (Lyapunov-, bounded-real-, positive-real-) balanced realization.*

If (A, B, C, D) *is a LQG- (Lyapunov-, bounded-real-, positive-real-) balanced realization of* G *with grammian* $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \dots, \sigma_k I_{n_k})$, $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$, *then all other LQG- (Lyapunov-, bounded-real-, positive-real-) balanced realizations over* $\mathcal{K}$ *are given by*

$$(QAQ^*, QB, CQ^*, D),$$

where Q *is a constant unitary matrix over* $\mathcal{K}$ *with* $Q = diag(Q_1, Q_2, \dots, Q_k)$, *with* $Q_j \in \mathcal{K}^{n_j \times n_j}$, $j = 1, \dots, k$. *Moreover, all other LQG- (Lyapunov-, bounded-real-, positive-real-) balanced realizations of* G *have the grammian* Σ.

Proof. Let $(A, B, C, D) \in L_n^{p,m}$ be a realization of the proper rational function G. Let Y, Z be the stabilizing solutions to the control and filter algebraic Riccati equations. Assume that T is a state space transformation of the system (A, B, C, D). It is easily verified that the stabilizing solutions to the control and filter algebraic Riccati equations of the system $(TAT^{-1}, TB, CT^{-1}, D)$ are given by TZT^* and $T^{-*}YT^{-1}$. To show that there exists an invertible T which simultaneously diagonalizes Z and Y, first note that since Z is positive and hermitian there exists T_1 invertible, such that

$$T_1 Z T_1^* = I.$$

Since Y is positive and hermitian there exists T_2 unitary such that

$$T_2^{-*}T_1^{-*}YT_1^{-1}T_2^{-1} =: \Sigma^2 =: diag(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2),$$

for some $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Now with

$$T := \Sigma^{1/2} T_2 T_1$$

we have

$$TZT^* = \Sigma^{1/2}T_2T_1ZT_1^*T_2^*\Sigma^{1/2} = \Sigma^{1/2}T_2IT_2^*\Sigma^{1/2} = \Sigma^{1/2}I\Sigma^{1/2} = \Sigma$$

and

$$T^{-*}YT^{-1} = \Sigma^{-1/2}T_2^{-*}T_1^{-*}YT_1^{-1}T_2^{-1}\Sigma^{-1/2} = \Sigma^{-1/2}\Sigma^2\Sigma^{-1/2} = \Sigma,$$

which shows the first statement.

Let $(A_1, B_1, C_1, D) \in L_n^{p,m}$ be another LQG-balanced realization of G with LQG-grammian Σ_1. Then there exists a state space transformation T such that

$$(A_1, B_1, C_1, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

and

$$\Sigma_1 = T\Sigma T^* = T^{-*}\Sigma T^{-1}.$$

Then $\Sigma_1^2 = T\Sigma^2T^{-1}$. Hence Σ_1^2 and Σ^2 are equivalent and since both are diagonal with decreasing diagonal entries we have that $\Sigma_1^2 = \Sigma^2$. Since $\Sigma_1 = T\Sigma T^*$ is diagonal we have that T is unitary. As $\Sigma_1^2T = T\Sigma^2$ it is easily verified that T has the required structure. If a state space transformation Q is given as in the statement of the Theorem it can be checked in a straightforward way that the transformed system is again LQG-balanced.

The statements for the other types of balancing follow analogously. □

Since balanced realizations are not unique they do not define a canonical form. Much of this paper will be devoted to the introduction of further constraints on balanced realizations to obtain canonical forms.

One of the interesting facts of the canonical forms that are presented here is that they all have a very similar structure, called the balanced form. This is irrespective of the class of systems for which they are derived. It will be shown that balanced forms also provide parametrizations of the respective classes of systems. For example, a canonical form will be derived for the class of stable minimal systems $S_n^{p,m}$ which is given in terms of the so-called Lyapunov balanced form. Conversely, it will be shown that each system which is in Lyapunov balanced form is necessarily minimal and stable. This provides for a parametrization of the class $S_n^{p,m}$ using Lyapunov balanced realizations. Analogous results will be derived for the classes $L_n^{p,m}$, $B_n^{p,m}$ and P_n^m using the corresponding balanced systems.

In many areas of applications models of dynamic processes are given in terms of high dimensional linear systems. Often however the dimension of the model is too high for an efficient analysis of the system. If a high dimensional model is obtained e.g. of an electrical circuit cost considerations may prohibit the implementation. In these and similar situations the question arises

whether the high dimensional system can be approximated well by a low dimensional system. The 'goodness' of an approximation will of course depend on the application and a norm-based criterion will be discussed later.

A principal requirement of a model reduction scheme should be that important qualitative properties of the high order system be retained in the approximant. If for example stability is an important feature of the system, then the approximant should also be stable. Balanced realizations provide model reduction schemes that have such properties. For example the balanced model reduction scheme for positive real systems guarantees that the approximating system is again positive real. The *balanced model reduction scheme* is defined as follows. Let (A, B, C, D) be a LQG (Lyapunov, bounded-real, positive-real) system over $\mathcal{K}$ that is partitioned as follows,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

with $A_{11} \in \mathcal{K}^{r\times r}$, $B_1 \in \mathcal{K}^{r\times m}$, $C_1 \in {}^{p\times r}$ and $r < n$. The r–dimensional system (A_{11}, B_1, C_1, D) is then called the *r-dimensional LQG (Lyapunov, bounded-real, positive-real) balanced approximant* of (A, B, C, D).

One of the reasons for the interest that balanced realizations received is due their model reduction properties which will be discussed in later sections and their connection to Hankel norm approximation (see e.g. [7], [5]).

Lyapunov balanced realizations play a particularly important role in our development. Many properties of the other types of balanced realizations can be deduced from properties of Lyapunov balanced realizations. The necessary machinery for this process will be introduced in Sect. 3.. Model reduction properties of Lyapunov balanced realizations will be investigated in Sect. 2.. The introduction of canonical forms for various classes of systems is the topic of Sections 4. and 5..

Having analyzed a canonical form, parametrization and model reduction for stable minimal systems in Sect. 3., a bijection between $S_n^{p,m}$ and $L_n^{p,m}$ will be introduced in Sect. 4.. This bijection will be used to carry the results for $S_n^{p,m}$ over to $L_n^{p,m}$. In a similar way canonical forms, parametrizations and model reduction results will be derived for bounded-real and positive real systems in Sect. 5..

If the system (A, B, C, D) is a realization of the proper rational function G, i.e. $G(s) = C(sI - A)^{-1}B + D$, for $s \in \mathcal{C}$, then we write $G \stackrel{r}{=} (A, B, C, D)$. Occasionally we will also write $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ to denote the system (A, B, C, D).

This work was supported by NSF Grant DMS-9304696.

2. Lyapunov Balanced Realizations and Model Reduction

In this section we are going to introduce basic facts concerning model reduction properties of Lyapunov balanced realizations. We are going to prove a result by Pernebo and Silverman [30] that shows that if (A,B,C,D) is a Lyapunov balanced system and (A_{11},B_1,C_1,D) is its r dimensional balanced approximant then the approximant is stable, minimal and again Lyapunov balanced. This is the case if a mild condition on the singular values is satisfied. In Theorem 3.3, it will be shown that this condition can be dropped if the system is in Lyapunov balanced canonical form.

The following Lemma will be needed frequently.

Lemma 2.1. *Let (A,B,C,D) be a continuous-time system and let P (Q) be a positive definite solution to the Lyapunov equation,*

$$AP+PA^*=-BB^* \quad (A^*Q+QA=-C^*C),$$

Then the system is stable if and only if it is reachable (observable).

Proof. That reachability (observability) implies stability is a standard result. Let now the system be stable and let x be an eigenvector of A^* with eigenvalue λ, i.e. $A^*x=\lambda x$. Then

$$0\geq -<BB^*x,x>=<(AP+PA^*)x,x>=<Px,A^*x>+<PA^*x,x>$$

$$=\bar\lambda<Px,x>+\lambda<Px,x>=2Re(\lambda)<Px,x>.$$

Since P is positive definite $<Px,x>>0$. The stability of the system implies that

$$Re(\lambda)<0.$$

Hence $<B^*x,B^*x>>0$ which implies that $B^*x\neq 0$. This implies the reachability of the system. The corresponding implication on observability of the system follows analogously. □

Theorem 2.1. **(Pernebo-Silverman)** *Let $(A,B,C,D)\in S_n^{p,m}$ be a n-dimensional Lyapunov balanced system with Lyapunov grammian $\Sigma=diag(\sigma_1,\sigma_2,\ldots,\sigma_n)$, $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_n\geq 0$. Let $r<n$ be such that $\sigma_{r+1}\neq\sigma_n$. Then the r-dimensional balanced approximant (A_{11},B_1,C_1,D) of (A,B,C,D) is stable, minimal and Lyapunov balanced with Lyapunov grammian $\Sigma_1=diag(\sigma_1,\sigma_2,\ldots,\sigma_r)$.*

Proof. Let $\Sigma_1=diag(\sigma_1,\ldots,\sigma_k)$. Then it is easily observed that Σ_1 solves the Lyapunov equations for the approximant (A_{11},B_1,C_1,D), i.e.

$$A_{11}\Sigma_1+\Sigma_1A_{11}^*=-B_1B_1^*,$$

$$A_{11}^*\Sigma_1+\Sigma_1A_{11}=-C_1^*C_1.$$

We need to show that A_{11} is stable. Let λ be an eigenvalue of A_{11}^* with eigenvector x, i.e.

$$A_{11}^*x=\lambda x.$$

Then

$$0\geq -<B_1^*x,B_1^*x>=-<B_1B_1^*x,x>=<(A_{11}\Sigma_1+\Sigma_1A_{11}^*)x,x>$$

$$=\lambda<\Sigma_1x,x>+\bar\lambda<\Sigma_1x,x>$$

$$=2Re(\lambda)<\Sigma_1x,x>.$$

Since $<\Sigma_1x,x>\geq 0$ this shows that $Re(\lambda)\leq 0$. Assume now that $\lambda=iy$, $y\in\Re$. Then the above calculation shows that

$$-<B_1^*x,B_1^*x>=2Re(\lambda)<\Sigma_1x,x>=0,$$

and therefore that $B_1^*x=0$. Multiplying the Lyapunov equation on the right by x we obtain,

$$0=-B_1B_1^*x=A_{11}\Sigma_1x+\Sigma_1A_{11}^*x=A_{11}\Sigma_1x+\lambda\Sigma_1x.$$

Hence,

$$A_{11}\Sigma_1x=-\lambda\Sigma_1x.$$

Using the equation

$$A_{11}^*\Sigma_1+\Sigma_1A_{11}=-C_1^*C_1,$$

we have,

$$-<C_1\Sigma_1x,C_1\Sigma_1x>=-<C_1^*C_1\Sigma_1x,\Sigma_1x>$$

$$=<(A_{11}^*\Sigma_1+\Sigma_1A_{11})\Sigma_1x,\Sigma_1x>=<\Sigma_1^2x,A_{11}\Sigma_1x>+<\Sigma_1A_{11}\Sigma_1x,\Sigma_1x>$$

$$=-\bar\lambda<\Sigma_1^2x,\Sigma_1x>-\lambda<\Sigma_1^2x,\Sigma_1x>=0,$$

since $\lambda+\bar\lambda=0$. Therefore $C_1\Sigma_1x=0$. Multiplying the Lyapunov equation $A_{11}^*\Sigma_1+\Sigma_1A_{11}=-C_1^*C_1$ on the right by Σ_1x we have

$$0=-C_1^*C_1\Sigma_1x=(A_{11}^*\Sigma_1+\Sigma_1A_{11})\Sigma_1x=A_{11}^*\Sigma_1^2x-\lambda\Sigma_1^2x,$$

i.e.

$$A_{11}^*\Sigma_1^2x=\lambda\Sigma_1^2x.$$

This shows that the eigenspace of A_{11}^* with eigenvalue λ is invariant under Σ_1^2. If $\Sigma_1=diag(\tilde\sigma_1I_{n_1},\tilde\sigma_2I_{n_2},\ldots,\tilde\sigma_lI_{n_l})$, where $\tilde\sigma_1>\tilde\sigma_2>\cdots>\tilde\sigma_l>0$, then the invariant subspaces of Σ_1^2 are subspaces of

$$E_i := diag(0I_{n_1+\cdots+n_{i-1}}, I_{n_i}, 0I_{n_{i+1}+\cdots+n_l})\mathcal{K}^r, \quad 1 \le i \le l.$$

Hence $x \in E_{i_0}$ for some $1 \le i_0 \le l$, i.e. $\Sigma_1^2 x = \sigma_{i_0}^2 x$. Now consider the (2,1) blocks of the Lyapunov equations of the original system (A, B, C, D), i.e.

$$A_{21}\Sigma_1 + \Sigma_2 A_{12}^* = -B_2 B_1^*,$$

$$A_{12}^*\Sigma_1 + \Sigma_2 A_{21} = -C_2^* C_1.$$

Multiplying the first equation on the right by x, on the left by Σ_2, and the second equation on the right by $\Sigma_1 x$ we obtain,

$$\Sigma_2 A_{21}\Sigma_1 x + \Sigma_2^2 A_{12}^* x = -\Sigma_2 B_2 B_1^* x = 0$$

$$A_{12}^*\Sigma_1^2 x + \Sigma_2 A_{21}\Sigma_1 x = -C_2^* C_1 \Sigma_1 x = 0.$$

Therefore,

$$\Sigma_2^2 A_{12}^* x = -\Sigma_2 A_{21}\Sigma_1 x = A_{12}^*\Sigma_1^2 x = \sigma_{i_0}^2 A_{12}^* x.$$

Since by assumption the diagonal entries of Σ_2^2 are distinct from $\sigma_{i_0}^2$ we have that $A_{12}^* x = 0$. Then with $\tilde{x} = (x^*, 0I_{n-r})^*$ we have

$$A^*\tilde{x} = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = iy\begin{pmatrix} x \\ 0 \end{pmatrix},$$

which is a contradiction to the stability of A^* and hence of A.

Reachability and observability of the system now follow from Lemma 2.1. □

The following result will give a quantitative bound on the size of the error that is incurred by the approximation process. The error is in term of the so-called H^∞ norm of the error function. The H^∞-norm of a stable proper rational transfer function F is given by

$$\|F\|_\infty = \sup_{\omega \in \Re} \|F(i\omega)\|.$$

This norm is of particular relevance in robust control (see e.g. [4]). This result was derived independently by Enns ([3]) and Glover ([7]).

Theorem 2.2. *Let G be the transfer function of a continuous-time n-dimensional Lyapunov balanced system with Lyapunov grammian $\Sigma = diag(\tilde{\sigma}_1 I_{n_1}, \tilde{\sigma}_2 I_{n_2}, \ldots, \tilde{\sigma}_k I_{n_k})$, with $\tilde{\sigma}_1 > \tilde{\sigma}_2 > \cdots > \tilde{\sigma}_k$. Let $r := n_1 + n_2 + \cdots + n_l$, $1 \le l \le k$. Let $\hat{G}_r$ be the transfer function of the r-dimensional balanced approximant. Then*

$$\|G - \hat{G}_r\|_\infty \le 2(\tilde{\sigma}_{l+1} + \cdots + \tilde{\sigma}_{n_k}).$$

3. A Lyapunov Balanced Canonical Form for Stable Continuous-Time Systems

In this section we derive a canonical form for the set $S_n^{p,m}$ of stable minimal continuous-time systems of fixed McMillan degree. As in the previous sections we only consider continuous-time systems and refer to the Concluding Remarks for a discussion of the discrete-time case. The canonical form of a stable minimal system will be Lyapunov balanced. Before addressing the derivation of a canonical form for the whole set of systems we will consider the subset of systems with identical singular values. A canonical form for this subset will serve as a building block for the canonical form for the full set $S_n^{p,m}$.

The following type of matrix will be used repeatedly. A matrix $M \in \mathcal{K}^{n\times l}$ with $rank(M) = n \le l$ is called *positive upper triangular* if it is of the form

$$M_0 := Q_0 M =$$

$$\begin{pmatrix} 0 & \cdots & 0 & m_{1i_1} & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & m_{2i_2} & * & \cdots & * & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & m_{ni_n} & * & \cdots & * \end{pmatrix},$$

with $m_{ji_j} \in \Re, m_{ji_j} > 0$ for all $j = 1, 2, \ldots, n$. The indices $i_1, \ldots, i_n$ are called the *independence indices.*

Note that positive upper triangular matrices are of full rank. Also, if $M_1 \in \mathcal{K}^{m\times n}$ and $M_2 \in \mathcal{K}^{n\times r}$ are both positive upper triangular then $M_1 M_2$ is positive upper triangular.

The following Lemma will be of importance.

Lemma 3.1. *Let $M \in \mathcal{K}^{n\times l}$ with $n = rank(M) \le l$. Then there exists a unitary matrix $Q_0 \in \mathcal{K}^{n\times n}$ such that $M_0 := Q_0 M$ is positive upper triangular. The matrices M_0 and Q_0 are unique, i.e. if $\tilde{M}_0 = \tilde{Q}_0 M$ is also positive upper triangular and $\tilde{Q}_0$ unitary then $\tilde{M}_0 = M_0$ and $\tilde{Q}_0 = Q_0$.*

Proof. Write $M = (m_1, m_2, \ldots, m_l)$, $m_j \in \mathcal{K}^n$, $1 \le j \le l$. Let i_1 be such that $m_{i_1} \ne 0$ and $m_j = 0$ for all $1 \le j < i_1$. Then there exists a unitary matrix $Q_1 \in \mathcal{K}^{n\times n}$, such that

$$Q_1 m_{i_1} = \begin{pmatrix} m_{1i_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with $m_{1i_1} > 0$. Hence

$$M_1 := Q_1 M = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & m_{1i_1} & * & \cdots & * \\ 0 & \cdots & 0 & 0 & & & \\ \vdots & & \vdots & \vdots & & \tilde{M}_2 & \\ 0 & \cdots & 0 & 0 & & & \end{array}\right),$$

with $\tilde{M}_2 \in \mathcal{K}^{(n-1)\times(l-i_1)}$ and $rank(\tilde{M}_2) = n-1$. It follows in a straightforward way that all unitary Q such that QM_1 has the same structure as M_1 are given by $Q = diag(1, \tilde{Q}_2)$, with $Q_2 \in \mathcal{K}^{(n-1)\times(n-1)}$ unitary and otherwise arbitrary. Find now in the same way as above $\tilde{Q}_{2,0}$ such that with $Q_2 = diag(1, \tilde{Q}_{2,0})$,

$$M_2 := Q_2 Q_1 M = Q_2 M_1 = \left(\begin{array}{cccc|c} 0 & \cdots & 0 & m_{1i_1} & \\ 0 & \cdots & 0 & 0 & * \quad \cdots \quad * \\ \vdots & & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & \tilde{Q}_{2,0}\tilde{M}_2 \end{array}\right),$$

$$= \begin{pmatrix} 0 & \cdots & 0 & m_{1i_1} & * & \cdots & * & * & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & m_{2i_2} & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}.$$

Proceeding inductively we can obtain the desired unique structure. □

A m-input p-output system (A,B,C,D) of dimension n is said to be in *σ-block form* if there exist integers, the so-called *step sizes* of the system, $m = \tau_0 \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_l > 0$ with $\sum_{i=1}^{l} \tau_i = n$ and

1. D is an arbitrary matrix in $\mathcal{K}^{p\times m}$,
2. $B = \begin{pmatrix} \overline{B} \\ 0 \end{pmatrix}$, where $\overline{B}$ is a $\tau_1 \times \tau_0$ positive upper triangular matrix,
3.

$$A = \begin{pmatrix} \tilde{A} + \mathcal{S}_1 & -\mathcal{A}_1^* & 0 & 0 & 0 & \cdots & 0 \\ \mathcal{A}_1 & \mathcal{S}_2 & -\mathcal{A}_2^* & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & \mathcal{S}_3 & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & -\mathcal{A}_{l-2}^* & 0 \\ \vdots & \vdots & & 0 & \mathcal{A}_{l-2} & \mathcal{S}_{l-1} & -\mathcal{A}_{l-1}^* \\ 0 & 0 & \cdots & 0 & 0 & \mathcal{A}_{l-1} & \mathcal{S}_l \end{pmatrix},$$

   where
   a) $\mathcal{S}_i$ is a $\tau_i \times \tau_i$ skew-hermitian matrix, $i = 1, 2, \ldots, l$,
   b) $\mathcal{A}_i$ is a positive upper triangular $\tau_{i+1} \times \tau_i$ matrix, $i = 1, \ldots, l-1$.
   c) $\tilde{A} \in \mathcal{K}^{\tau_1\times\tau_1}$, is a function of σ, $\overline{B}$ and D, with $\sigma > 0$.
4. $C = (\overline{C}, 0)$ where $\overline{C} \in \mathcal{K}^{p\times\tau_1}$ is a function of $\overline{B}, D$ and the matrix $U \in \mathcal{K}^{p\times\tau_1}$ is such that $U^*U = I_{\tau_1}$.

Depending on the class of systems that we consider and the type of balancing that we study a different function will be chosen in 3.c.) and 4.). If the system (A,B,C,D) is in σ-block form with $\tilde{A} = -\frac{1}{2\sigma}\overline{BB}^*$ and $\overline{C} = U(\overline{BB}^*)^{\frac{1}{2}}$ then the system is said to be in *Lyapunov σ-block form.* Clearly a system in Lyapunov σ-block form is uniquely specified by

1. the *discrete parameters* $n, p, m, l, \tau_1, \tau_2, \ldots, \tau_l$.
2. the *continuous parameters* σ, $\overline{B}$, U, D, $\mathcal{A}_1, \ldots, \mathcal{A}_{l-1}$, $\mathcal{S}_1, \ldots, \mathcal{S}_l$. As specified before the fine structure of the matrices $\mathcal{A}_1, \ldots, \mathcal{A}_{l-1}$ is determined by the independence indices.

The relevance of the above systems becomes clear in the following results. The next proposition shows that a system in Lyapunov σ-block form is necessarily stable and minimal. A square transfer function G is called *σ- inner,* $\sigma > 0$, if $\frac{1}{\sigma}G$ is inner, i.e. G is σ-inner if it is stable and $G^*G = \sigma^2 I$. Similarly, a minimal system is called σ-inner if its transfer function is σ-inner.

We will need the following characterization of inner functions (see e.g. [6]).

Lemma 3.2. *Let G be a not necessarily square proper rational function. Then the following two statements are equivalent.*

1. *G is inner,*
2. *If $G \stackrel{r}{=} (A,B,C,D)$ is a minimal state space realization, then there exists $Y = Y^* > 0$ such that*
   a) $A^*Y + YA = -C^*C$,
   b) $C^*D + YB = 0$,
   c) $D^*D = I$.

Proof. Consider first the case of inner functions. Assume that 1.) is true. Let $G \stackrel{r}{=} (A,B,C,D)$ be a minimal realization of the inner function G. Then $G^* \stackrel{r}{=} (-A^*, -C^*, B^*, D^*)$ is also a minimal realization of G^*. Since G is inner

$$I = G^*G \stackrel{r}{=} \left(\begin{array}{cc|c} -A^* & -C^*C & -C^*D \\ 0 & A & B \\ \hline B^* & D^*C & D^*D \end{array}\right).$$

This implies that $D^*D = I$. Since (A,B,C,D) is stable there exists $Y = Y^* > 0$ such that

$$A^*Y + YA = -C^*C.$$

Performing a state space transformation of this realization of G^*G with $\begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}$ we obtain

$$I = G^*G \stackrel{r}{=} \left(\begin{array}{cc|c} -A^* & -A^*Y - YA - C^*C & -C^*D - YB \\ 0 & A & B \\ \hline B^* & B^*Y + D^*C & I \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -A^* & 0 & -C^*D - YB \\ 0 & A & B \\ \hline B^* & B^*Y + D^*C & I \end{array} \right)$$

$$= \left(\begin{array}{c|c} -A^* & -C^*D - YB \\ \hline B^* & I \end{array} \right) + \left(\begin{array}{c|c} A & B \\ \hline B^*Y + D^*C & 0 \end{array} \right).$$

Since the first system in this decomposition is antistable and the second system is stable, the addition of these systems can only be I if the first system is I and the second system is the 0 system. Since (A, B, C, D) is minimal it follows that $(A, B, B^*Y + D^*C, 0)$ is reachable. Hence this system is the 0 system only if $B^*Y + D^*C = 0$. But if $B^*Y + D^*C = 0$ then also $(-A^*, -C^*D - YB, B^*, I) \stackrel{r}{=} I$. This shows 2.).

That 2.) implies 1.) is easily verified. □

Proposition 3.1. *Let (A, B, C, D) be a m-input, p-output continuous-time system in Lyapunov σ-block form, then $(A, B, C, D) \in S_n^{p,m}$, i.e. the system is minimal and stable. Moreover, it is Lyapunov balanced with Lyapunov grammian $\Sigma_S = \sigma I$. If $p \geq m$ and if D is such that $D^*D = \sigma^2 I$ and $C^*D + \Sigma_S B = 0$, then the system is σ-inner.*

Proof. Let (A, B, C, D) be a n-dimensional system in Lyapunov σ-block form. The specific structures of A and B imply that $[\, B \;\; AB \;\; \cdots \;\; A^{n-1}B \,]$ is positive upper triangular and hence of rank n. Therefore the system is reachable. It is also easily checked that the Lyapunov equation

$$AP + PA^* = -BB^*,$$

has the positive definite solution $P = \sigma I$. Hence the reachability of the system implies by Lemma 2.1 that the system is stable. As by construction $BB^* = C^*C$, the Lyapunov equation

$$A^*Q + QA = -C^*C$$

has the solution $Q = \sigma I$. Applying Lemma 2.1 again we have that the stability of the system implies that it is observable. Hence $(A, B, C, D) \in S_n^{p,m}$. Since $P = Q = \sigma I$ the system is Lyapunov balanced with Lyapunov grammian $\Sigma_S = \sigma I$.

If G is the transfer function of the system, then $(A, \frac{1}{\sigma}B, C, \frac{1}{\sigma}D)$ is a minimal realization of $\frac{1}{\sigma}G$. It is checked easily that this system satisfies the conditions of Lemma 3.2. Hence $\frac{1}{\sigma}G$ is inner. □

In the previous Proposition it was shown that a system in Lyapunov σ-block form is Lyapunov balanced and its grammian is a multiple of the identity matrix. We now need to show a converse result which states that if a stable minimal system has a Lyapunov grammian that is a multiple of the identity matrix then there exists an equivalent system which is in Lyapunov σ-block form. The following Lemma will be needed to show that the 'step sizes' decrease.

Lemma 3.3. *Let (A, B, C, D) be a n-dimensional system. Let $t_i := rank([\, B \;\; AB \;\; \cdots \;\; A^{i-1}B \,])$, $i = 2, 3, \ldots, n$; $t_1 := rank(B)$. Let $\tau_i = t_i - t_{i-1}$, $i = 1, \ldots, n$, with $t_0 = 0$. Then*

$$\tau_0 \geq \tau_1 \geq \cdots \geq \tau_n \geq 0 \text{ and } \sum_{i=1}^{n} \tau_i = n.$$

Proof. For some r such $1 \leq r \leq n$, let $c_1, \ldots, c_k$ be the columns of $A^{r-1}B$ that are linearly dependent on the columns of $[\, B \;\; AB \;\; \cdots A^{r-2}B \,]$. Then $Ac_1, Ac_2, \ldots, Ac_k$ are linearly dependent on the columns of $A[\, B \;\; AB \;\; \cdots A^{r-2}B \,]$ and therefore are linearly dependent on the columns of $[\, B \;\; AB \;\; \cdots A^{r-1}B \,]$. Hence

$$\begin{aligned} 0 \leq \tau_r &= rank\,[\, B \;\; AB \;\; \cdots A^rB \,] - rank\,[\, B \;\; AB \;\; \cdots A^{r-1}B \,] \\ &\leq rank\,[\, B \;\; AB \;\; \cdots A^{r-1}B \,] - rank\,[\, B \;\; AB \;\; \cdots A^{r-2}B \,] \\ &= \tau_{r-1}. \end{aligned}$$

Moreover

$$\sum_{i=1}^{n} \tau_i = \sum_{i=1}^{n} t_i - t_{i-1} = t_n - t_0 = n.$$

□

Theorem 3.1. *Let $(A, B, C, D) \in S_n^{p,m}$ and let Y, Z be the positive definite solutions to the Lyapunov equations*

$$0 = AZ + ZA^* + BB^*, \quad 0 = A^*Y + YA + C^*C.$$

Assume that $ZY = \sigma^2 I$, $\sigma > 0$. Then there exists a state space transformation T such that

$$(A_b, B_b, C_b, D_b) := (TAT^{-1}, TB, CT^{-1}, D)$$

is in Lyapunov σ-block form. Therefore (A_b, B_b, C_b, D_b) is Lyapunov balanced with Lyapunov grammian $\Sigma_S = \sigma I$. Moreover, the map Γ_S that assigns to such a system (A, B, C, D) the system (A_b, B_b, C_b, D_b) is a canonical form.

Proof. Since $ZY = \sigma^2 I$ we can assume that (A,B,C,D) is Lyapunov balanced with Lyapunov grammian $\Sigma_S = \sigma I$. By Theorem 1.1 all other Lyapunov balanced realizations are given by a state space transformation with an orthogonal state space transformation Q. By the minimality of the system the $n \times nm$ matrix $R := \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ has rank n. Let now Q_0 be the unique unitary matrix such that $Q_0 R$ is positive upper triangular (Lemma 3.1). Consider the Lyapunov balanced system

$$(A_b, B_b, C_b, D_b) := (Q_0 A Q_0^*, Q_0 B, C Q_0^*, D).$$

Since the matrix

$$R_b := Q_0 R = \begin{bmatrix} B_b & A_b B_b & \cdots & A_b^{n-1} B_b \end{bmatrix}$$

is positive upper triangular we can write it as

$$R_b = \begin{pmatrix} R_{11} & * & * & \cdots & * & \cdots & * \\ 0 & R_{22} & * & \cdots & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 0 & R_{ll} & * & \cdots & * \end{pmatrix},$$

where R_{ii} is a $\tau_i \times m$, $1 \leq s_i \leq m$, $i = 1, \dots, l$, positive upper triangular full rank matrix. The indices τ_i, $i = 1, \dots, l$, are all strictly larger than 0 because of the special structure of the matrix R_b. That $\tau_{i+1} \leq \tau_i$, $i = 1, \dots, l-1$, and $\sum_{i=1}^{l} \tau_i = n$, follows from Lemma 3.3.

Set $\overline{B} := R_{11}$. Then it can be seen that

$$B_b = \begin{pmatrix} \overline{B} \\ 0 \end{pmatrix} = \begin{pmatrix} R_{11} \\ 0 \end{pmatrix}$$

which has the required structure. The second block column of R_b is given by

$$\begin{pmatrix} * \\ R_{22} \\ 0 \end{pmatrix} = A_b B_b = A_b \begin{pmatrix} \overline{B} \\ 0 \end{pmatrix}.$$

This implies that A_b is of the form

$$A_b = \left(\begin{array}{c|c} A_{11} & * \\ A_{21} & * \\ 0 & * \end{array} \right),$$

where $A_{11} \in \mathcal{K}^{\tau_0 \times \tau_0}$, $A_{21} \in \mathcal{K}^{\tau_1 \times \tau_0}$. As $R_{22} = A_{21} B_1$ we have that necessarily A_{21} is positive upper triangular, since both R_{22} and B_1 are positive upper triangular. Considering stepwise all other block columns of R_b shows that A_b has the structure,

$$A_b = \begin{pmatrix} A_{11} & * & * & \cdots & * & \cdots & * \\ A_{21} & * & * & \cdots & * & \cdots & * \\ 0 & A_{32} & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & A_{l,l-1} & * & \cdots & * \end{pmatrix},$$

where $A_{i+1,i} \in \mathcal{K}^{\tau_{i+1} \times \tau_i}$ and positive upper triangular, $i = 1, \dots, l-1$.

Since the system (A_b, B_b, C_b, D_b) is Lyapunov balanced with Lyapunov grammian $\Sigma_S = \sigma I$ we have that

$$A_b + A_b^* = \frac{-1}{\sigma} B_b B_b^* = \frac{-1}{\sigma} diag(\overline{B}\overline{B}^*, 0, \dots, 0).$$

This implies immediately the required structure for A_b. Also, $A_b + A_b^* = \frac{-1}{\sigma} B_b B_b^* = \frac{-1}{\sigma} C_b^* C_b$. Hence $B_b B_b^* = C_b^* C_b$ and therefore $C_b = (\overline{C}, 0)$, for some $\overline{C} \in \mathcal{K}^{p \times \tau_1}$. Thus $\overline{B}\overline{B}^* = \overline{C}^* \overline{C}$. Writing $U = \overline{C}(\overline{B}\overline{B}^*)^{-\frac{1}{2}}$, we have

$$U^* U = (\overline{B}\overline{B}^*)^{-\frac{1}{2}} \overline{C}^* \overline{C} (\overline{B}\overline{B}^*)^{-\frac{1}{2}} = (\overline{B}\overline{B}^*)^{-\frac{1}{2}} \overline{B}\overline{B}^* (\overline{B}\overline{B}^*)^{-\frac{1}{2}} = I_{\tau_1}.$$

Therefore $\overline{C} = U(\overline{B}\overline{B}^*)^{\frac{1}{2}}$ has the required structure.

It remains to show that the map Γ is a canonical form. By construction, Definition 1.1, part 1 is satisfied. It remains to show Definition 1.1, part 2. This is the case if we can show that two equivalent systems (A_i, B_i, C_i, D_i), $i = 1, 2$, in Lyapunov σ-block are identical. Both systems (A_i, B_i, C_i, D_i), $i = 1, 2$, are Lyapunov balanced and since both systems are equivalent they have the same Lyapunov grammian $\Sigma_S = \sigma I$ and there exists an orthogonal Q such that $(A_1, B_1, C_1, D_1) = (QA_2Q^*, QB_2, C_2Q^*, D_2)$. Hence for $R_i = \begin{bmatrix} B_i & A_i B_i & \cdots & A_i^{n-1} B_i \end{bmatrix}$, $i = 1, 2$, we have $R_1 = QR_2$. Note that by construction both R_i, $i = 1, 2$, are positive upper triangular. The uniqueness statement of Lemma 3.1, therefore implies that $R_1 = R_2$ and $Q = I$, and hence both systems are identical. □

In ([21]) a similar result was given for systems with real coefficients. The work by Hanzon ([11]) contains this result in its version for systems with real coefficients. As a Corollary to Proposition 3.1 and to Theorem 3.1 we obtain the following state space characterization of square σ-inner systems.

To prove the Corollary we need the following Lemma.

Lemma 3.4. *Let (A,B,C,D) be a minimal realization of the square inner rational function G. Let Y, Z be such that*

$$A^*Y + YA = -C^*C, \quad AZ + ZA^* = -BB^*.$$

Then $YZ = I$.

Proof. Since G is inner, by Lemma 3.2, $C^*D+YB=0$. Since $DD^*=D^*D=I$, $CY^{-1}=-DB^*$. As $A^*Y+YA=-C^*C$ we have

$$AY^{-1}+Y^{-1}A^*=-Y^{-1}C^*CY^{-1}=-(CY^{-1})^*(CY^{-1})$$
$$=-BD^*DB^*=-BB^*.$$

By the uniqueness of the solution to this Lyapunov equation it follows that $Y^{-1}=Z$, which implies the claim. □

Corollary 3.1. *Let $(A_\sigma,B_\sigma,C_\sigma,D_\sigma)$ be a m-input, m-output continuous-time system of dimension n. The following statements are equivalent.*

1. *$(A_\sigma,B_\sigma,C_\sigma,D_\sigma)$ is a minimal σ-inner system.*
2. *$(A_\sigma,B_\sigma,C_\sigma,D_\sigma)=(TAT^{-1},TB,CT^{-1},D)$ for some invertible matrix T, where (A,B,C,D) is in the following Lyapunov σ-block form: there exist indices $m=\tau_0\geq\tau_1\geq\cdots\geq\tau_l>0$, with $\sum_{i=1}^{l}\tau_i=n$, $\sigma>0$, such that*

   a) $B=\begin{pmatrix}\overline{B}\\0\end{pmatrix}$, *where $\overline{B}$ is a $\tau_1\times\tau_0$ positive upper triangular matrix,*

   b)

$$A=\begin{pmatrix}\tilde{A}+\mathcal{S}_1 & -\mathcal{A}_1^* & 0 & 0 & 0 & \cdots & 0\\ \mathcal{A}_1 & \mathcal{S}_2 & -\mathcal{A}_2^* & 0 & 0 & \cdots & 0\\ 0 & \mathcal{A}_2 & \mathcal{S}_3 & \ddots & \ddots & & \vdots\\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & -\mathcal{A}_{l-2}^* & 0\\ \vdots & \vdots & & 0 & \mathcal{A}_{l-2} & \mathcal{S}_{l-1} & -\mathcal{A}_{l-1}^*\\ 0 & 0 & \cdots & \cdots & 0 & \mathcal{A}_{l-1} & \mathcal{S}_l\end{pmatrix},$$

   where

   $\mathcal{S}_i$, is a $\tau_i\times\tau_i$ skew-hermitian matrix, $i=1,2,\ldots,l$,

   $\mathcal{A}_i$ is a positive upper triangular $\tau_{i+1}\times\tau_i$ matrix, $i=1,\ldots,l-1$;

$$\tilde{A}=-\frac{1}{2\sigma}\overline{BB}^*,$$

   c)

   D is a $m\times m$-matrix such that $D^*D=\sigma^2I$

   d)

$$C=-\frac{1}{\sigma}DB^*.$$

Moreover, the system (A,B,C,D) as defined in 2.) is Lyapunov balanced with Lyapunov grammian $\Sigma_S=\sigma I$. The map Γ that assigns to each σ-inner system $(A_\sigma,B_\sigma,C_\sigma,D_\sigma)$ in 1.) the realization in 2.) is a canonical form.

Proof. Let $(A_\sigma,B_\sigma,C_\sigma,D_\sigma)$ is a minimal square σ-inner system. Let Y,Z be the solutions to the Lyapunov equations

$$A_\sigma^*Y+YA_\sigma=-C_\sigma^*C_\sigma,\quad A_\sigma Z+ZA_\sigma^*=-B_\sigma B_\sigma^*.$$

Lemma 3.4 implies that $YZ=\sigma^2I$. Hence by the Theorem there exists a unique equivalent system (A,B,C,D) that is in Lyapunov σ-block form. This implies that A and B have the stated structure. Since the system is square and σ-inner, it follows by Lemma 3.2 that $D^*D=DD^*=\sigma^2I$ and $C\Sigma+DB^*=0$, where $\Sigma=\sigma I$ is the Lyapunov grammian of the system. Hence $C=-\frac{1}{\sigma}DB^*$ and therefore 2.)

Let now (A,B,C,D) be as in 2.). It is necessary to show that the system is in Lyapunov σ-block form. Since $C=-\frac{1}{\sigma}DB^*$, it follows that $C=(\overline{C},0)$, where $\overline{C}\in\mathcal{K}^{p\times\tau_1}$ is such that $\overline{C}=-\frac{1}{\sigma}D\overline{B}^*=U(\overline{BB}^*)^{\frac{1}{2}}$, with $U:=-\frac{1}{\sigma}DB^*(\overline{BB}^*)^{-\frac{1}{2}}$ such that $U^*U=I_{\tau_1}$. Hence (A,B,C,D) is in Lyapunov σ-block form and by Proposition 3.1 the system is minimal and σ-inner. This implies 1.) The remaining statements follow from the Theorem. □

In the single-input single-output case this representation simplifies substantially.

Corollary 3.2. *Let $(A_\sigma,b_\sigma,c_\sigma,d_\sigma)$ be a single-input single-output continuous-time system of dimension n. The following statements are equivalent:*

1. *$(A_\sigma,b_\sigma,c_\sigma,d_\sigma)$ is a minimal σ-inner system.*
2. *$(A_\sigma,b_\sigma,b_\sigma,d_\sigma)=(TAT^{-1},Tb,cT^{-1},d)$ for some invertible matrix T, where (A,b,c,d) is in the following σ-inner form:*

$$A=\begin{pmatrix}a_{11}+i\beta_1 & -\alpha_1 & 0 & \cdots & \cdots & 0\\ \alpha_1 & i\beta_2 & -\alpha_2 & 0 & & \vdots\\ 0 & \alpha_2 & i\beta_3 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & -\alpha_{n-2} & 0\\ \vdots & & 0 & \alpha_{n-2} & i\beta_{n-1} & -\alpha_{n-1}\\ 0 & \cdots & \cdots & 0 & \alpha_{n-1} & i\beta_n\end{pmatrix},$$

$$b=\begin{pmatrix}b_1\\0\\\vdots\\0\end{pmatrix},\quad c=(\,s_1b_1,0,\cdots,0\,),\quad d=-s_1\sigma,$$

with $\sigma > 0$, $b_1 > 0$, $s_1 \in \mathcal{K}, |s_1| = 1$, $\alpha_i > 0$, $i = 1, \ldots, n-1$, *and* $a_{11} := -\frac{b_1^2}{2\sigma}$. *For* $\mathcal{K} = \Re$, $\beta_i = 0$, $i = 1, 2, \ldots, n$, *and* $s_1 = \pm 1$.

Moreover, the system (A, b, c, d) *as defined in 2.) is Lyapunov balanced with Lyapunov grammian* $\Sigma_S = \sigma I$. *The map that assigns to each* σ*-inner system* $(A_\sigma, b_\sigma, c_\sigma, d_\sigma)$ *in 1.) the realization in 2.) is a canonical form.*

Having analyzed in some depth the canonical form for systems with identical Lyapunov singular values we can now proceed to develop a canonical form for the class $S_n^{p,m}$ of stable minimal systems. The approach will be to reduce the canonical form problem for general systems to the canonical form problem for subsystems with identical singular values. To this end we need to introduce the following definitions.

An m-input p-output system (A, B, C, D) of dimension n is said to be in *balanced form* if there exist so-called *block indices* $n_1, n_2, \ldots, n_k$, $\sum_{i=1}^k n_i = n$ such that if (A, B, C, D) is partitioned as

$$A = \begin{pmatrix} A(1,1) & \cdots & A(1,j) & \cdots & A(1,k) \\ \vdots & & \vdots & & \vdots \\ A(i,1) & \cdots & A(i,j) & \cdots & A(i,k) \\ \vdots & & \vdots & & \vdots \\ A(k,1) & \cdots & A(k,j) & \cdots & A(k,k) \end{pmatrix}, \quad B = \begin{pmatrix} B(1) \\ \vdots \\ B(i) \\ \vdots \\ B(k) \end{pmatrix},$$

$$C = \begin{pmatrix} C(1) & \cdots & C(i) & \cdots & C(k) \end{pmatrix},$$

where $A(i,j) \in \mathcal{K}^{n_i \times n_j}$, $B(j) \in \mathcal{K}^{n_j \times m}$ and $C(i) \in \mathcal{K}^{p \times n_i}$, $1 \le i, j \le k$, we have that

1. the block diagonal systems $(A(i,i), B(i), C(i), D)$ are in σ_i-block form with $\sigma_1 > \cdots > \sigma_i > \cdots > \sigma_k > 0$, and step sizes $m = \tau_0^j \ge \tau_1^j \ge \tau_2^j \ge \cdots \ge \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \le j \le k$.
2. the block entries $A(i,j)$, $1 \le i, j \le k$, $i \ne j$, are given by

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{A}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ is a function of σ_i, σ_j, $U(i)$, $U(j)$, $\overline{B}(i)$, $\overline{B}(j)$ and D, $1 \le i, j \le k$, $i \ne j$.

The specific function that is chosen in 2.) will depend on the type of balancing that we study. If a system (A, B, C, D) is in balanced form such that the block diagonal systems $(A(i,i), B(i), C(i), D)$ are in Lyapunov σ_i-block form and

$$\tilde{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j \overline{B}_i \overline{B}_j^* - \sigma_i (\overline{B}_i \overline{B}_i^*)^{\frac{1}{2}} U_i^* U_j (\overline{B}_j \overline{B}_j^*)^{\frac{1}{2}}),$$

for $1 \le i, j \le k$, $i \ne j$, then the system is said to be in *Lyapunov balanced form* with Lyapunov singular values $\sigma_1, \ldots, \sigma_k$.

The following proposition states an interesting *augmentation property* of linear systems whose corresponding Lyapunov equations have positive definite solutions. If the system is partitioned into two sets of states and the corresponding block diagonal subsystems are stable and minimal, then the system itself is stable and minimal provided a weak technical condition is satisfied.

Note that if the system (A, B, C, D) is in Lyapunov balanced form then the block diagonal subsystems $(A(i,i), B(i), C(i), D)$ are stable, minimal and Lyapunov balanced with Lyapunov grammian $\sigma_i I_{n_i}$, $i = 1, \ldots, k$. As will be shown in the proof of Theorem 3.2, the definition of Lyapunov balanced form is such that the Proposition can be immediately applied to a system being in Lyapunov balanced form.

Proposition 3.2. *(Kabamba [15]) Let*

$$(A, B, C, D) = \left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D \right),$$

be a n*-dimensional linear system that is conformally partitioned, i.e.* (A_{11}, B_1, C_1, D) *is a* k*-dimensional continuous-time system,* $0 < k < n$. *Assume that there exist positive definite* $n \times n$ *matrices* $P = diag(P_1, P_2)$, $Q = diag(Q_1, Q_2)$, *where* P_1, Q_1 *are* $k \times k$, *such that the sets of eigenvalues of* $P_1 Q_1$ *and of* $P_2 Q_2$ *have zero intersection. If further*

$$AP + PA^* = -BB^*, \quad A^*Q + QA = -C^*C,$$

and (A_{ii}, B_i, C_i, D), $i = 1, 2$, *are minimal then* (A, B, C, D) *is stable and minimal.*

Proof. Note that for $i = 1, 2$,

$$A_{ii} P_i + P_i A_{ii}^* = -B_i B_i^*,$$

and therefore by Lemma 2.1 the reachability of (A_{ii}, B_i, C_i, D), $i = 1, 2$, also implies the stability of the systems.

Assume now that A is not stable. Therefore there exists $x \in \mathcal{K}^n$, $x \ne 0$, $\lambda \in \mathcal{C}$, with $Re(\lambda) \ge 0$ such that

$$A^* x = \lambda x.$$

Now consider

$$\begin{aligned} 0 &\ge -< BB^*x, x > = -< B^*x, B^*x > \\ &= < (AP + PA^*)x, x > = < PA^*x, x > + < x, PA^*x > \\ &= 2Re(\lambda) < x, Px > . \end{aligned}$$

Since $P > 0$ this inequality can only hold if $Re(\lambda) = 0$ and $B^*x = 0$. Hence $\lambda = i\omega$ for some $\omega \in \Re$. Multiplying the Lyapunov equation on the right by x we obtain

$$0 = BB^*x = APx + PA^*x = APx + \lambda x.$$

Hence

$$APx = -\lambda Px,$$

i.e. Px is an eigenvector of A with eigenvalue $-\lambda$. Using the second Lyapunov equation the above reasoning applied to (A, C) shows that $C^*Px = 0$ and

$$A^*QPx = -QAPx = \lambda QPx,$$

i.e. QPx is also an eigenvector of A^* with the same eigenvalue λ. Hence the eigenspace of A^* corresponding to the eigenvalue λ is invariant under the transformation $QP = diag(Q_1P_1, Q_2P_2)$.

Let $\mathcal{K}^n = X_1 \oplus X_2$ be the orthogonal decomposition of the state-space that gives rise to the block partitioning of the state-space matrices and the matrices P and Q. Due to the block structure of QP and the fact that the eigenvalues of Q_1P_1 and of Q_2P_2 are different, we have that if E is an invariant subspace of QP, then either $E \subseteq X_1$ or $E \subseteq X_2$. Therefore also the eigenspace of A^* with eigenvalue λ has this form. Hence $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, for some $x_1 \in X_1$, $x_1 \neq 0$, or $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, for some $x_2 \in X_2$, $x_2 \neq 0$. If $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ then

$$A^*x = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11}^*x_1 \\ A_{12}^*x_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ 0 \end{pmatrix}$$

and hence $A_{11}^*x_1 = \lambda x_1$, which is a contradiction to the stability of A_{11}. Similarly if $x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ we obtain a a contradiction to the stability of A_{22}. Hence we have the stability of A. Since the Lyapunov equations corresponding to the system have the positive definite solutions P and Q we therefore also have by Lemma 2.1 the reachability and the observability of the system (A, B, C, D). □

We can now give a canonical form and a parametrization result for minimal and stable systems of dimension n, i.e. for systems in the class $S_n^{p,m}$. This canonical form is called *Lyapunov balanced canonical form.* The other implication of the Theorem shows the parametrization result that each system in Lyapunov balanced form is automatically stable, minimal and Lyapunov balanced.

Theorem 3.2. *Let (A_s, B_s, C_s, D_s) be a m-input p-output continuous-time system of dimension n. Then the following are equivalent:*

1. *(A_s, B_s, C_s, D_s) is a stable and minimal system, i.e. is in $S_n^{p,m}$.*
2. *$(A_s, B_s, C_s, D_s) = (TAT^{-1}, TB, CT^{-1}, D)$ for some invertible T, where (A, B, C, D) is in Lyapunov balanced form, i.e. there exist block indices $n_1, \ldots, n_k$, $\sum_{j=1}^k n_j = n$, parameters $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$ and k families of step sizes $m = \tau_0^j \geq \tau_1^j \geq \tau_2^j \geq \cdots \geq \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \leq j \leq k$, such that*
   a) $B = (\underbrace{\overline{B}_1^T, 0}_{n_1}, \underbrace{\overline{B}_2^T, 0}_{n_2}, \ldots, \underbrace{\overline{B}_k^T, 0}_{n_k})^T$, where $\overline{B}_j \in \mathcal{K}^{\tau_1^j \times \tau_0^j}$ is positive upper triangular, $1 \leq j \leq k$.
   b) $C = (\underbrace{U_1(\overline{B}_1\overline{B}_1^*)^{\frac{1}{2}}, 0}_{n_1}, \underbrace{U_2(\overline{B}_2\overline{B}_2^*)^{\frac{1}{2}}, 0}_{n_2}, \ldots, \underbrace{U_k(\overline{B}_k\overline{B}_k^*)^{\frac{1}{2}}, 0}_{n_k})$, where $U_j \in \mathcal{K}^{p \times \tau_1^j}$, $U_j^*U_j = I_{\tau_1^j}$, $1 \leq j \leq k$.
   c)

$$A = \left(\begin{array}{c|c|c|c|c} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline A(i,1) & \cdots & A(i,i) & \cdots & A(i,k) \\ \hline \vdots & & \vdots & \ddots & \vdots \\ \hline A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \leq i \leq k$

$$A(i,i) = \begin{pmatrix} \tilde{A}_{ii} + \mathcal{S}_1^i & (-\mathcal{A}_1^i)^* & & 0 \\ \mathcal{A}_1^i & \mathcal{S}_2^i & \ddots & \\ & \ddots & \ddots & (-\mathcal{A}_{l_i-1}^i)^* \\ 0 & & \mathcal{A}_{l_i-1}^i & \mathcal{S}_{l_i}^i \end{pmatrix},$$

for $1 \leq i, j \leq k$, $i \neq j$

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

i. $\mathcal{S}_i^j$ is a $\tau_i^j \times \tau_i^j$ skew-hermitian matrix, $i = 1, 2, \ldots, l_j$, $1 \leq j \leq k$.
ii. $\mathcal{A}_i^j$ is a positive upper triangular $\tau_{i+1}^j \times \tau_i^j$ matrix, $i = 1, 2, \ldots, l_j - 1$, $1 \leq j \leq k$.
iii. $\tilde{A}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ is given by

$$\tilde{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j \overline{B}_i \overline{B}_j^* - \sigma_i (\overline{B}_i \overline{B}_i^*)^{\frac{1}{2}} U_i^* U_j (\overline{B}_j \overline{B}_j^*)^{\frac{1}{2}}),$$

$$\text{for } 1 \le i,j \le k, \quad i \ne j, \quad \text{and}$$

$$\tilde{A}_{ii} = -\frac{1}{2\sigma_i}\overline{B}_i\overline{B}_i^* \quad \text{for } 1 \le i \le k.$$

Moreover, the system (A,B,C,D) as defined in (2) is Lyapunov balanced with Lyapunov grammian $\Sigma_S = diag(\sigma_1 I_{n_1},\ldots,\sigma_k I_{n_k})$, where $n_1,\ldots,n_k$ are the block indices and $\sigma_1 > \cdots > \sigma_k > 0$ are the Lyapunov singular values of the system in (2). The map Γ_S that assigns to each system in $S_n^{p,m}$ the realization in (2) is a canonical form.

Proof. Assume that (A_s,B_s,C_s,D_s) is a stable and minimal system. We can assume without loss of generality that the system is Lyapunov balanced with Lyapunov grammian

$$\Sigma_S = diag(\sigma_1 I_{n_1},\sigma_2 I_{n_2},\ldots,\sigma_k I_{n_k}), \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

By Theorem 1.1 all other Lyapunov balanced realizations can be obtained by a state space transformation of the form $Q = diag(Q_1,Q_2,\ldots,Q_k)$, $Q_j \in \mathcal{K}^{n_j\times n_j}$, $j = 1,2,\ldots,k$.

Let (A_s,B_s,C_s,D_s) be partitioned according to the block indices $n_1,n_2,\ldots,n_k$, such that

$$A_s = (A_s(i,j))_{1\le i,j\le k}, \qquad A_s(i,j) \in \mathcal{K}^{n_i\times n_j}, \quad 1 \le i,j \le k,$$

$$B_s = (B_s(1)^T,B_s(2)^T,\cdots,B_s(k)^T)^T, \qquad B_s(j) \in \mathcal{K}^{n_j\times m}, \quad 1 \le j \le k,$$

$$C_s = (C_s(1),C_s(2),\cdots,C_s(k)), \qquad C_s(i) \in \mathcal{K}^{p\times n_i}, \quad 1 \le i \le k.$$

By Theorem 2.1 the block diagonal subsystems $(A_s(j,j),B_s(j),C_s(j),D_s)$ are stable, minimal and Lyapunov balanced with Lyapunov grammian $\sigma_j I_{n_j}$, $1 \le j \le k$. By Theorem 3.1 there exists a unique unitary $Q_j \in \mathcal{K}^{n_j\times n_j}$, such that

$$(A(j,j),B(j),C(j),D) := (Q_jA_s(j,j)Q_j^*,Q_jB_s(j),C_s(j)Q_j^*,D_s)$$

is in Lyapunov σ_j-block form, $1 \le j \le k$. Then $(A,B,C,D) := (QA_sQ^*,QB_s,C_sQ^*,D_s)$ where $Q := diag(Q_1,Q_2,\ldots,Q_k)$ is uniquely determined, and the block diagonal subsystems have the desired structure. The system is Lyapunov balanced with Lyapunov grammian $\Sigma = \Sigma_S$. To conclude this part of the proof it remains to be shown that the off-diagonal block matrices of A have the stated representation. Let (A,B,C,D) be partitioned according to the block indices $n_1,n_2,\ldots,n_k$, i.e.

$$A = (A(i,j))_{1\le i,j\le k}, \qquad A(i,j) \in \mathcal{K}^{n_i\times n_j}, \quad 1 \le i,j \le k,$$

$$B = (B(1)^T,B(2)^T,\cdots,B(k)^T)^T, \qquad B(j) \in \mathcal{K}^{n_j\times m}, \quad 1 \le j \le k,$$

$$C = (C(1),C(2),\cdots,C(k)), \qquad C(i) \in \mathcal{K}^{p\times n_i}, \quad 1 \le i \le k.$$

Let $1 \le i,j \le k, i \ne j$, and consider the (i,j) block entry of the Lyapunov equations

$$A^*\Sigma + \Sigma A = -C^*C, \qquad A\Sigma + \Sigma A^* = -BB^*,$$

i.e.

$$A(j,i)^*\sigma_j + \sigma_i A(i,j) = -C(i)^*C(j), \quad A(i,j)\sigma_j + \sigma_i A(j,i)^* = -B(i)B(j)^*,$$

or

$$\begin{bmatrix} \sigma_i & \sigma_j \\ \sigma_j & \sigma_i \end{bmatrix}\begin{bmatrix} A(i,j) \\ A(j,i)^* \end{bmatrix} = -\begin{bmatrix} C(i)^*C(j) \\ B(i)B(j)^* \end{bmatrix}.$$

Since by assumption $\sigma_i \ne \sigma_j$, this equation can be solved to give

$$A(i,j) = \frac{1}{\sigma_i^2 - \sigma_j^2}[\sigma_j B(i)B(j)^* - \sigma_i C(i)^*C(j)].$$

The structure of $B(i),B(j),C(i)$ and $C(j)$ shows that

$$A(i,j) = \begin{bmatrix} \tilde{A}_{ij} & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$\tilde{A}_{ij} := \frac{1}{\sigma_i^2 - \sigma_j^2}[\sigma_j\overline{B}_i\overline{B}_j^* - \sigma_i\overline{C}_i^*\overline{C}_j] \in \mathcal{K}^{\tau_1^i\times\tau_1^j}.$$

Hence 2.)

Now assume 2.). By construction of (A,B,C,D), $\Sigma_S = diag(\sigma_1 I_{n_1},\sigma_2 I_{n_2},\cdots,\sigma_k I_{n_k})$ solves the Lyapunov equations

$$A^*\Sigma + \Sigma A = -C^*C, \qquad A\Sigma + \Sigma A^* = -BB^*.$$

Let (A,B,C,D) be partitioned according to the block indices. By construction the block diagonal subsystems are in Lyapunov σ_i-block form, $1 \le i \le k$. Hence Proposition 3.1 implies that they are stable and minimal. Hence the system (A,B,C,D) is stable and minimal by Proposition 3.2. Since Σ_S solves the two Lyapunov equations the system is balanced with Lyapunov grammian Σ_S.

□

Specialization of the theorem to the single input single output case gives the following Corollary.

Corollary 3.3. *Let (A_s,b_s,c_s,d_s) be a single-input single-output continuous-time system of dimension n. Then the following statements are equivalent:*

1. *(A_s,b_s,c_s,d_s) is a stable minimal system.*
2. *$(A_s,b_s,c_s,d_s) = (TAT^{-1},Tb,cT^{-1},d)$ for some invertible matrix T, where (A,b,c,d) is in the following Lyapunov balanced form with block indices $n_1,n_2,\ldots,n_k$, $\sum_{i=1}^k n_i = n$.*

a) $b = (\underbrace{b_1, 0, \ldots, 0}_{n_1}, \underbrace{b_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n_k})^T$, *where* $b_i > 0$, $1 \leq i \leq k$.

b) $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n_1}, \underbrace{s_2 b_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n_k})$, *where* $s_i \in \mathcal{K}$, $|s_i| = 1$, $1 \leq i \leq k$.

c)

$$A = \left(\begin{array}{c|c|c|c} A(1,1) & A(1,2) & \cdots & A(1,k) \\ \hline A(2,1) & A(2,2) & \cdots & A(2,k) \\ \hline \vdots & & \ddots & \vdots \\ \hline A(k,1) & A(k,2) & \cdots & A(k,k) \end{array} \right),$$

where for $1 \leq j \leq k$,

$$A(j,j) = \begin{pmatrix} a_{jj} + i\beta_1^j & -\alpha_1^j & & 0 \\ \alpha_1^j & i\beta_2^j & \ddots & \\ & \ddots & \ddots & -\alpha_{n_j-1}^j \\ 0 & & \alpha_{n_j-1}^j & i\beta_{n_j}^j \end{pmatrix},$$

for $1 \leq i, j \leq k$, $i \neq j$,

$$A(i,j) = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\alpha_j^i > 0 \text{ for } 1 \leq j \leq n_i - 1, \quad 1 \leq i \leq k,$$

$$a_{ij} = \frac{\sigma_j - \bar{s}_i s_j \sigma_i}{\sigma_i^2 - \sigma_j^2} b_i b_j \text{ for } 1 \leq i, j \leq k, \quad i \neq j,$$

$$a_{ii} = -\frac{1}{2\sigma_i} b_i^2 \text{ for } 1 \leq i \leq k.$$

$$\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

If $\mathcal{K} = \Re$, *then*

$$s_i = \pm 1, \text{ for } i = 1, 2, \ldots, k,$$

$$\beta_j^i = 0, \text{ for } 1 \leq j \leq n_i, \quad 1 \leq i \leq k,$$

$$a_{ij} = \frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}, \text{ for } 1 \leq i, j \leq k.$$

Moreover, the system (A, b, c, d) *as defined in* (2) *is Lyapunov balanced with Lyapunov grammian* $\Sigma_S = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. *The map* Γ_S *that assigns to each system in* $S_n^{1,1}$ *the realization in* (2) *is a canonical form.*

Proof. The proof follows by specializing the previous theorem to the case $m = p = 1$ and renaming the parameters $\overline{B}_i$ by b_i, $1 \leq i \leq k$; S_i^j by $\sqrt{-1}\beta_i^j$, $i = 1, \ldots, l_j$, $1 \leq j \leq k$, and U_i by S_i, $i = 1, \ldots, k$. □

We can now reconsider the model reduction problem for Lyapunov balanced systems. In Theorem 2.1 we showed that a k-dimensional balanced approximant of a Lyapunov balanced system with Lyapunov grammian $\Sigma = diag(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n)$ is stable minimal and Lyapunov balanced if $\sigma_r > \sigma_{r+1}$. If the system is in Lyapunov balanced canonical form the same result holds without the condition on the point of truncation.

Theorem 3.3. *Let* $(A, B, C, D) \in S_n^{p,m}$ *be in Lyapunov balanced canonical form with Lyapunov grammian* Σ. *Let* $1 \leq r < n$. *Then the* r*-dimensional balanced approximant* (A_1, B_1, C_1, D) *is in* $S_r^{p,m}$ *and is in Lyapunov balanced canonical form with Lyapunov grammian* Σ_1 *where* $\Sigma = diag(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathcal{K}^{r \times r}$.

Proof. Let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ be the r-dimensional balanced approximant of (A, B, C, D). Let $\hat{\Sigma}$ be the $r \times r$ principal submatrix of Σ. It is easily checked that

$$\hat{A}\hat{\Sigma} + \hat{\Sigma}\hat{A}^* = -\hat{B}\hat{B}^*,$$

$$\hat{A}^*\hat{\Sigma} + \hat{\Sigma}\hat{A} = -\hat{C}^*\hat{C}.$$

Let $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$, $\sigma_1 > \sigma_2 > \cdots > \sigma_k$. If $r = n_1 + \cdots + n_l$ for some $1 \leq l < k$, then the result follows by Theorem 2.1 or the immediately verified fact that $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ admits a Lyapunov balanced parametrization. If $n_1 + \cdots + n_l < r < n_1 + \cdots + n_{l+1}$, for some $1 \leq l < k - 1$, we will also show that $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ admits a Lyapunov balanced parametrization.

Partition $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ such that

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \end{pmatrix},$$

$$\hat{\Sigma} = diag(\hat{\Sigma}_1, \hat{\Sigma}_2),$$

such that $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ is a $n_1 + \cdots + n_l$-dimensional system and $\hat{\Sigma}_2 = \sigma_{l+1} I_{r-(n_1+\cdots+n_l)}$. Then

$$\hat{A}_{ii}\hat{\Sigma}_i + \hat{\Sigma}_i\hat{A}_{ii}^* = -\hat{B}_i\hat{B}_i^*,$$

$$\hat{A}_{ii}^*\hat{\Sigma}_i + \hat{\Sigma}_i\hat{A}_{ii} = -\hat{C}_i^*\hat{C}_i,$$

for $i = 1, 2$, and $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ admits a Lyapunov balanced parametrization. If we show that $(\hat{A}_{22}, \hat{B}_2, \hat{C}_2, \hat{D})$ is in Lyapunov σ_{l+1}-block form, then $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is in Lyapunov balanced form since $\hat{\Sigma}$ solves the observability and reachability Lyapunov equation for $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Then the result will follow from Theorem 3.2.

It follows by inspection that $\hat{B}_2$ has the structure of a B-matrix of a system in Lyapunov σ_{l+1}-block form. Also by inspection and since

$$\hat{A}_{22}\hat{\Sigma}_2 + \hat{\Sigma}_2\hat{A}_{22}^* = -\hat{B}_2\hat{B}_2^*$$

it follows that $\hat{A}_{22}$ has the structure of a A-matrix of a system in Lyapunov σ_{l+1}-block form. Hence $(\hat{A}_{22}, \hat{B}_2, \hat{C}_2, \hat{D})$ is stable and reachable, by Proposition 3.1. Since

$$\hat{A}_{22}^*\hat{\Sigma}_2 + \hat{\Sigma}_2\hat{A}_{22} = -\hat{C}_2\hat{C}_2$$

it follows by Corollary 2.1 that $(\hat{A}_{22}, \hat{B}_2, \hat{C}_2, \hat{D})$ is also observable. Hence $(\hat{A}_{22}, \hat{B}_2, \hat{C}_2, \hat{D})$ is stable, minimal and in Lyapunov balanced with Lyapunov grammian $\hat{\Sigma}_2$. Moreover, because of the structure of $\hat{A}_{22}$ and $\hat{B}_2$, the system is in Lyapunov σ_{l+1}-block form. □

4. L-Characteristic, LQG-Balanced Canonical Form and Model Reduction for Minimal Systems

In this section a canonical form will be given for the class $L_n^{p,m}$ of minimal systems of fixed McMillian degree. The canonical form is defined in terms of LQG-balanced realizations. This canonical form is derived from the Lyapunov-balanced canonical form for stable systems using the L-characteristic, a bijection between the class $L_n^{p,m}$ of minimal systems of McMillian degree n and the class $S_n^{p,m}$ of stable minimal systems of McMillian degree n. A substantial part of the section will be devoted to the introduction and analysis of this bijection. This bijection is here introduced using a state space formulation. The analysis of the L-characteristic will be followed by the derivation of the LQG-balanced canonical form, a parametrization result for $L_n^{p,m}$ and an investigation of the model reduction properties of LQG balanced systems.

In the following definition the L-characteristic is introduced for a system in $L_n^{p,m}$. For a system in $S_n^{p,m}$ the inverse L-characteristic will be defined. Part of this section is devoted to show that the L-characteristic defines a bijection from $L_n^{p,m}$ to $S_n^{p,m}$ whose inverse is the inverse L-characteristic.

Throughout this section we will use the following abbreviations. Let (A, B, C, D) be a system, then set $R_L := I + D^*D$, $S_L := I + DD^*$ and $A_L := A - BR_L^{-1}D^*C$.

Definition 4.1. *1. Let (A, B, C, D) be a minimal system. Let Y be the stabilizing solution of the control algebraic Riccati equation and let Z be the stabilizing solution of the filter algebraic Riccati equation, i.e.*

$$0 = A_L^*Y + YA_L - YBR_L^{-1}B^*Y + C^*S_L^{-1}C$$

$$0 = A_LZ + ZA_L^* - ZC^*S_L^{-1}CZ + BR_L^{-1}B^*$$

with $A_L - BR_L^{-1}B^*Y$ and $A_L - ZC^*S_L^{-1}C$ stable. Then the system

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) := \chi_L((A, B, C, D))$$

$$:= (A_L - BR_L^{-1}B^*Y, BR_L^{-\frac{1}{2}}, S_L^{-\frac{1}{2}}C(I + ZY), D)$$

$$= ((I + ZY)^{-1}(A_L - ZC^*S_L^{-1}C)(I + ZY), BR_L^{-\frac{1}{2}}, S_L^{-\frac{1}{2}}C(I + ZY), D)$$

is called the L-characteristic *of the system.*

2. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a stable minimal system and let P and Q be the solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then the system

$$(A, B, C, D) := I\chi_L((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}))$$

$$= (\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}, \mathcal{B}R_L^{\frac{1}{2}}, S_L^{\frac{1}{2}}\mathcal{C}(I + PQ)^{-1}, \mathcal{D})$$

$$= \Big((I + PQ)(\mathcal{A} + (I + PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C})(I + PQ)^{-1}, \mathcal{B}R_L^{\frac{1}{2}}, S_L^{\frac{1}{2}}\mathcal{C}(I + PQ)^{-1}, \mathcal{D}\Big)$$

is called the inverse L-characteristic *of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Here $R_L := I + \mathcal{D}^*\mathcal{D}$ and $S_L := I + \mathcal{D}\mathcal{D}^*$.*

Note that both expressions for the L-characteristic are identical because of the Bucy relations given in the following Lemma.

Lemma 4.1. *Let (A, B, C, D) be a minimal system. Then*

$$(I + ZY)(A_L - BR^{-1}B^*Y) = (A_L - ZC^*S^{-1}C)(I + ZY),$$

where Y is the stabilizing solution to the control algebraic Riccati equation and Z is the stabilizing solution to the filter algebraic Riccati equation.

Proof. Consider the two Riccati equation,

$$0 = A_L^*Y + YA_L - YBR^{-1}B^*Y + C^*S^{-1}C,$$
$$0 = A_LZ + ZA_L^* - ZC^*R^{-1}CZ + BS^{-1}B^*.$$

Multiplying the first equation on the left by Z and the second equation on the right by Y, equating both equations and adding A_L to both sides we obtain

$$A_L + ZA_L^*Y + ZYA_L - ZYBR^{-1}B^*Y + ZC^*S^{-1}C$$
$$= A_L + A_LZY + ZA_L^*Y - ZC^*S^{-1}CZY + BR^{-1}B^*Y.$$

Canceling the term ZA_L^*Y from either side and collecting terms, we obtain

$$(I + ZY)(A_L - BR^{-1}B^*Y) = (A_L - ZC^*S^{-1}C)(I + ZY).$$

□

That the two expressions for the inverse L-characteristic are identical follows from the following Lemma.

Lemma 4.2. *Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in S_n^{p,m}$ and let P and Q be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

$$[\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}][I + PQ] =$$
$$[I + PQ][\mathcal{A} + (I + PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}]$$

and

$$[\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1}][I + PQ] = [I + PQ][\mathcal{A} + (I + PQ)^{-1}P\mathcal{C}^*\mathcal{C}]$$

Proof. We have

$$[\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}][I + PQ] = \mathcal{A}(I + PQ) + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})$$
$$= \mathcal{A} + \mathcal{A}PQ + \mathcal{B}\mathcal{B}^*Q + \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} + (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q + \mathcal{B}\mathcal{D}^*\mathcal{C}$$
$$= \mathcal{A} + (-P\mathcal{A}^*)Q + \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} - P(-Q\mathcal{A} - \mathcal{C}^*\mathcal{C}) + \mathcal{B}\mathcal{D}^*\mathcal{C}$$
$$= \mathcal{A} + PQ\mathcal{A} + (P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}$$
$$= [I + PQ][\mathcal{A} + (I + PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}].$$

which shows the first identity. Subtracting $\mathcal{B}\mathcal{D}^*\mathcal{C}$ from either side implies the second identity. □

The following Lemma shows that both the L-characteristic and inverse L-characteristic preserve the minimality and equivalence of systems.

Lemma 4.3. *1. The L-characteristic of a minimal system is stable and minimal. The L-characteristics of two equivalent systems are equivalent.*
2. The inverse L-characteristic of a stable minimal system is minimal. The inverse L-characteristics of two equivalent systems are equivalent.

Proof. 1.) Since Y is the stabilizing solution of the control algebraic Riccati equation the matrix $A_L - BR_L^{-1}B^*Y$ is stable by definition. It is easily seen that the characteristic system is reachable. The observability of the system follows by the second representation of the characteristic.

Let $(A, B, C, D) \in L_n^{p,m}$. If Z is the stabilizing solution to the Riccati equation,

$$A_LZ + ZA_L^* - ZC^*S_L^{-1}CZ + BR_L^{-1}B^* = 0,$$

then TZT^* is the stabilizing solution to this Riccati equation for the system $(TAT^{-1}, TB, CT^{-1}, D)$, where T is non-singular. Similarly, if Y is the stabilizing solution to

$$A_L^*Y + YA_L - YBR_L^{-1}B^*Y + C^*S_L^{-1}C = 0,$$

then $T^{-*}YT^{-1}$ is the stabilizing solution to this Riccati equation for the system $(TAT^{-1}, TB, CT^{-1}, D)$. Using this fact it is easily seen that the L-characteristics of two equivalent systems are equivalent.

2.) The proof is similar to the proof of 1.). □

The L-characteristic and the inverse L-characteristic have interesting properties concerning the way solutions of Riccati equations respectively Lyapunov equations are mapped under the characteristic maps.

Proposition 4.1. *1. Let (A, B, C, D) be a minimal system and let Y be the stabilizing solution to the control algebraic Riccati equation and let Z be the stabilizing solution to the filter algebraic Riccati equation. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ be the L-characteristic of (A, B, C, D). Then the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

have solutions

$$P := (I + ZY)^{-1}Z = Z(I + YZ)^{-1}, \quad Q := Y + YZY.$$

2. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ be a stable minimal system. Let P, Q be the positive definite solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Let (A, B, C, D) be the inverse L-characteristic of $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$. Then

$$Y := Q(I + PQ)^{-1} = (I + QP)^{-1}Q$$

is the stabilizing solution to the control algebraic Riccati equation and

$$Z := P + PQP$$

is the stabilizing solution to the filter algebraic Riccati equation of the system (A, B, C, D).

Proof. 1.) We want to show that with $P = (I+ZY)^{-1}Z = Z(I+YZ)^{-1}$ we have,

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*.$$

To do this consider

$$(I+ZY)[\mathcal{A}P + P\mathcal{A}^*](I+YZ)$$
$$= (I+ZY)[(A_L - BR_L^{-1}B^*Y)P + P(A_L - BR_L^{-1}B^*Y)^*](I+YZ)$$
$$= (I+ZY)(A_L - BR_L^{-1}B^*Y)Z + Z(A_L - BR_L^{-1}B^*Y)^*(I+YZ)$$
$$= A_LZ + ZA_L^* + Z(YA_L + A_L^*Y)Z$$
$$-2ZYBR_L^{-1}B^*YZ - BR_L^{-1}B^*YZ - ZYBR_L^{-1}B^*.$$

Using the two Riccati equations this gives,

$$(I+ZY)[\mathcal{A}P + P\mathcal{A}^*](I+YZ)$$
$$= ZC^*S_L^{-1}CZ - BR_L^{-1}B^* + Z[YBR_L^{-1}B^*Y - C^*S_L^{-1}C)]Z$$
$$-2ZYBR_L^{-1}B^*YZ - BR_L^{-1}B^*YZ - ZYBR_L^{-1}B^*$$
$$= -(I+ZY)BR_L^{-1}B^*(I+YZ)$$
$$= -(I+ZY)\mathcal{B}\mathcal{B}^*(I+YZ),$$

which implies the claim. Now with $Q = Y + YZY$, we have

$$\mathcal{A}^*Q + Q\mathcal{A} = \mathcal{A}^*Y(I+ZY) + (I+YZ)Y\mathcal{A}$$
$$= (A_L^*Y - YBR_L^{-1}B^*Y)(I+ZY) + (I+YZ)(YA_L - YBR_L^{-1}B^*Y).$$

Using the Riccati equation, we have

$$\mathcal{A}^*Q + Q\mathcal{A} = (-YA_L - C^*S_L^{-1}C)(I+ZY) + (I+YZ)(-A_L^*Y - C^*S_L^{-1}C)$$
$$= -C^*S_L^{-1}C(I+ZY) - (I+YZ)C^*S_L^{-1}C - A_L^*Y - YA_L - Y(A_LZ + ZA_L^*)Y$$
$$= -C^*S_L^{-1}C(I+ZY) - (I+YZ)C^*S_L^{-1}C$$
$$-YBR_L^{-1}B^*Y + C^*S_L^{-1}C - Y(ZC^*S_L^{-1}CZ)Y + YBR_L^{-1}B^*Y$$
$$= -(I+YZ)C^*S_L^{-1}C(I+ZY)$$
$$= -\mathcal{C}^*\mathcal{C}.$$

2.) First note that

$$A_L = A - BR_L^{-1}D^*C$$

$$= \mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I+PQ)^{-1} - \mathcal{B}R_L^{1/2}R_L^{-1}\mathcal{D}^*S_L^{1/2}\mathcal{C}(I+PQ)^{-1}$$
$$= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1},$$

where we have used that $R_L^{1/2}\mathcal{D}^* = \mathcal{D}^*S_L^{1/2}$. Since,

$$(I+QP)[A_L^*Y + YA_L - YBR_L^{-1}B^*Y + C^*S_L^{-1}C](I+PQ)$$
$$= (I+QP)[(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1})^*Q(I+PQ)^{-1}$$
$$+(I+QP)^{-1}Q(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1})$$
$$-(I+QP)^{-1}Q\mathcal{B}R_L^{1/2}R_L^{-1}R_L^{1/2}\mathcal{B}^*Q(I+PQ)^{-1}$$
$$+(I+PQ)^{-*}\mathcal{C}^*S_L^{1/2}S_L^{-1}S_L^{1/2}\mathcal{C}(I+PQ)^{-1}](I+PQ)$$
$$= (I+QP)\mathcal{A}^*Q + Q\mathcal{B}\mathcal{B}^*Q + Q\mathcal{A}(I+PQ) + Q\mathcal{B}\mathcal{B}^*Q - Q\mathcal{B}\mathcal{B}^*Q + \mathcal{C}^*\mathcal{C}$$
$$= \mathcal{A}^*Q + Q\mathcal{A} + \mathcal{C}^*\mathcal{C} + Q(P\mathcal{A}^* + \mathcal{A}P + \mathcal{B}\mathcal{B}^*)Q$$
$$= 0,$$

we have verified the first identity. Now with $Z := P(I+QP)$ we have

$$A_LZ + ZA_L^* - ZC^*S_L^{-1}CZ + BR_L^{-1}B^*$$
$$= (\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1})(I+PQ)P + P(I+QP)(\mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1})^*$$
$$-P(I+QP)(I+PQ)^{-*}\mathcal{C}^*S_L^{1/2}S_L^{-1}S_L^{1/2}\mathcal{C}(I+PQ)^{-1}(I+PQ)P$$
$$+\mathcal{B}R_L^{1/2}R_L^{-1}R_L^{1/2}\mathcal{B}^*$$
$$= \mathcal{A}P + \mathcal{A}PQP + \mathcal{B}\mathcal{B}^*QP + P\mathcal{A}^* + PQP\mathcal{A}^* + PQ\mathcal{B}\mathcal{B}^* - P\mathcal{C}^*\mathcal{C}P + \mathcal{B}\mathcal{B}^*$$
$$= \mathcal{A}^*P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^* + (\mathcal{A}P + \mathcal{B}\mathcal{B}^*)QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$
$$= 0 - P\mathcal{A}^*QP - P\mathcal{C}^*\mathcal{C}P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$
$$= -P(\mathcal{A}^*Q + \mathcal{C}^*\mathcal{C})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$
$$= P(Q\mathcal{A})P + PQ(P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$
$$= PQ(\mathcal{A}P + P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*)$$
$$= 0,$$

which shows the second identity. Since

$$A_L - BR_L^{-1}B^*Y$$
$$= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} - \mathcal{B}R_L^{1/2}R_L^{-1}R_L^{1/2}\mathcal{B}^*Q(I+PQ)^{-1}$$
$$= \mathcal{A},$$

which is stable and

$$A_L - ZC^*S_L^{-1}C$$
$$= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} - P(I+QP)(I+PQ)^{-*}\mathcal{C}^*S_L^{1/2}S_L^{-1}S_L^{1/2}\mathcal{C}(I+PQ)^{-1}$$
$$= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I+PQ)^{-1} - P\mathcal{C}^*\mathcal{C}(I+PQ)^{-1}$$
$$= (I+PQ)[\mathcal{A} + (I+PQ)^{-1}P\mathcal{C}^*\mathcal{C}](I+PQ)^{-1} - P\mathcal{C}^*\mathcal{C}(I+PQ)^{-1}$$

$$= (I + PQ)\mathcal{A}(I + PQ)^{-1},$$

is stable, where we have used Lemma 4.2, we have shown that Z, Y are the stabilizing solutions to the Riccati equations. □

That the L-characteristic indeed induces a bijection between $L_n^{p,m}$ and $S_n^{p,m}$ is established in the following theorem.

Theorem 4.1. *The map*

$$\chi_L : L_n^{p,m} \to S_n^{p,m}$$

is a bijection that preserves system equivalence, with inverse $I\chi_L$.

Proof. It is first shown that χ_L is injective with left inverse $I\chi_L$, i.e. that $I\chi_L \cdot \chi_L$ is the identity map on $L_n^{p,m}$.

Let $(A, B, C, D) \in L_n^{p,m}$ and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in S_n^{p,m}$ be its L-characteristic, i.e.

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = (A_L, -BR_L^{-1}B^*Y, BR_L^{-1/2}, S_L^{-1/2}C(I + ZY), D),$$

where Y and Z are the stabilizing solutions to the respective Riccati equations. We know by Proposition 4.1 that the solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

are given by $P = (I+ZY)^{-1}Z = Z(I+YZ)^{-1}$, $Q = Y+YZY$. Hence we can see that $PQ = ZY$. Now apply $I\chi_L$ to $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ and set $(A_1, B_1, C_1, D_1) := I\chi_L((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}))$, then $D_1 = D$ and

$$B_1 = \mathcal{B}R_L^{1/2} = BR_L^{-1/2}R_L^{1/2} = B,$$

$$C_1 = S_L^{1/2}\mathcal{C}(I + PQ)^{-1} = S_L^{1/2}S_L^{-1/2}C(I + ZY)(I + ZY)^{-1} = C,$$

$$\begin{aligned} A_1 &= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1} + \mathcal{B}\mathcal{D}^*\mathcal{C}(I + PQ)^{-1} \\ &= A - BR_L^{-1}(D^*C + B^*Y) + BR_L^{-1}B^*Y(I + ZY)(I + ZY)^{-1} \\ &\quad + BR_L^{-1}D^*C(I + ZY)(I + ZY)^{-1} \\ &= A, \end{aligned}$$

i.e. $I\chi_L \cdot \chi_L((A, B, C, D)) = (A, B, C, D)$ for $(A, B, C, D) \in L_n^{p,m}$.

It is now shown that χ_L is surjective with right inverse $I\chi_L$, i.e. that $\chi_L \cdot I\chi_L$ is the identity map on $S_n^{p,m}$.

Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in S_n^{p,m}$ and let $(A, B, C, D) = (\mathcal{A} + \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I + PQ)^{-1}, \mathcal{B}R_L^{1/2}, S_L^{1/2}\mathcal{C}(I+PQ)^{-1}, \mathcal{D})$ be its inverse L-characteristic. Now consider

$$(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}) := \chi_L \cdot I\chi_L((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}))$$

$$= \chi_L((A, B, C, D)) = (A_L - BR_L^{-1}B^*Y, BR_L^{-1/2}, S_L^{-1/2}C(I + ZY), D),$$

where Y, Z are the stabilizing solutions to the control respectively filter algebraic Riccati equations of the system (A, B, C, D). By Proposition 4.1

$$Y = Q(I + PQ)^{-1} = (I + QP)^{-1}Q, \quad Z = P + PQP.$$

where Q, P are the positive definite solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \ \mathcal{A}^*Q + Q\mathcal{A} = \mathcal{C}^*\mathcal{C}.$$

Now, using that $A_L = \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1}$, we have $\mathcal{D}_1 = D$, and

$$\begin{aligned} \mathcal{A}_1 &= A_L - BR_L^{-1}B^*Y \\ &= \mathcal{A} + \mathcal{B}\mathcal{B}^*Q(I + QP)^{-1} - \mathcal{B}R_L^{1/2}R_L^{-1}R_L^{1/2}\mathcal{B}Q(I + QP)^{-1} \\ &= \mathcal{A}, \end{aligned}$$

$$\mathcal{B}_1 = BR_L^{-1/2} = \mathcal{B}R_L^{1/2}R_L^{-1/2} = \mathcal{B},$$

$$\mathcal{C}_1 = S_L^{-1/2}C(I + ZY) = S_L^{-1/2}S_L^{1/2}\mathcal{C}(I + PQ)^{-1}(I + PQ) = \mathcal{C},$$

which shows the claim that $\chi_L \cdot I\chi_L$ is the identity map. Therefore χ_L is a bijection with inverse $\chi_L^{-1} = I\chi_L$. That χ_L preserves system equivalence was shown in Lemma 4.3. □

This theorem was first shown in ([27]) where it was used to show that the manifolds $L_n^{p,m}/\sim$ and $S_n^{p,m}/\sim$ are diffeomorphic.

In the following corollary it is shown that the L-characteristic maps LQG balanced systems to stable minimal systems whose reachability and observability grammians are diagonal.

Corollary 4.1. *Let $(A, B, C, D) \in L_n^{p,m}$ and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be its L-characteristic. Let $Y(Z)$ be the stabilizing solution to the control (filter) algebraic Riccati equation and let P, Q be the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \ \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then

1. *if (A, B, C, D) is LQG balanced with LQG grammian Σ_L then*

$$P = \Sigma_L(I + \Sigma_L^2)^{-1}, \ Q = \Sigma_L(I + \Sigma_L^2).$$

2. *if $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is Lyapunov balanced with Lyapunov grammian Σ_S, then*

$$Y = \Sigma_S(I + \Sigma_S^2)^{-1}, \ Z = \Sigma_S(I + \Sigma_S^2).$$

Proof. The statements follow immediately from Proposition 4.1. □

If $\Gamma_S : \ S_n^{p,m} \rightarrow S_n^{p,m}$ is a canonical form, then a canonical form can be defined on $L_n^{p,m}$ using the bijection χ_L by setting

$$\Gamma := \chi_L^{-1} \circ \Gamma_S \circ \chi_L.$$

If Γ_S is the Lyapunov balanced canonical form, it follows from Corollary 4.1 that a system in the Γ canonical form is close to being LQG-balanced. In fact a simple diagonal state space transformation will LQG-balance the system.

Lemma 4.4. *1. If $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in S_n^{p,m}$ is such that the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \ \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

have solutions

$$P = \Sigma(I+\Sigma^2)^{-1}, \ Q = \Sigma(I+\Sigma^2)$$

for some positive diagonal matrix Σ, then

$$\Delta_S((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})) := (T\mathcal{A}T^{-1}, T\mathcal{B}, \mathcal{C}T^{-1}, \mathcal{D})$$

with $T = (I+\Sigma^2)^{\frac{1}{2}}$ is Lyapunov balanced with Lyapunov grammian Σ.

2. If $(A,B,C,D) \in L_n^{p,m}$ is such that the control (filter) algebraic Riccati equation has the stabilizing solution $Y(Z)$ with

$$Y = \Sigma(I+\Sigma^2)^{-1}, \ Z = \Sigma(I+\Sigma^2)$$

for some positive diagonal matrix Σ, then

$$\Delta_L((A,B,C,D)) := (TAT^{-1}, TB, CT^{-1}, D)$$

with $T = (I+\Sigma^2)^{\frac{1}{2}}$ is LQG balanced with LQG grammian Σ.

Proof. 1.) If the system $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ has reachability grammian P and observability grammian Q then the system $(T\mathcal{A}T^{-1}, T\mathcal{B}, \mathcal{C}T^{-1}, \mathcal{D})$ has reachability grammian TPT^* and observability grammian $T^{-*}QT^{-1}$. With P, Q and T as in the statement of the Lemma we therefore have

$$TPT^* = (I+\Sigma^2)^{\frac{1}{2}}\Sigma(I+\Sigma^2)^{-1}(I+\Sigma^2)^{\frac{1}{2}} = \Sigma,$$

$$T^{-*}QT^{-1} = (I+\Sigma^2)^{-\frac{1}{2}}\Sigma(I+\Sigma^2)(I+\Sigma^2)^{-\frac{1}{2}} = \Sigma,$$

which implies 1.).
2.) This is shown analogously to 1.). □

With the diagonal scaling map Δ_L as defined in the previous Lemma set

$$\Gamma_L := \Delta_L \circ \chi_L^{-1} \circ \Gamma_S \circ \chi_L.$$

The diagonal scaling now assures that Γ_L is a canonical form for $L_n^{p,m}$ in terms of LQG balanced systems. This fact will be used in the following Theorem to derive a canonical form and parametrization for $L_n^{p,m}$. This canonical form is called the *LQG balanced canonical form.*

Theorem 4.2. *Let (A_l, B_l, C_l, D_l) be a m-input p-output continuous-time system of dimension n. Then the following are equivalent:*

1. *(A_l, B_l, C_l, D_l) is a minimal system, i.e in $L_n^{p,m}$.*
2. *$(A_l, B_l, C_l, D_l) = (TAT^{-1}, TB, CT^{-1}, D)$ for some invertible T, where (A,B,C,D) is in LQG balanced form, i.e. there exist block indices $n_1, \ldots, n_k$, $\sum_{j=1}^k n_j = n$, parameters $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$ and k families of step sizes $m = \tau_0^j \geq \tau_1^j \geq \tau_2^j \geq \cdots \geq \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \leq j \leq k$, such that*
   a) $B = (\underbrace{\overline{B}_1^T, 0}_{n_1}, \underbrace{\overline{B}_2^T, 0}_{n_2}, \ldots, \underbrace{\overline{B}_k^T, 0}_{n_k})^T R_L^{\frac{1}{2}}$, where $\overline{B}_j \in \mathcal{K}^{\tau_1^j \times \tau_0^j}$ is positive upper triangular, $1 \leq j \leq k$.
   b) $C = S_L^{\frac{1}{2}}(\underbrace{U_1(\overline{B}_1\overline{B}_1^*)^{\frac{1}{2}}, 0}_{n_1}, \underbrace{U_2(\overline{B}_2\overline{B}_2^*)^{\frac{1}{2}}, 0}_{n_2}, \ldots, \underbrace{U_k(\overline{B}_k\overline{B}_k^*)^{\frac{1}{2}}, 0}_{n_k})$, where $U_j \in \mathcal{K}^{p \times \tau_1^j}$, $U_j^* U_j = I_{\tau_1^j}$, $1 \leq j \leq k$.
   c)

$$A = \left(\begin{array}{c|c|c|c|c} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline A(i,1) & \cdots & A(i,i) & \cdots & A(i,k) \\ \hline \vdots & & \vdots & \ddots & \vdots \\ \hline A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \leq i \leq k$

$$A(i,i) = \begin{pmatrix} \tilde{A}_{ii} + \mathcal{S}_1^i & (-\mathcal{A}_1^i)^* & & 0 \\ \mathcal{A}_1^i & \mathcal{S}_2^i & \ddots & \\ & \ddots & \ddots & (-\mathcal{A}_{l_i-1}^i)^* \\ 0 & & \mathcal{A}_{l_i-1}^i & \mathcal{S}_{l_i}^i \end{pmatrix},$$

for $1 \leq i,j \leq k$, $i \neq j$

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

i. $\mathcal{S}_i^j$ *is a* $\tau_i^j \times \tau_i^j$ *skew-hermitian matrix,* $i = 1, 2, \ldots, l_j$, $1 \le j \le k$.
ii. $\mathcal{A}_i^j$ *is a positive upper triangular* $\tau_{i+1}^j \times \tau_i^j$ *matrix,* $i = 1, 2, \ldots, l_j - 1$, $1 \le j \le k$.
iii. $\tilde{A}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ *is given by*

$$\tilde{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}\left(\sigma_j(1+\sigma_i^2)\overline{B}_i\overline{B}_j^* - \sigma_i(1+\sigma_j^2)(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} U_i^* U_j (\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}\right)$$

$$+\overline{B}_i D^* U_j (\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}},$$

for $1 \le i, j \le k, \quad i \ne j,$ *and*

$$\tilde{A}_{ii} = -\frac{1}{2\sigma_i}(1-\sigma_i^2)\overline{B}_i\overline{B}_i^* + \overline{B}_i D^* U_i (\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} \quad \text{for} \quad 1 \le i \le k.$$

Moreover, the system (A, B, C, D) *as defined in (2) is LQG balanced with LQG grammian* $\Sigma_L = diag(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$*, where* $n_1, \ldots, n_k$ *are the block indices and* $\sigma_1 > \cdots > \sigma_k > 0$ *are the LQG singular values of the system in (2). The map* Γ_L *that assigns to each system in* $L_n^{p,m}$ *the realization in (2) is a canonical form.*

Proof. Let $(A_l, B_l, C_l, D_l) \in L_n^{p,m}$. Then $\chi_L((A_l, B_l, C_l, D_l))$ is in $S_n^{p,m}$. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) := \Gamma_S(\chi_L((A_l, B_l, C_l, D_l)))$ be the Lyapunov balanced canonical form of $\chi_L((A_l, B_l, C_l, D_l))$. Since χ_L and χ_L^{-1} respects system equivalence the system

$$(A, B, C, D) = \Delta_L(\chi_L^{-1}(\Gamma_S(\chi_L((A_l, B_l, C_l, D_l)))))$$

is equivalent to (A_l, B_l, C_l, D_l). Moreover the system is LQG balanced. It is straightforward to check that $\Gamma_L := \Delta_L \circ \chi_L^{-1} \circ \Gamma_L \circ \chi_L$ defines a canonical form for $L_n^{p,m}$.

It is necessary to show that (A, B, C, D) admits the stated parametrization. Consider now the Lyapunov balanced parametrization of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, i.e. let $n_1, \ldots, n_k$, $\sum_{j=1}^k n_j = n$ be the block sizes, let $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$ the Lyapunov singular values, let $m = \tau_0^j \ge \tau_1^j \ge \tau_2^j \ge \cdots \ge \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \le j \le k$ the families of step sizes such that

1. $\mathcal{B} = (\underbrace{\overline{\mathcal{B}}_1^T, 0}_{n_1}, \underbrace{\overline{\mathcal{B}}_2^T, 0}_{n_2}, \ldots, \underbrace{\overline{\mathcal{B}}_k^T, 0}_{n_k})^T$,
   where $\overline{\mathcal{B}}_j \in \mathcal{K}^{\tau_1^j \times \tau_0^j}$ is positive upper triangular, $1 \le j \le k$.
2. $\mathcal{C} = (\underbrace{\mathcal{U}_1(\overline{\mathcal{B}}_1\overline{\mathcal{B}}_1^*)^{\frac{1}{2}}, 0}_{n_1}, \underbrace{\mathcal{U}_2(\overline{\mathcal{B}}_2\overline{\mathcal{B}}_2^*)^{\frac{1}{2}}, 0}_{n_2}, \ldots, \underbrace{\mathcal{U}_k(\overline{\mathcal{B}}_k\overline{\mathcal{B}}_k^*)^{\frac{1}{2}}, 0}_{n_k})$,
   where $\mathcal{U}_j \in \mathcal{K}^{p \times \tau_1^j}, \mathcal{U}_j^*\mathcal{U}_j = I_{\tau_1^j}$, $1 \le j \le k$.

3.

$$A = \left(\begin{array}{c|c|c|c|c} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline A(i,1) & \cdots & A(i,i) & \cdots & A(i,k) \\ \hline \vdots & & \vdots & \ddots & \vdots \\ \hline A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \le i \le k$

$$A(i,i) = \begin{pmatrix} \tilde{A}_{ii} + \mathcal{S}_1^i & (-\mathcal{A}_1^i)^* & & 0 \\ \mathcal{A}_1^i & \mathcal{S}_2^i & \ddots & \\ & \ddots & \ddots & (-\mathcal{A}_{l_i-1}^i)^* \\ 0 & & \mathcal{A}_{l_i-1}^i & \mathcal{S}_{l_i}^i \end{pmatrix},$$

for $1 \le i, j \le k$, $i \ne j$

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and
a) $\mathcal{S}_i^j$ is a $\tau_i^j \times \tau_i^j$ skew-hermitian matrix, $i = 1, 2, \ldots, l_j$, $1 \le j \le k$.
b) $\mathcal{A}_i^j$ is a positive upper triangular $\tau_{i+1}^j \times \tau_i^j$ matrix, $i = 1, 2, \ldots, l_j - 1$, $1 \le j \le k$.
c) $\tilde{\mathcal{A}}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ is given by

$$\tilde{\mathcal{A}}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}\left(\sigma_j \overline{\mathcal{B}}_i\overline{\mathcal{B}}_j^* - \sigma_i(\overline{\mathcal{B}}_i\overline{\mathcal{B}}_i^*)^{\frac{1}{2}}\mathcal{U}_i^*\mathcal{U}_j(\overline{\mathcal{B}}_j\overline{\mathcal{B}}_j^*)^{\frac{1}{2}}\right),$$

$$\text{for } 1 \le i, j \le k, \quad i \ne j, \quad \text{and } \tilde{\mathcal{A}}_{ii} = -\frac{1}{2\sigma_i}\overline{\mathcal{B}}_i\overline{\mathcal{B}}_i^* \quad \text{for } 1 \le i \le k.$$

The system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is Lyapunov balanced with Lyapunov grammian $\Sigma = diag(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$. Then by Corollary 4.1 and Lemma 4.4

$$\begin{aligned} (A, B, C, D) \;=\; & ((I + \Sigma^2)^{-\frac{1}{2}}(\mathcal{A} + \mathcal{B}(\mathcal{B}^*\Sigma + \mathcal{D}^*\mathcal{C})(I + \Sigma^2)^{-1})(I + \Sigma^2)^{\frac{1}{2}}, \\ & (I + \Sigma^2)^{-\frac{1}{2}}\mathcal{B}R_L^{\frac{1}{2}}, S_L^{\frac{1}{2}}\mathcal{C}(I + \Sigma^2)^{-\frac{1}{2}}, \mathcal{D}). \end{aligned}$$

Setting

$$\overline{B_j} := \frac{1}{\sqrt{1 + \sigma_j^2}}\overline{\mathcal{B}}_j, \quad 1 \le i \le k,$$

it follows that $B = (I+\Sigma^2)^{-\frac{1}{2}}\mathcal{B}R_L^{\frac{1}{2}}$ has the required structure. Since with $U_j := \mathcal{U}_j$

$$S_L^{\frac{1}{2}}\mathcal{U}_j(\overline{\mathcal{B}}_j\overline{\mathcal{B}}_j^*)^{\frac{1}{2}}\frac{1}{\sqrt{1+\sigma_j^2}} = S_L^{\frac{1}{2}}U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$

for $1 \le i \le k$, it follows that $C = S_L^{\frac{1}{2}}\mathcal{C}(I+\Sigma^2)^{-\frac{1}{2}}$ also has the required structure. Note that

$$A = (I+\Sigma^2)^{-\frac{1}{2}}(\mathcal{A}+\mathcal{B}(\mathcal{B}^*\Sigma+\mathcal{D}^*\mathcal{C})(I+\Sigma^2)^{-1})(I+\Sigma^2)^{\frac{1}{2}}$$
$$= (I+\Sigma^2)^{-\frac{1}{2}}\mathcal{A}(I+\Sigma^2)^{\frac{1}{2}} + BR_L^{-1}B^*\Sigma + BR_L^{-1}D^*BS_L^{-1}C.$$

If (A,B,C,D) is partitioned according to the block indices $n_1,\dots,n_k$, then for $1 \le i,j \le k$, $i \ne j$,

$$A_{ij} = \frac{\sqrt{1+\sigma_j^2}}{\sqrt{1+\sigma_i^2}}\mathcal{A}_{ij} + B_iR_L^{-1}B_j^*\sigma_j + B_jR_L^{-1}D^*S_L^{-1}C_j.$$

This shows that all entries of A_{ij} are zero with the exception of the principal $\tau_1^i \times \tau_1^j$-subblock $\tilde{A}_{ij}$ which is given by

$$\tilde{A}_{ij} = \frac{\sqrt{1+\sigma_j^2}}{\sqrt{1+\sigma_i^2}}\tilde{\mathcal{A}}_{ij} + \overline{B}_i\overline{B}_j^*\sigma_j + \overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{\sqrt{1+\sigma_j^2}}{\sqrt{1+\sigma_i^2}}\left(\frac{1}{\sigma_i^2-\sigma_j^2}(\sigma_j\overline{\mathcal{B}}_i\overline{\mathcal{B}}_j^* - \sigma_i(\overline{\mathcal{B}}_i\overline{\mathcal{B}}_i^*)^{\frac{1}{2}}\mathcal{U}_i^*\mathcal{U}_j(\mathcal{B}_j\overline{\mathcal{B}}_j^*)^{\frac{1}{2}})\right)$$
$$+\overline{B}_i\overline{B}_j^*\sigma_j + \overline{B}_iD^*U_j(\overline{B}_i\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{1+\sigma_j^2}{\sigma_i^2-\sigma_j^2}\left(\sigma_j\overline{B}_i\overline{B}_j^* - \sigma_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}U_i^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}\right)$$
$$+\overline{B}_i\overline{B}_j^*\sigma_j + \overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{1}{\sigma_i^2-\sigma_j^2}\left(\sigma_j(1+\sigma_i^2)\overline{B}_i\overline{B}_j^* - \sigma_i(1+\sigma_j^2)(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}U_i^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}\right)$$
$$+\overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$

for $1 \le i,j \le k$, $i \ne j$. For $1 \le i \le k$, we have

$$A_{ii} = \mathcal{A}_{ii} + B_iR_L^{-1}(B_i^*\sigma_i + D^*S_L^{-1}C_i).$$

The principal $\tau_1^i \times \tau_1^i$ submatrix of A_{ii}, $1 \le i \le k$, is given by

$$\tilde{\mathcal{A}}_{ii} + \mathcal{S}_1^i + \overline{B}_iR_L^{\frac{1}{2}}R_L^{-1}(R_L^{\frac{1}{2}}\overline{B}_i^*\sigma_i + D^*S_L^{\frac{1}{2}}U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}})$$

$$= -\frac{1}{2\sigma_i}\overline{B}_i\overline{B}_i^* + \mathcal{S}_1^i + \overline{B}_i\overline{B}_i^*\sigma_i + \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}$$
$$= -\frac{1}{2\sigma_i}(1+\sigma_i^2)\overline{B}_i\overline{B}_i^* + \mathcal{S}_1^i + \overline{B}_i\overline{B}_i^*\sigma_i + \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}$$
$$= -\frac{1}{2\sigma_i}(1-\sigma_i^2)\overline{B}_i\overline{B}_i^* + \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} + \mathcal{S}_1^i,$$

where we set $S_i^j = \mathcal{S}_i^j$ for $i = 1,2,\dots,l_j$, $1 \le j \le k$. This shows that A has the stated form. Therefore (A,B,C,D) is in the stated canonical form. Hence Γ_L has the claimed properties.

To complete the proof it remains to be shown that if a system (A,B,C,D) has the parametrization stated in 2.) then the system is minimal and LQG balanced with LQG grammian $\Sigma_L = diag(\sigma_1 I_{n_1},\dots,\sigma_k I_{n_k})$. Let

$$(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) := ((I+\Sigma^2)^{\frac{1}{2}}(A_L - BR_L^{-1}B^*\Sigma)(I+\Sigma^2)^{-\frac{1}{2}},$$
$$(I+\Sigma^2)^{\frac{1}{2}}BR_L^{-\frac{1}{2}}, S_L^{-\frac{1}{2}}C(I+\Sigma^2)^{-\frac{1}{2}}, D).$$

Straightforward algebraic manipulations, similar to those in the first part of the proof, show that $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ admits the parametrization for Lyapunov balanced systems as given in Theorem 3.2. Hence $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in \mathcal{S}_n^{p,m}$ is Lyapunov balanced with Lyapunov grammian Σ_L. Then it follows from the construction of $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ as the 'formal' pre-image of (A,B,C,D) under $\Delta_L \circ \chi_L^{-1}$, that $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) = \chi_L(\Delta_L^{-1}((A,B,C,D)))$. Hence (A,B,C,D) is minimal and LQG balanced with LQG grammian Σ_L. □

It is worth commenting that as was established in the proof of the theorem, if $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ is a Lyapunov balanced canonical form and $(A,B,C,D) = \Delta_L(\chi_L^{-1}((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})))$ then up to some scaling of the $\overline{B}_i$ parameters, the parameters of (A,B,C,D) are the same as those for $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$. The only essential difference between the parametrization of $\mathcal{S}_n^{p,m}$ given in Theorem 3.2 and the parametrization of $\mathcal{L}_n^{p,m}$ given in the previous Theorem is the way in which the system parameters go into the entries $\tilde{A}_{ij}$, $1 \le i,j \le k$.

Specialization of the theorem to the single input single output case gives the following corollary.

Corollary 4.2. *Let (A_l,b_l,c_l,d_l) be a single-input single-output continuous-time system of dimension n. Then the following statements are equivalent:*

1. *(A_l,b_l,c_l,d_l) is a minimal system.*
2. *$(A_l,b_l,c_l,d_l) = (TAT^{-1},Tb,cT^{-1},d)$ for some invertible matrix T, where (A,b,c,d) is in the following LQG balanced form with block indices $n_1,n_2,\dots,n_k$, $\sum_{i=1}^k n_i = n$:*
   a) $b = (\underbrace{b_1,0,\dots,0}_{n_1},\underbrace{b_2,0,\dots,0}_{n_2},\dots,\underbrace{b_k,0,\dots,0}_{n_k})^T$,
   where $b_i > 0$, $1 \le i \le k$.

b) $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n_1}, \underbrace{s_2 b_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n_k}),$
where $s_i \in \mathcal{K}$, $|s_i| = 1$, $1 \le i \le k$.

c)

$$A = \left(\begin{array}{c|c|c|c} A(1,1) & A(1,2) & \cdots & A(1,k) \\ \hline A(2,1) & A(2,2) & \cdots & A(2,k) \\ \hline \vdots & & \ddots & \vdots \\ \hline A(k,1) & A(k,2) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \le j \le k$

$$A(j,j) = \begin{pmatrix} a_{jj} + i\beta_1^j & -\alpha_1^j & & 0 \\ \alpha_1^j & i\beta_2^j & \ddots & \\ & \ddots & \ddots & -\alpha_{n_j-1}^j \\ 0 & & \alpha_{n_j-1}^j & i\beta_{n_j}^j \end{pmatrix},$$

for $1 \le i,j \le k$, $i \ne j$,

$$A(i,j) = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\alpha_j^i > 0 \text{ for } 1 \le j \le n_i - 1, \quad 1 \le i \le k,$$

$$a_{ij} = \tfrac{-b_i b_j}{1+|d|^2}\left[\tfrac{\bar{s}_i s_j \sigma_i (1+\sigma_j^2) - \sigma_j (1+\sigma_i^2)}{\sigma_i^2 - \sigma_j^2} - s_j d^*\right] \quad \text{for } 1 \le i,j \le k, \quad i \ne j,$$

$$a_{ii} = -\frac{b_i^2}{1+|d|^2}\left[\frac{1}{2\sigma_i}(1-\sigma_i^2) - s_i d^*\right], \text{ for } 1 \le i \le k,$$

$$\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

If $\mathcal{K} = \Re$, *then*

$$s_i = \pm 1, \quad 1 \le i \le k,$$

$$\beta_j^i = 0, \text{ for } 1 \le j \le n_i, \quad 1 \le i \le k,$$

$$a_{ij} = \frac{-b_i b_j}{1+d^2}\left[\frac{1 - s_i s_j \sigma_i \sigma_j}{s_i s_j \sigma_i + \sigma_j} - s_j d\right], \text{ for } 1 \le i,j \le k.$$

Moreover, the system (A, b, c, d) *as defined in* (2) *is LQG balanced with LQG grammian* $\Sigma_L = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. *The map* Γ_L *that assigns to each system in* $L_n^{1,1}$ *the realization in* (2) *is a canonical form.*

Proof. The result follows immediately from the previous theorem by absorbing $R_L^{\frac{1}{2}} = S_L^{\frac{1}{2}}$ into $\overline{B}_i$ parameters and setting $b_i := \overline{B}_i R_L^{\frac{1}{2}}$, $1 \le i \le k$. The remaining parameter replacements are as in the proof of Corollary 3.3. □

Having established a parametrization of $L_n^{p,m}$ in terms of LQG balanced realizations, LQG balanced model reduction can now be analyzed.

Theorem 4.3. *Let* $(A, B, C, D) \in L_n^{p,m}$ *be in LQG balanced canonical form with LQG grammian* Σ. *Let* $1 \le r < n$. *Then the r-dimensional balanced approximant* (A_{11}, B_1, C_1, D) *is in* $L_r^{p,m}$ *and is in LQG balanced canonical form with LQG grammian* Σ_1, *where* $\Sigma = diag(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathcal{K}^{r \times r}$.

Proof. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = \Delta_L(\chi_L((A, B, C, D)))$. It follows by Corollary 4.1 and Lemma 4.4 that if $(A, B, C, D) \in L_n^{p,m}$ is in LQG balanced canonical form with LQG grammian Σ, then $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is in Lyapunov balanced canonical form with Lyapunov grammian Σ.

Let $P_r \in \mathcal{K}^{r \times n}$ be the projection matrix given by

$$P_r = (\delta_{ij})_{1 \le i \le r, 1 \le j \le n},$$

then

$$(A_{11}, B_1, C_1, D) = (P_r A P_r^*, P_r B, C P_r^*, D).$$

If $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D})$ is the r-dimensional balanced approximant of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ then

$$B_1 = P_r B = P_r (I + \Sigma^2)^{-\frac{1}{2}} \mathcal{B} R_L^{\frac{1}{2}} = (I + \Sigma_1^2)^{-\frac{1}{2}} P_r \mathcal{B} R_L^{\frac{1}{2}} = (I + \Sigma_1^2)^{-\frac{1}{2}} \mathcal{B}_1 R_L^{\frac{1}{2}},$$

$$C_1 = C P_r^* = S_L^{\frac{1}{2}} \mathcal{C} (I + \Sigma^2)^{-\frac{1}{2}} P_r^* = S_L^{\frac{1}{2}} \mathcal{C} P_r^* (I + \Sigma_1^2)^{-\frac{1}{2}} = S_L^{\frac{1}{2}} \mathcal{C}_1 (I + \Sigma_1^2)^{-\frac{1}{2}},$$

$$\begin{aligned} A_{11} = P_r A P_r^* &= P_r (I + \Sigma^2)^{-\frac{1}{2}} (\mathcal{A} + \mathcal{B}(\mathcal{B}^* \Sigma + \mathcal{D}^* \mathcal{C})(I + \Sigma^2)^{-1})(I + \Sigma^2)^{\frac{1}{2}} P_r^* \\ &= (I + \Sigma_1^2)^{-\frac{1}{2}} (P_r \mathcal{A} P_r^* + P_r \mathcal{B}(\mathcal{B}^* P_r^* \Sigma_1 + \mathcal{D}^* \mathcal{C} P_r)(I + \Sigma_1^2)^{-1})(I + \Sigma_1^2)^{\frac{1}{2}} \\ &= (I + \Sigma_1^2)^{-\frac{1}{2}} (\mathcal{A}_{11} + \mathcal{B}_1(\mathcal{B}_1^* \Sigma_1 + \mathcal{D}^* \mathcal{C}_1)(I + \Sigma_1^2)^{-1})(I + \Sigma_1^2)^{\frac{1}{2}}. \end{aligned}$$

Since Σ_1 is the Lyapunov grammian of the system $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D})$ which is in Lyapunov balanced canonical form, this shows that

$$(A_{11}, B_1, C_1, D) = \Delta_L(\chi_L^{-1}((\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D})))$$

and hence by the properties of $\Delta_L \circ \chi_L^{-1}$, the system (A_{11}, B_1, C_1, D) is minimal and in LQG balanced canonical form with LQG grammian Σ_1. □

As a corollary we also obtain a model reduction result for general LQG balanced systems ([14]). Analogously to Theorem 2.1 a condition has to be placed on the point at which truncation occurs.

Corollary 4.3. *Let (A,B,C,D) be a n-dimensional LQG balanced system in $L_n^{p,m}$ with LQG grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$, $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$. Let $r = n_1 + n_2 + \cdots + n_l$ for some $1 \leq l \leq k$. Then the r-dimensional balanced approximant (A_{11}, B_1, C_1, D) of (A,B,C,D) is minimal and LQG balanced with LQG grammian $\Sigma_1 = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_l I_{n_l})$.*

Proof. Let (A,B,C,D) be as in the statement. By Theorem 1.1 all equivalent LQG balanced systems are given by (QAQ^*, QB, CQ^*, D) with $Q = diag(Q_1, Q_2, \ldots, Q_k)$, $Q_i \in \mathcal{K}^{n_i \times n_i}$ unitary. Let $Q_0 = diag(Q_1^0, Q_2^0, \ldots, Q_k^0)$, $Q_i^0 \in \mathcal{K}^{n_i \times n_i}$ unitary, be such that $(Q_0 A Q_0^*, Q_0 B, C Q_0^*, D)$ is in LQG balanced canonical form. By Theorem 4.3 the r-dimensional balanced approximant of this system is in $L_r^{p,m}$ and LQG balanced with LQG grammian $\Sigma_1 = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_l I_{n_l})$. Because of the block diagonal structure of Q_0 and the assumption on r, the approximant can be written as $(\hat{Q}_0 A_{11} \hat{Q}_0^*, \hat{Q}_0 B_1, C_1 \hat{Q}_0^*, D)$ where $\hat{Q}_0 = (\hat{Q}_1^0, \hat{Q}_2^0, \ldots, \hat{Q}_l^0)$. Since $\hat{Q}_0$ is unitary this shows that (A_{11}, B_1, C_1, D) has the claimed properties. □

5. Characteristics, Canonical Forms and Model Reduction for Bounded-Real and Positive-Real Systems

The purpose of this section is to derive canonical forms for bounded-real and positive-real systems in terms of bounded-real respectively positive-real balanced systems. The approach that is taken is analogous to the one used in the previous section to derive the LQG balanced canonical form for minimal systems in $L_n^{p,m}$.

Analogously to the discussion in the previous section we will introduce characteristic maps χ_B and χ_P for bounded-real and positive-real systems. These characteristic maps are used to carry the Lyapunov balanced canonical form for systems in $S_n^{p,m}$ over to $B_n^{p,m}$ respectively P_n^m to introduce a canonical form for bounded real and positive real systems. In contrast to the characteristic map χ_L, the range space of χ_B and χ_P will no longer be the set $S_n^{p,m}$ of stable minimal systems of given McMillan degree, but rather the subsets $US_{n,B}^{p,m}$ and $US_{n,P}^{m,m}$. A system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in S_n^{p,m} (S_n^{m,m})$ is in $US_{n,B}^{p,m}$ $(US_{n,P}^{m,m})$ if

1.) $\lambda_{max}(PQ) < 1$, where P, Q are the solutions to the Lyapunov equation

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

2.) $I - \mathcal{D}^*\mathcal{D} > 0 \quad (\mathcal{D} + \mathcal{D}^* > 0)$.

The results on the parameterization of bounded-real/positive-real systems will be used to analyze the balanced model reduction method for bounded-real and positive-real systems.

Throughout this section we will use the following abbreviations. Let (A,B,C,D) be a system. If $I - D^*D > 0$ set $R_B := I - D^*D$, $S_B := I - DD^*$ and $A_B := A + BR_B^{-1}D^*C$. If the system has equal input and output dimensions and $D + D^* > 0$ set $R_P := D + D^*$ and $A_P := A - BR_P^{-1}C$.

The characteristic map and inverse characteristic map for bounded-real systems is defined as follows.

Definition 5.1. *1.) Let (A,B,C,D) be a bounded-real system. Let Y be the stabilizing solution of the control bounded-real Riccati equation and let Z be the stabilizing solution of the filter bounded real Riccati equation, i.e.*

$$0 = A_B^*Y + YA_B + YBR_B^{-1}B^*Y + C^*S_B^{-1}C$$

$$0 = A_BZ + ZA_B^* + ZC^*S_B^{-1}CZ + BR_B^{-1}B^*$$

with $A_B + BR_B^{-1}B^*Y$ and $A_B + ZC^*S_B^{-1}C$ stable. Then the system

$$\chi_B((A,B,C,D)) := (A_B + BR_B^{-1}B^*Y, BR_B^{-\frac{1}{2}}, S_B^{-\frac{1}{2}}C(I - ZY), D)$$

$$= ((I - ZY)^{-1}(A_B + ZC^*S_B^{-1}C)(I - ZY), BR_B^{-\frac{1}{2}}, S_B^{-\frac{1}{2}}C(I - ZY), D)$$

is called the B-characteristic *of the system (A,B,C,D).*

2.) Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a stable minimal system in $US_{n,B}^{p,m}$ and let P and Q be the solutions of the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then the system

$$I\chi_B((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})) := (\mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1}, \mathcal{B}R_B^{\frac{1}{2}}, S_B^{\frac{1}{2}}\mathcal{C}(I - PQ)^{-1}, \mathcal{D})$$

$$= ((I - PQ)(\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C})(I - PQ)^{-1}, \mathcal{B}R_B^{\frac{1}{2}}, S_B^{\frac{1}{2}}\mathcal{C}(I - PQ)^{-1}, \mathcal{D})$$

is called the inverse B-characteristic *of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Here $R_B := I - \mathcal{D}^*\mathcal{D}$ and $S_B := I - \mathcal{D}\mathcal{D}^*$.*

Since the analysis of the characteristic map for bounded-real systems is quite similar to that of the characteristic map for positive-real systems, the definition of the positive real characteristic will be given now.

Definition 5.2. *1.) Let (A,B,C,D) be a positive-real system. Let Y be the stabilizing solution of the control positive real Riccati equation and let Z be the stabilizing solution of the filter positive real Riccati equation, i.e.*

$$0 = A_P^*Y + YA_P + YBR_P^{-1}B^*Y + C^*R_P^{-1}C$$

$$0 = A_PZ + ZA_P^* + ZC^*R_P^{-1}CZ + BR_P^{-1}B^*$$

with $A_P + BR_P^{-1}B^*Y$ and $A_P + ZC^*R_P^{-1}C$ stable. Then the system

$$\chi_P((A,B,C,D)) := (A_P + BR_P^{-1}B^*Y, BR_P^{-\frac{1}{2}}, R_P^{-\frac{1}{2}}C(I - ZY), D)$$

$$= ((I - ZY)^{-1}(A_P + ZC^* R_P^{-1} C)(I - ZY), BR_P^{-\frac{1}{2}}, R_P^{-\frac{1}{2}} C(I - ZY), D)$$

is called the P-characteristic *of the system* (A, B, C, D).

2.) Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ *be a stable minimal system in* $US^{m,m}_{n,P}$ *and let* P *and* Q *be the solutions of the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^* Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Then the system

$$I\chi_P((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})) := (\mathcal{A} - \mathcal{B}(\mathcal{B}^* Q - \mathcal{C})(I - PQ)^{-1}, \mathcal{B}R_P^{\frac{1}{2}}, R_P^{\frac{1}{2}}\mathcal{C}(I - PQ)^{-1}, \mathcal{D})$$

$$= ((I - PQ)(\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B})\mathcal{C})(I - PQ)^{-1}, \mathcal{B}R_P^{\frac{1}{2}}, R_P^{\frac{1}{2}}\mathcal{C}(I - PQ)^{-1}, \mathcal{D})$$

is called the inverse P-characteristic *of the system* $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. *Here* $R_P = \mathcal{D} + \mathcal{D}^*$.

Note that both expressions for the B-characteristic and P-characteristic are identical because of the following Lemma.

Lemma 5.1. *1. With the assumptions as in the Definition 5.1,*

$$(I - ZY)(A_B + BR_B^{-1}B^* Y) = (A_B + ZC^* S_B^{-1} C)(I - ZY),$$

where Y *is the stabilizing solution to the control bounded-real Riccati equation and* Z *is the stabilizing solution to the filter bounded-real Riccati equation.*

2. With the assumptions as in the Definition 5.2,

$$(I - ZY)(A_P + BR_P^{-1}B^* Y) = (A_P + ZC^* S_P^{-1} C)(I - ZY),$$

where Y *is the stabilizing solution to the control positive-real Riccati equation and* Z *is the stabilizing solution to the filter positive-real Riccati equation.*

Proof. 1.) Consider the two Riccati equations,

$$0 = A_B^* Y + Y A_B - Y B R_B^{-1} B^* Y + C^* S_B^{-1} C,$$

$$0 = A_B Z + Z A_B^* - Z C^* R_B^{-1} C Z + B S_B^{-1} B^*.$$

Multiplying the first equation on the left by $-Z$ and the second equation on the right by $-Y$, equating both equations and adding A_B to both sides we obtain

$$A_B - Z A_B^* Y - Z Y A_B - Z Y B R_B^{-1} B^* Y - Z C^* S_B^{-1} C$$

$$= A_B - A_B Z Y - Z A_B^* Y - Z C^* S_B^{-1} C Z Y - B R_B^{-1} B^* Y.$$

Canceling the term $Z A_B^* Y$ from either side and collecting terms, we obtain

$$(I - ZY)(A_B + BR_B^{-1}B^* Y) = (A_B + ZC^* S_B^{-1} C)(I - ZY).$$

2.) The statement follows analogously to 1.).

□

That $I - ZY$ is invertible follows by general results on the *gap* between the maximal and minimal solution to the bounded-real Riccati equation (see e.g. [34]). The analogous argument shows that the two expressions for the positive real characteristic are identical and well-defined.

It is now necessary to analyze the bounded-real and positive-real characteristic. The main result is Theorem 5.1 in which it will be shown that these two characteristic maps are bijections whose inverses are the corresponding inverse characteristic maps. To this end the following Lemmas and Propositions need to be established.

That the two expressions for the inverse B-characteristic and the inverse P-characteristic are identical follows from the following Lemma.

Lemma 5.2. *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ *be a stable minimal system and let* P *and* Q *be such that*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^* Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

1. If $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in US^{p,m}_{n,B}$ *then*

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^* Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1}][I - PQ]$$

$$= [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}],$$

and

$$[\mathcal{A} - \mathcal{B}\mathcal{B}^* Q(I - PQ)^{-1}][I - PQ] = [I - PQ][\mathcal{A} - (I - PQ)^{-1} P\mathcal{C}^*\mathcal{C}].$$

2. If $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in US^{m,m}_{n,P}$ *then*

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^* Q + \mathcal{C})(I - PQ)^{-1}][I - PQ] = [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B})\mathcal{C}],$$

and

$$[\mathcal{A} - \mathcal{B}\mathcal{B}^* Q(I - PQ)^{-1}][I - PQ] = [I - PQ][\mathcal{A} - (I - PQ)^{-1} P\mathcal{C}^*\mathcal{C}].$$

Proof. 1.) We have

$$[\mathcal{A} - \mathcal{B}(\mathcal{B}^* Q + \mathcal{D}^*\mathcal{C})(I - PQ)^{-1}][I - PQ]$$

$$= \mathcal{A}(I - PQ) - \mathcal{B}(\mathcal{B}^* Q + \mathcal{D}^*\mathcal{C}) = \mathcal{A} - \mathcal{A}PQ - \mathcal{B}\mathcal{B}^* Q - \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= \mathcal{A} - [\mathcal{A}P + \mathcal{B}\mathcal{B}^*]Q - \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= \mathcal{A} - [-P\mathcal{A}^*]Q - \mathcal{B}\mathcal{D}^*\mathcal{C} = \mathcal{A} - PQ\mathcal{A} - P\mathcal{C}^*\mathcal{C} - \mathcal{B}\mathcal{D}^*\mathcal{C}$$

$$= [I - PQ][\mathcal{A} - (I - PQ)^{-1}(P\mathcal{C}^* + \mathcal{B}\mathcal{D}^*)\mathcal{C}].$$

Adding $\mathcal{B}\mathcal{D}^*\mathcal{C}$ to both sides gives the second identity.

2.) The proof is analogous to the proof of 1.). □

In the following lemma it is shown that the characteristic and inverse characteristic maps preserve minimality and respect system equivalence.

Lemma 5.3. *1. The B-characteristic (P-characteristic) of a bounded real (positive real) system is stable and minimal. The B-characteristics (P-characteristics) of two equivalent systems are equivalent.*

2. The inverse B-characteristic (inverse P-characteristic) of a system in $US_{n,B}^{p,m}$ ($US_{n,P}^{m,m}$) is minimal. The inverse B-characteristics (inverse P-characteristics) of two equivalent systems are equivalent.

Proof. The proof is analogous to the proof of Lemma 4.3. □

The following Proposition shows important connections between the solutions of Riccati equations corresponding to a bounded-real (positive-real) system and the solutions to the Lyapunov equations of its bounded-real (positive-real) characteristic.

Proposition 5.1. *1. Let (A,B,C,D) be a bounded-real (positive-real) system and let Y be the stabilizing solution to the control bounded-real (positive-real) Riccati equation and let Z be the stabilizing solution to the filter bounded-real (positive-real) Riccati equation. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ be its B-characteristic (P-characteristic). Then the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

have solutions

$$P = (I-ZY)^{-1}Z = Z(I-YZ)^{-1}, \quad Q = Y - YZY.$$

2. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ be a stable minimal system in $US_{n,B}^{p,m}$ ($US_{n,P}^{m,m}$). Let P, Q be the solutions to the Lyapunov equations

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \quad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}.$$

Let (A,B,C,D) be its inverse B-characteristic (P-characteristic). Then

$$Y = Q(I-PQ)^{-1} = (I-QP)^{-1}Q$$

is the stabilizing solution to the control bounded-real (positive-real) Riccati equation and $Z = P - PQP$ is the stabilizing solution to the filter bounded-real (positive-real) Riccati equation.

Proof. We only consider the case for bounded-real systems. The proof of the results for positive-real systems is analogous.

1.) We first show that $\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*$ with $P = (I-ZY)^{-1}Z = Z(I-YZ)^{-1}$. Since $I - ZY$ is invertible this follows from,

$$\begin{aligned}
(I-ZY)(\mathcal{A}P+P\mathcal{A}^*)(I-YZ) &= (I-ZY)\mathcal{A}Z + Z\mathcal{A}^*(I-YZ)\\
&= (I-ZY)[A_B + BR_B^{-1}B^*Y]Z + Z[A_B + BR_B^{-1}B^*Y]^*(I-YZ)\\
&= A_BZ + ZA_B^* - Z(YA_B + A_B^*Y)Z\\
&\quad + (I-ZY)BR_B^{-1}B^*YZ + ZYBR_B^{-1}B^*(I-YZ)\\
&= -ZC^*S_B^{-1}CZ - BR_B^{-1}B^* - Z[-YBR_B^{-1}B^*Y - C^*S_B^{-1}C]Z\\
&\quad + (I-ZY)BR_B^{-1}B^*YZ + ZYBR_B^{-1}B^*(I-YZ)\\
&= -BR_B^{-1}B^* + ZYBR_B^{-1}B^*YZ\\
&\quad + (I-ZY)BR_B^{-1}B^*YZ + ZYBR_B^{-1}B^*(I-YZ)\\
&= -(I-ZY)BR_B^{-1}B^*(I-YZ)\\
&= -(I-ZY)\mathcal{B}\mathcal{B}^*(I-YZ).
\end{aligned}$$

Let now $Q = Y - YZY$. We are going to show that $\mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$. We consider

$$\begin{aligned}
&\mathcal{A}^*Q + Q\mathcal{A}\\
&= [A_B + BR_B^{-1}B^*Y]^*Y(I-ZY) + (I-YZ)Y[A_B + BR_B^{-1}B^*Y]\\
&= [A_B^*Y + YBR_B^{-1}B^*Y](I-ZY) + (I-YZ)[YA_B + YBR_B^{-1}B^*Y].
\end{aligned}$$

Using the bounded-real Riccati equation this gives,

$$\begin{aligned}
\mathcal{A}^*Q + Q\mathcal{A} &= [-YA_B - C^*S_B^{-1}C](I-ZY) + (I-YZ)[-A_B^*Y - C^*S_B^{-1}C]\\
&= -C^*S_B^{-1}C(I-ZY) - (I-YZ)C^*S_B^{-1}C - YA_B - A_B^*Y + Y(A_BZ + ZA_B^*)Y\\
&= -C^*S_B^{-1}(I-ZY) - (I-YZ)C^*S_B^{-1}C + YBR_B^{-1}B^*Y + C^*S_B^{-1}C\\
&\quad + Y[-ZC^*S_B^{-1}CZ - BR_B^{-1}B^*]Y\\
&= -(I-YZ)C^*S_B^{-1}C(I-ZY)\\
&= -\mathcal{C}^*\mathcal{C}.
\end{aligned}$$

2.) First note that

$$\begin{aligned}
A_B &= A + BR_B^{-1}D^*C\\
&= \mathcal{A} - \mathcal{B}(\mathcal{B}^*Q + \mathcal{D}^*\mathcal{C})(I-PQ)^{-1} + \mathcal{B}R_B^{1/2}R_B^{-1}\mathcal{D}^*S_B^{1/2}\mathcal{C}(I-PQ)^{-1}\\
&= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1}.
\end{aligned}$$

Since $I - QP$ is invertible, we have for $Y = Q(I-PQ)^{-1} = (I-QP)^{-1}Q$,

$$\begin{aligned}
&(I-QP)[A_B^*Y + YA_B + YBR_B^{-1}B^*Y + C^*S_B^{-1}C](I-PQ)\\
&= (I-QP)[(\mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1})^*Q(I-PQ)^{-1}
\end{aligned}$$

$$+(I-QP)^{-1}Q(\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1})$$
$$+(I-QP)^{-1}Q\mathcal{B}R_B^{1/2}R_B^{-1}R_B^{1/2}\mathcal{B}^*Q(I-PQ)^{-1}$$
$$+(I-PQ)^{-*}\mathcal{C}^*S_B^{1/2}S_B^{-1}S_B^{1/2}\mathcal{C}(I-PQ)^{-1}](I-PQ)$$
$$=(I-QP)\mathcal{A}^*Q-Q\mathcal{B}\mathcal{B}^*Q+Q\mathcal{A}(I-PQ)-Q\mathcal{B}\mathcal{B}^*Q+Q\mathcal{B}\mathcal{B}^*Q+\mathcal{C}^*\mathcal{C}$$
$$=\mathcal{A}^*Q+Q\mathcal{A}+\mathcal{C}^*\mathcal{C}-Q(P\mathcal{A}^*+\mathcal{A}P+\mathcal{B}\mathcal{B}^*)Q$$
$$=0,$$

we have verified the first identity. Now with $Z=P(I-QP)$ we have

$$A_BZ+ZA_B^*+ZC^*S_B^{-1}CZ+BR_B^{-1}B^*$$
$$=(\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1})(I-PQ)P+P(I-QP)(\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1})^*$$
$$+P(I-QP)(I-PQ)^{-*}\mathcal{C}^*S_B^{1/2}S_B^{-1}S_B^{1/2}\mathcal{C}(I-PQ)^{-1}(I-PQ)P$$
$$+\mathcal{B}R_B^{1/2}R_B^{-1}R_B^{1/2}\mathcal{B}^*$$
$$=\mathcal{A}P-\mathcal{A}PQP-\mathcal{B}\mathcal{B}^*QP+P\mathcal{A}^*-PQP\mathcal{A}^*-PQ\mathcal{B}\mathcal{B}^*+P\mathcal{C}^*\mathcal{C}P+\mathcal{B}\mathcal{B}^*$$
$$=\mathcal{A}^*P+P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*-(\mathcal{A}P+\mathcal{B}\mathcal{B}^*)QP+P\mathcal{C}^*\mathcal{C}P-PQ(P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*)$$
$$=0+P\mathcal{A}^*QP+P\mathcal{C}^*\mathcal{C}P-PQ(P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*)$$
$$=P(\mathcal{A}^*Q+\mathcal{C}^*\mathcal{C})P-PQ(P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*)$$
$$=-P(Q\mathcal{A})P-PQ(P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*)$$
$$=-PQ(\mathcal{A}P+P\mathcal{A}^*+\mathcal{B}\mathcal{B}^*)$$
$$=0,$$

which shows the second identity. Since

$$A_B+BR_B^{-1}B^*Y=\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1}+\mathcal{B}R_B^{1/2}R_B^{-1}R_B^{1/2}\mathcal{B}^*Q(I-PQ)^{-1}$$
$$=\mathcal{A},$$

is stable and

$$A_B+ZC^*S_B^{-1}C$$
$$=\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1}+P(I-QP)(I-PQ)^{-*}\mathcal{C}^*S_B^{1/2}S_B^{-1}S_B^{1/2}\mathcal{C}(I-PQ)^{-1}$$
$$=\mathcal{A}-\mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1}+P\mathcal{C}^*\mathcal{C}(I-PQ)^{-1}$$
$$=(I-PQ)[\mathcal{A}-(I-PQ)^{-1}P\mathcal{C}^*\mathcal{C}](I-PQ)^{-1}+P\mathcal{C}^*\mathcal{C}(I-PQ)^{-1}$$
$$=(I-PQ)\mathcal{A}(I-PQ)^{-1},$$

is stable, where we have used Lemma 5.2, we have shown that Z, Y are the stabilizing solutions to the bounded-real Riccati equations.

□

We are now in a position to show that the B-characteristic and P-characteristic are bijections.

Theorem 5.1. *The maps*

$$\chi_B: \quad B_n^{p,m}\to US_{n,B}^{p,m}$$
$$\chi_P: \quad P_n^m\to US_{n,P}^{m,m}$$

are bijections that preserve system equivalence with inverses $\chi_B^{-1}=I\chi_B$ *and* $\chi_P^{-1}=I\chi_P$.

Proof. We are only going to consider the B-characteristic. The proof for the P-characteristic is analogous. We first show that $\chi_B(B_n^{p,m})\subseteq US_{n,B}^{p,m}$. Let $(A,B,C,D)\in B_n^{p,m}$. It follows that $\lambda_{max}(ZY)<1$, where Z, Y are the stabilizing solutions to the bounded-real Riccati equations (see e.g. [34]). If $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})=\chi_B((A,B,C,D))$ and P, Q are the positive definite solutions to the Lyapunov equations

$$\mathcal{A}P+P\mathcal{A}^*=-\mathcal{B}\mathcal{B}^*,\quad \mathcal{A}^*Q+Q\mathcal{A}=-\mathcal{C}^*\mathcal{C},$$

then as a consequence of Proposition 5.1 we have $PQ=ZY$ and therefore that $\lambda_{max}(PQ)<1$. Clearly, $I-\mathcal{D}^*\mathcal{D}=I-D^*D>0$ and hence $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\in US_{n,B}^{p,m}$.

We show next that $I\chi_B(US_{n,B}^{p,m})\subseteq B_n^{p,m}$. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\in US_{n,B}^{p,m}$ and let $(A,B,C,D)=I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$. By Lemma 5.3 the system (A,B,C,D) is minimal. In Proposition 5.1 it was shown that the two bounded real Riccati equations for the system (A,B,C,D) have positive definite stabilizing solutions Y, Z. This together with the fact that $\lambda_{max}(ZY)=\lambda_{max}(PQ)<1$ implies (see e.g. [34]) that (A,B,C,D) is bounded-real.

That χ_B preserves system equivalence was established in Lemma 5.3. We are next going to show that χ_B is injective, or more precisely that $I\chi_B\cdot\chi_B$ is the identity map. Let $(A,B,C,D)\in B_n^{p,m}$, let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})=\chi_B((A,B,C,D))$ and set $(A_1,B_1,C_1,D_1):=I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$. We have

$$D_1=D,$$
$$B_1=\mathcal{B}R_B^{1/2}=BR_B^{-1/2}R_B^{1/2}=B,$$
$$C_1=S_B^{1/2}\mathcal{C}(I-PQ)^{-1}=S_B^{-1/2}S_B^{1/2}C(I-ZY)(I-ZY)^{-1}=C,$$
$$A_1=\mathcal{A}-\mathcal{B}(\mathcal{B}^*Q+\mathcal{D}^*\mathcal{C})(I-PQ)^{-1}$$
$$=A_B+BR_B^{-1}B^*Y-BR_B^{-1/2}R_B^{-1/2}B^*Y$$
$$-BR_B^{-1/2}D^*S_B^{-1/2}C(I-ZY)(I-ZY)^{-1}$$
$$=A+BR_B^{-1}D^*C-BR_B^{-1}D^*C$$
$$=A,$$

which shows that $I\chi_B\cdot\chi_B((A,B,C,D))=(A,B,C,D)$ and hence that χ_B is injective. We now show that χ_B is surjective by showing that $\chi_B\cdot I\chi_B$ is the identity map. Let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in US^{p,m}_{n,B}$, let $(A,B,C,D) := I\chi_B((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}))$ and set $(\mathcal{A}_1,\mathcal{B}_1,\mathcal{C}_1,\mathcal{D}_1) := \chi_B((A,B,C,D))$. Then

$$\mathcal{D}_1 = \mathcal{D},$$
$$\mathcal{C}_1 = R_B^{-1/2}C(I-ZY) = S_B^{-1/2}S_B^{1/2}\mathcal{C}(I-PQ)^{-1}(I-PQ) = \mathcal{C},$$
$$\mathcal{B}_1 = BR_B^{-1/2} = \mathcal{B}R_B^{1/2}R_B^{-1/2} = \mathcal{B},$$
$$\mathcal{A}_1 = A_B + BR_B^{-1}B^*Y$$
$$= \mathcal{A} - \mathcal{B}\mathcal{B}^*Q(I-PQ)^{-1} + \mathcal{B}R_B^{1/2}R_B^{-1}R_B^{1/2}\mathcal{B}^*Q(I-PQ)^{-1}$$
$$= \mathcal{A}.$$

This shows that χ_B is surjective. Hence we have that χ_B is bijective with inverse $\chi_B^{-1} = I\chi_B$. □

Analogous to the situation for the L-characteristic we can also show that the B-characteristic (P-characteristic) maps bounded-real balanced (positive-real balanced) systems to stable minimal systems whose reachability and observability grammians are diagonal.

Corollary 5.1. *Let $(A,B,C,D) \in B^{p,m}_n$ (P^m_n) and let $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ be its B-characteristic (P-characteristic). Let Y be the stabilizing solution to the control bounded-real (positive-real) Riccati equation and let Z be the stabilizing solutions to the filter bounded-real (positive-real) Riccati equation. If P, Q are the solutions to the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \qquad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C},$$

and

1. *if (A,B,C,D) is bounded-real (positive-real) balanced with bounded-real (positive-real) grammian Σ, then*
$$P = \Sigma(I-\Sigma^2)^{-1}, \qquad Q = \Sigma(I-\Sigma^2).$$
2. *if $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ is Lyapunov balanced with Lyapunov grammian Σ, then*
$$Y = \Sigma(I-\Sigma^2)^{-1}, \qquad Z = \Sigma(I-\Sigma^2).$$

If $\Gamma_S: \; S^{p,m}_n \to S^{p,m}_n$ is the Lyapunov balanced canonical form, then

$$\Gamma_B := \chi_B^{-1} \circ \Gamma_S \circ \chi_B, \qquad \Gamma_P := \chi_P^{-1} \circ \Gamma_S \circ \chi_P$$

define canonical forms for $B^{p,m}_n$ and P^m_n. But the canonical forms are not in terms of the respective balanced realizations as is clear from the previous corollary. Analogously to the construction of the LQG-balanced canonical form diagonal scaling will produce the desired result.

Lemma 5.4. *1. If $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}) \in S^{p,m}_n$ is such that the Lyapunov equations*

$$\mathcal{A}P + P\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \qquad \mathcal{A}^*Q + Q\mathcal{A} = -\mathcal{C}^*\mathcal{C}$$

have solutions

$$P = \Sigma(I-\Sigma^2)^{-1}, \; Q = \Sigma(I-\Sigma^2)$$

for some positive diagonal matrix $\Sigma < 1$, then

$$\Delta_S((\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})) := (T\mathcal{A}T^{-1}, T\mathcal{B}, \mathcal{C}T^{-1}, \mathcal{D})$$

with $T = (I-\Sigma^2)^{\frac{1}{2}}$ is Lyapunov balanced with Lyapunov grammian Σ.

2. If $(A,B,C,D) \in B^{p,m}_n$ is such that the control (filter) bounded-real Riccati equation has the stabilizing solution $Y(Z)$ with

$$Y = \Sigma(I-\Sigma^2)^{-1}, \; Z = \Sigma(I-\Sigma^2)$$

for some positive diagonal matrix $\Sigma < 1$, then

$$\Delta_B((A,B,C,D)) := (TAT^{-1}, TB, CT^{-1}, D)$$

with $T = (I-\Sigma^2)^{\frac{1}{2}}$ is bounded-real balanced with bounded-real grammian Σ.

3. If $(A,B,C,D) \in P^m_n$ is such that the control (filter) positive-real Riccati equation has the stabilizing solution $Y(Z)$ with

$$Y = \Sigma(I-\Sigma^2)^{-1}, \; Z = \Sigma(I-\Sigma^2)$$

for some positive diagonal matrix $\Sigma < 1$, then

$$\Delta_P((A,B,C,D)) := (TAT^{-1}, TB, CT^{-1}, D)$$

with $T = (I-\Sigma^2)^{\frac{1}{2}}$ is positive-real balanced with positive-real grammian Σ.

Proof. The proof is analogous to the proof of Lemma 4.4. □

With the diagonal scaling maps Δ_B, Δ_P set

$$\Gamma_B := \Delta_B \circ \chi_B^{-1} \circ \Gamma_S \circ \chi_B,$$
$$\Gamma_P := \Delta_P \circ \chi_P^{-1} \circ \Gamma_S \circ \chi_P.$$

Then Γ_B (Γ_P) defines a canonical form for bounded-real (positive-real) systems in terms of bounded-real (positive-real) realizations.

The *bounded-real balanced canonical form* and *bounded-real balanced parametrization* result is given in the following Theorem.

Theorem 5.2. *Let (A_b, B_b, C_b, D_b) be a m-input p-output continuous-time system of dimension n. Then the following are equivalent:*

1. *(A_b, B_b, C_b, D_b) is a bounded-real system, i.e in $B_n^{p,m}$.*
2. *$(A_b, B_b, C_b, D_b) = (TAT^{-1}, TB, CT^{-1}, D)$ for some invertible T, where (A, B, C, D) is in bounded-real balanced form, i.e. there exist block indices $n_1, \dots, n_k$, $\sum_{j=1}^k n_j = n$, parameters $1 > \sigma_1 > \sigma_2 > \dots > \sigma_k > 0$ and k families of step sizes $m = \tau_0^j \geq \tau_1^j \geq \tau_2^j \geq \dots \geq \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \leq j \leq k$, such that*
   a) *$B = (\underbrace{\overline{B}_1^T, 0}_{n_1}, \underbrace{\overline{B}_2^T, 0}_{n_2}, \dots, \underbrace{\overline{B}_k^T, 0}_{n_k})^T R_B^{\frac{1}{2}}$, where $\overline{B}_j \in \mathcal{K}^{\tau_1^j \times \tau_0^j}$ is positive upper triangular, $1 \leq j \leq k$.*
   b) *$C = S_B^{\frac{1}{2}}(\underbrace{U_1(\overline{B}_1\overline{B}_1^*)^{\frac{1}{2}}, 0}_{n_1} \underbrace{U_2(\overline{B}_2\overline{B}_2^*)^{\frac{1}{2}}, 0}_{n_2}, \dots, \underbrace{U_k(\overline{B}_k\overline{B}_k^*)^{\frac{1}{2}}, 0}_{n_k})$, where $U_j \in \mathcal{K}^{p \times \tau_1^j}$, $U_j^* U_j = I_{\tau_1^j}$, $1 \leq j \leq k$.*
   c)

$$A = \left(\begin{array}{c|c|c|c|c} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline A(i,1) & \cdots & A(i,i) & \cdots & A(i,k) \\ \hline \vdots & & \vdots & \ddots & \vdots \\ \hline A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \leq i \leq k$

$$A(i,i) = \begin{pmatrix} \tilde{A}_{ii} + \mathcal{S}_1^i & (-\mathcal{A}_1^i)^* & & 0 \\ \mathcal{A}_1^i & \mathcal{S}_2^i & \ddots & \\ & \ddots & \ddots & (-\mathcal{A}_{l_i-1}^i)^* \\ 0 & & \mathcal{A}_{l_i-1}^i & \mathcal{S}_{l_i}^i \end{pmatrix},$$

for $1 \leq i, j \leq k$, $i \neq j$

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

i. $\mathcal{S}_i^j$ is a $\tau_i^j \times \tau_i^j$ skew-hermitian matrix, $i = 1, 2, \dots, l_j$, $1 \leq j \leq k$.
ii. $\mathcal{A}_i^j$ is a positive upper triangular $\tau_{i+1}^j \times \tau_i^j$ matrix, $i = 1, 2, \dots, l_j - 1$, $1 \leq j \leq k$.

iii. $\tilde{A}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ is given by

$$\tilde{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j(1-\sigma_i^2)\overline{B}_i\overline{B}_j^* - \sigma_i(1-\sigma_j^2)(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} U_i^* U_j (\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}})$$
$$- \overline{B}_i D^* U_j (\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}},$$

for $1 \leq i, j \leq k$, $i \neq j$ and

$$\tilde{A}_{ii} = -\frac{1}{2\sigma_i}(1+\sigma_i^2)\overline{B}_i\overline{B}_i^* - \overline{B}_i D^* U_i (\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}, \quad \text{for } 1 \leq i \leq k.$$

Moreover, the system (A, B, C, D) as defined in (2) is bounded-real balanced with bounded-real grammian $\Sigma_B = diag(\sigma_1 I_{n_1}, \dots, \sigma_k I_{n_k})$, where $n_1, \dots, n_k$ are the block indices and $1 > \sigma_1 > \dots > \sigma_k > 0$ are the bounded-real singular values of the system in (2). The map Γ_B that assigns to each system in $B_n^{p,m}$ the realization in (2) is a canonical form.

Proof. Let $(A_b, B_b, C_b, D_b) \in B_n^{p,m}$. Then $\chi_B((A_b, B_b, C_b, D_b))$ is in $US_n^{p,m}$. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) := \Gamma_S(\chi_B((A_b, B_b, C_b, D_b)))$ be the Lyapunov balanced canonical form of $\chi_B((A_b, B_b, C_b, D_b))$. Since $\Gamma_S := \chi_B^{-1} \circ \Gamma_S \circ \chi_B$ defines a canonical form for $B_n^{p,m}$, it follows that

$$(A, B, C, D) := \Delta_B(\chi_B^{-1}((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})))$$

is a bounded-real balanced system which is equivalent to (A_b, B_b, C_b, D_b). To check that (A, B, C, D) is in the stated form, consider the Lyapunov balanced parametrization of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Assume that the parametrization of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is given using the notation that was introduced in the proof of Theorem 4.2. The system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is Lyapunov balanced with Lyapunov grammian $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \dots, \sigma_k I_{n_k})$ with $1 > \sigma_1 > \sigma_2 > \dots > \sigma_k > 0$. By Corollary 5.1 and Lemma 5.3

$$\begin{aligned}(A, B, C, D) &= ((I - \Sigma^2)^{-\frac{1}{2}}(\mathcal{A} - \mathcal{B}(\mathcal{B}^*\Sigma - \mathcal{C})(I - \Sigma^2)^{-1})(I - \Sigma^2)^{\frac{1}{2}}, \\ &\quad (I - \Sigma^2)^{-\frac{1}{2}}\mathcal{B}R_B^{\frac{1}{2}}, S_B^{\frac{1}{2}}\mathcal{C}(I - \Sigma^2)^{-\frac{1}{2}}, \mathcal{D}))\end{aligned}$$

Setting $\overline{B}_j := \frac{1}{\sqrt{1-\sigma_j^2}}\overline{\mathcal{B}}_j$, $1 \leq j \leq k$, it follows that $B = (I - \Sigma^2)^{-\frac{1}{2}}\mathcal{B}R_B^{\frac{1}{2}}$ has the required structure. Since with $U_j := \mathcal{U}_j$,

$$S_B^{\frac{1}{2}}\mathcal{U}_j(\overline{\mathcal{B}}_j\overline{\mathcal{B}}_j^*)^{\frac{1}{2}}\frac{1}{\sqrt{1-\sigma_j^2}} = S_B^{\frac{1}{2}}U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$

for $1 \leq j \leq k$, it follows that $C = S_B^{\frac{1}{2}}\mathcal{C}(I - \Sigma^2)^{-\frac{1}{2}}$ has the required structure. Note that

$$A = (I - \Sigma^2)^{-\frac{1}{2}}(\mathcal{A} - \mathcal{B}(\mathcal{B}^*\Sigma + \mathcal{D}^*\mathcal{C})(I - \Sigma^2)^{-1})(I - \Sigma^2)^{\frac{1}{2}}$$

$$= (I-\Sigma^2)^{-\frac{1}{2}}\mathcal{A}(I-\Sigma^2)^{\frac{1}{2}} - (I-\Sigma^2)^{-\frac{1}{2}}\mathcal{B}(\mathcal{B}^*\Sigma+\mathcal{D}^*\mathcal{C})(I-\Sigma^2)^{-1})(I-\Sigma^2)^{\frac{1}{2}}$$
$$= (I-\Sigma^2)^{-\frac{1}{2}}\mathcal{A}(I-\Sigma^2)^{\frac{1}{2}} - BR_B^{-\frac{1}{2}}R_B^{-\frac{1}{2}}B^*\Sigma - BR_B^{-\frac{1}{2}}D^*S_B^{-\frac{1}{2}}C$$
$$= (I-\Sigma^2)^{-\frac{1}{2}}\mathcal{A}(I-\Sigma^2)^{\frac{1}{2}} - BR_B^{-1}B^*\Sigma - BR_B^{-\frac{1}{2}}D^*S_B^{-\frac{1}{2}}C.$$

If (A,B,C,D) is partitioned according to the block indices $n_1,n_2,\dots,n_k$, then for $1\le i,j\le k$, $i\neq j$,

$$A_{ij} = \frac{\sqrt{1-\sigma_j^2}}{\sqrt{1-\sigma_i^2}}\mathcal{A}_{ij} - B_iR_B^{-1}B_j^*\sigma_j - B_iR_B^{-\frac{1}{2}}D^*S_B^{-\frac{1}{2}}C_j.$$

This show that all entries A_{ij} are zero with the exception of the principal $\tau_1^i\times\tau_1^j$-subblock $\tilde{A}_{ij}$ which is given by

$$\tilde{A}_{ij} = \frac{\sqrt{1-\sigma_j^2}}{\sqrt{1-\sigma_i^2}}\tilde{\mathcal{A}}_{ij} - \overline{B}_i\overline{B}_j^*\sigma_j - \overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{\sqrt{1-\sigma_j^2}}{\sqrt{1-\sigma_i^2}}\left(\frac{1}{\sigma_i^2-\sigma_j^2}\left(\sigma_j\overline{\mathcal{B}}_i\overline{\mathcal{B}}_j^* - \sigma_i(\overline{\mathcal{B}}_i\overline{\mathcal{B}}_i^*)^{\frac{1}{2}}\mathcal{U}_i^*\mathcal{U}_j(\overline{\mathcal{B}}_j\overline{\mathcal{B}}_j^*)^{\frac{1}{2}}\right)\right)$$
$$-\overline{B}_i\overline{B}_j^*\sigma_j - \overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{1-\sigma_j^2}{\sigma_i^2-\sigma_j^2}\left(\sigma_j\overline{B}_i\overline{B}_j^* - \sigma_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}U_i^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}\right)$$
$$-\overline{B}_i\overline{B}_j^*\sigma_j - \overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$
$$= \frac{1}{\sigma_i^2-\sigma_j^2}\left(\sigma_j(1-\sigma_i^2)\overline{B}_i\overline{B}_j^* - \sigma_i(1-\sigma_j^2)(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}U_i^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}\right)$$
$$-\overline{B}_iD^*U_j(\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}}$$

for $1\le i,j\le k$, $i\neq j$. For $1\le i\le k$, we have

$$A_{ii} = \mathcal{A}_{ii} - B_iR_B^{-1}B_i^*\sigma_i - B_iR_B^{-\frac{1}{2}}D^*S_B^{-\frac{1}{2}}C_i$$

The principal $\tau_1^i\times\tau_1^i$ submatrix, $1\le i\le k$, is given by

$$\tilde{\mathcal{A}}_{ii} + \mathcal{S}_1^i - \overline{B}_i\overline{B}_i^*\sigma_i - \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}$$
$$= -\frac{1}{2\sigma_i}\overline{\mathcal{B}}_i\overline{\mathcal{B}}_i^* - \mathcal{S}_1^i - \overline{B}_i\overline{B}_i^*\sigma_i - \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}$$
$$= -\frac{1-\sigma_i^2}{2\sigma_i}\overline{B}_i\overline{B}_i^* + \mathcal{S}_1^i - \overline{B}_i\overline{B}_i^*\sigma_i - \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}$$
$$= -\frac{1+\sigma_i^2}{2\sigma_i}\overline{B}_i\overline{B}_i^* - \overline{B}_iD^*U_i(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} + S_1^i,$$

where we set $S_i^j := \mathcal{S}_i^j$, for $i=1,2,\dots,l_j$, $1\le j\le k$. This show that A has the stated form. Hence (A,B,C,D) admits the stated canonical form. The parameterization part of the theorem is shown similar to the analogous part of the proof of Theorem 4.2. □

Specializing the previous result to the single-input single-output case results in the following corollary.

Corollary 5.2. *Let (A_b,b_b,c_b,d_b) be a single-input single-output continuous-time system of dimension n. Then the following statements are equivalent:*

1. *(A_b,b_b,c_b,d_b) is a stable minimal system.*
2. *$(A_b,b_b,c_b,d_b) = (TAT^{-1},Tb,cT^{-1},d)$ for some invertible matrix T, where (A,b,c,d) is in the following bounded-real balanced form with block indices $n_1,n_2,\dots,n_k$, $\sum_{i=1}^k n_i = n$, $|d|<1$:*
   - *a) $b = (\underbrace{b_1,0,\dots,0}_{n_1},\underbrace{b_2,0,\dots,0}_{n_2},\dots,\underbrace{b_k,0,\dots,0}_{n_k})^T$, where $b_i>0$, $1\le i\le k$.*
   - *b) $c = (\underbrace{s_1b_1,0,\dots,0}_{n_1},\underbrace{s_2b_2,0,\dots,0}_{n_2},\dots,\underbrace{s_kb_k,0,\dots,0}_{n_k})$, where $s_i\in\mathcal{K}$, $|s_i|=1$, $1\le i\le k$.*
   - *c)*

$$A = \begin{pmatrix} A(1,1) & A(1,2) & \cdots & A(1,k) \\ A(2,1) & A(2,2) & \cdots & A(2,k) \\ \vdots & & \ddots & \vdots \\ A(k,1) & A(k,2) & \cdots & A(k,k) \end{pmatrix},$$

where for $1\le j\le k$,

$$A(j,j) = \begin{pmatrix} a_{jj}+i\beta_1^j & -\alpha_1^j & & 0 \\ \alpha_1^j & i\beta_2^j & \ddots & \\ & \ddots & \ddots & -\alpha_{n_j-1}^j \\ 0 & & \alpha_{n_j-1}^j & i\beta_{n_j}^j \end{pmatrix},$$

for $1\le i,j\le k$, $i\neq j$,

$$A(i,j) = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\alpha_j^i > 0 \textit{ for } 1 \le j \le n_i - 1, \quad 1 \le i \le k,$$

$$a_{ij} = \frac{-b_i b_j}{1-|d|^2}\left[\frac{\bar{s}_i s_j \sigma_i(1-\sigma_j^2) - \sigma_j(1-\sigma_i^2)}{\sigma_i^2-\sigma_j^2} + s_j d^*\right] \quad \textit{for } 1 \le i,j \le k, \quad i \ne j,$$

$$a_{ii} = -\frac{b_i^2}{1-|d|^2}\left[\frac{1}{2\sigma_i}(1+\sigma_i^2) + s_i d^*\right] \quad \textit{for } 1 \le i \le k,$$

$$1 > \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

If $\mathcal{K} = \Re$, then

$$s_i = \pm 1, \textit{ for } 1 \le i \le k,$$

$$\beta_j^i = 0, \textit{ for } 1 \le j \le n_i, \quad 1 \le i \le k,$$

$$a_{ij} = \frac{-b_i b_j}{1-d^2}\left[\frac{1 + s_i s_j \sigma_i \sigma_j}{s_i s_j \sigma_i + \sigma_j} + s_j d\right], \textit{ for } 1 \le i,j \le k.$$

Moreover, the system (A,b,c,d) as defined in (2) is bounded-real balanced with bounded-real grammian $\Sigma_B = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. The map Γ_B that assigns to each system in $B_n^{1,1}$ the realization in (2) is a canonical form.

The *positive-real balanced canonical form* and *positive-real balanced parametrization* result is given in the following Theorem.

Theorem 5.3. *Let (A_p, B_p, C_p, D_p) be a m-input m-output continuous-time system of dimension n. Then the following are equivalent:*

1. *(A_p, B_p, C_p, D_p) is a positive-real system, i.e in P_n^m.*
2. *$(A_p, B_p, C_p, D_p) = (TAT^{-1}, TB, CT^{-1}, D)$ for some invertible T, where (A, B, C, D) is in positive-real balanced form, i.e. there exist block indices $n_1, \ldots, n_k$, $\sum_{j=1}^k n_j = n$, parameters $1 > \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$ and k families of step sizes $m = \tau_0^j \ge \tau_1^j \ge \tau_2^j \ge \cdots \ge \tau_{l_j}^j > 0$, $\sum_{i=1}^{l_j} \tau_i^j = n_j$, $1 \le j \le k$, such that*
   a) *$B = (\underbrace{\overline{B}_1^T, 0}_{n_1}, \underbrace{\overline{B}_2^T, 0}_{n_2}, \ldots, \underbrace{\overline{B}_k^T, 0}_{n_k})^T R_P^{\frac{1}{2}}$, where $\overline{B}_j \in \mathcal{K}^{\tau_1^j \times \tau_0^j}$ is positive upper triangular, $1 \le j \le k$.*
   b) *$C = R_P^{\frac{1}{2}}(\underbrace{U_1(\overline{B}_1\overline{B}_1^*)^{\frac{1}{2}}, 0}_{n_1} \underbrace{U_2(\overline{B}_2\overline{B}_2^*)^{\frac{1}{2}}, 0}_{n_2}, \ldots, \underbrace{U_k(\overline{B}_k\overline{B}_k^*)^{\frac{1}{2}}, 0}_{n_k})$, where $U_j \in \mathcal{K}^{p \times \tau_1^j}$, $U_j^* U_j = I_{\tau_1^j}$, $1 \le j \le k$.*
   c)

$$A = \left(\begin{array}{c|c|c|c|c} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline A(i,1) & \cdots & A(i,i) & \cdots & A(i,k) \\ \hline \vdots & & \vdots & \ddots & \vdots \\ \hline A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \le i \le k$

$$A(i,i) = \begin{pmatrix} \tilde{A}_{ii} + \mathcal{S}_1^i & (-\mathcal{A}_1^i)^* & & 0 \\ \mathcal{A}_1^i & \mathcal{S}_2^i & \ddots & \\ & \ddots & \ddots & (-\mathcal{A}_{l_i-1}^i)^* \\ 0 & & \mathcal{A}_{l_i-1}^i & \mathcal{S}_{l_i}^i \end{pmatrix},$$

for $1 \le i,j \le k$, $i \ne j$

$$A(i,j) = \begin{pmatrix} \tilde{A}_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

i. *$\mathcal{S}_i^j$ is a $\tau_i^j \times \tau_i^j$ skew-hermitian matrix, $i = 1, 2, \ldots, l_j$, $1 \le j \le k$.*
ii. *$\mathcal{A}_i^j$ is a positive upper triangular $\tau_{i+1}^j \times \tau_i^j$ matrix, $i = 1, 2, \ldots, l_j - 1$, $1 \le j \le k$.*
iii. *$\tilde{A}_{ij} \in \mathcal{K}^{\tau_1^i \times \tau_1^j}$ is given by*

$$\tilde{A}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j(1-\sigma_i^2)\overline{B}_i\overline{B}_j^* - \sigma_i(1-\sigma_j^2)(\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}} U_i^* U_j (\overline{B}_j\overline{B}_j^*)^{\frac{1}{2}})$$

$$+ \overline{B}_i U_j (\overline{B}_j \overline{B}_j^*)^{\frac{1}{2}},$$

for $1 \le i,j \le k$, $i \ne j$ and

$$\tilde{A}_{ii} = -\frac{1}{2\sigma_i}(1+\sigma_i^2)\overline{B}_i\overline{B}_i^* + \overline{B}_i U_i (\overline{B}_i\overline{B}_i^*)^{\frac{1}{2}}, \quad \textit{for } 1 \le i \le k.$$

Moreover, the system (A, B, C, D) as defined in (2) is positive-real balanced with positive-real grammian $\Sigma_P = diag(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$, where $n_1, \ldots, n_k$ are the block indices and $1 > \sigma_1 > \cdots > \sigma_k > 0$ are the positive-real singular values of the system in (2). The map Γ_P that assigns to each system in P_n^p the realization in (2) is a canonical form.

Proof. The proof is analogous to the proof of Theorem 5.2. □

Specializing the previous result to the single-input single-output case results in the following corollary.

Corollary 5.3. *Let (A_p, b_p, c_p, d_p) be a single-input single-output continuous-time system of dimension n. Then the following statements are equivalent:*

1. *(A_p, b_p, c_p, d_p) is a stable minimal system.*

2. $(A_p, b_p, c_p, d_p) = (TAT^{-1}, Tb, cT^{-1}, d)$ *for some invertible matrix* T, *where* (A, b, c, d) *is in the following positive-real balanced form with block indices* $n_1, n_2, \ldots, n_k$, $\sum_{i=1}^{k} n_i = n$, $Re(d) > 0$:

a) $b = (\underbrace{b_1, 0, \ldots, 0}_{n_1}, \underbrace{b_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n_k})^T$, *where* $b_i > 0$, $1 \le i \le k$.

b) $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n_1}, \underbrace{s_2 b_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n_k})$, *where* $s_i \in \mathcal{K}$, $|s_i| = 1$, $1 \le i \le k$.

c)

$$A = \left(\begin{array}{c|c|c|c} A(1,1) & A(1,2) & \cdots & A(1,k) \\ \hline A(2,1) & A(2,2) & \cdots & A(2,k) \\ \hline \vdots & & \ddots & \vdots \\ \hline A(k,1) & A(k,2) & \cdots & A(k,k) \end{array}\right),$$

where for $1 \le j \le j$

$$A(j,j) = \begin{pmatrix} a_{jj} + i\beta_1^j & -\alpha_1^j & & 0 \\ \alpha_1^j & i\beta_2^j & \ddots & \\ & \ddots & \ddots & -\alpha_{n_j-1}^j \\ 0 & & \alpha_{n_j-1}^j & i\beta_{n_j}^j \end{pmatrix},$$

for $1 \le i, j \le k$, $i \ne j$,

$$A(i,j) = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\alpha_j^i > 0 \text{ for } 1 \le j \le n_i - 1, \quad 1 \le i \le k,$$

$$a_{ij} = \frac{-b_i b_j}{2Re(d)} \left[\frac{\bar{s}_i s_j \sigma_i (1-\sigma_j^2) - \sigma_j (1 - \sigma_i^2)}{\sigma_i^2 - \sigma_j^2} - s_j\right] \quad \text{for } 1 \le i, j \le k, \quad i \ne j,$$

$$a_{ii} = -\frac{b_i^2}{2Re(d)} \left[\frac{1}{2\sigma_i}(1 + \sigma_i^2) - s_i\right] \text{ for } 1 \le i \le k,$$

$$1 > \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0.$$

If $\mathcal{K} = \Re$, *then*

$$s_i = \pm 1, \quad \text{for } 1 \le i \le k,$$

$$\beta_j^i = 0, \text{ for } 1 \le j \le n_i, \quad 1 \le i \le k,$$

$$a_{ij} = \frac{-b_i b_j}{2d(s_i s_j \sigma_i + \sigma_j)} (1 - s_i \sigma_i)(1 - s_j \sigma_j), \text{ for } 1 \le i, j \le k.$$

Moreover, the system (A, b, c, d) *as defined in (2) is positive-real balanced with positive-real grammian* $\Sigma_P = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$. *The map* Γ_P *that assigns to each system in* P_n^1 *the realization in (2) is a canonical form.*

We can now use the previous parametrization results to prove a model reduction result for bounded-real (positive-real) balanced systems.

Theorem 5.4. *Let* $(A, B, C, D) \in B_n^{p,m}$ (P_n^m) *be in bounded-real (positive-real) balanced canonical form with bounded-real (positive-real) grammian* Σ. *Let* $1 \le r < n$. *Then* r*-dimensional balanced approximant* (A_{11}, B_1, C_1, D) *is in* $B_r^{p,m}$ (P_r^m) *and is in bounded-real (positive-real) balanced canonical form with bounded-real (positive-real) grammian* Σ_1, *where* $\Sigma = diag(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathcal{K}^{r \times r}$.

Proof. The proof of this theorem is analogous to the proof of Theorem 4.3. □

In the following Corollary a model reduction result is obtained for bounded-real (positive-real) balanced systems which are not necessarily in the respective canonical form.

Corollary 5.4. *Let* (A, B, C, D) *be a* n*-dimensional bounded-real (positive-real) balanced system in* $B_n^{p,m}$ (P_n^m) *with bounded-real (positive-real) grammian* $\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_k I_{n_k})$, $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$. *Let* $r = n_1 + n_2 + \cdots + n_l$ *for some* $1 \le l \le k$. *Then the* r*-dimensional balanced approximant* (A_{11}, B_1, C_1, D) *of* (A, B, C, D) *is in* $B_r^{p,m}$ (P_r^m) *and is bounded-real (positive-real) with bounded-real (positive-real) grammian* $\Sigma_1 = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_l I_{n_l})$.

Proof. The proof is analogous to the proof of Corollary 4.3. □

6. Concluding Remarks

In this paper we discussed canonical forms and parametrization results for minimal, stable, bounded-real and positive-real continuous-time systems. In many applications, such as system identification, it is however desirable to have the analogous results for discrete-time systems. Balanced realizations for corresponding classes of discrete-time systems can be defined in a completely analogous way to the continuous-time setting, by balancing solutions to the respective discrete-time Riccati equations (see e.g. [19], [2]). If $DS_n^{p,m}$ is the set of discrete-time stable minimal n-dimensional systems with p-dimensional output space and m-dimensional input space, the *bilinear transform*

$$M: \quad S_n^{p,m} \to DS_n^{p,m}$$
$$(A_c, B_c, C_c, D_c) \mapsto (A_d, B_d, C_d, D_d)$$

where

$$A_d = (I - A_c)^{-1}(I + A_c)$$
$$B_d = \sqrt{2}(I - A_c)^{-1}B_c$$
$$C_d = \sqrt{2}C_c(I - A_c)^{-1}$$
$$D_d = C_c(I - A_c)^{-1}B_c + D_c$$

is a bijection which preserves system equivalence. It also maps bijectively continuous-time bounded-real (positive-real) systems to discrete-time bounded-real (positive-real) systems. Since the bilinear transform also preserves the various notations of balancing, it can be used to carry the continuous-time results over to the discrete-time case to define balanced canonical forms for discrete-time systems (see [21], [26]). In particular

$$D\Gamma_S := M \circ \Gamma_S \circ M^{-1}$$

defines a balanced canonical form for the class of stable discrete-time systems $DS_n^{p,m}$ and

$$D\Gamma_B := M \circ \Gamma_B \circ M^{-1}$$
$$D\Gamma_P := M \circ \Gamma_P \circ M^{-1}$$

define canonical forms for the class $DB_n^{p,m}$ respectively DP_n^m of discrete-time bounded-real systems in $DS_n^{p,m}$ and for the class DP_n^m of discrete-time positive-real systems in $DS_n^{m,m}$.

In this paper only we did not present any results on canonical forms for minimum phase systems. Such results are, however, easily derived from those for positive-real systems using the state space formulae that relate a positive real system to the associated spectral factor ([26]). Such a parametrization for minimum phase systems is of importance in time series analysis, where based on an observed time series an innovative model is to be identified (see e.g. [1], [9], [18], [17]). That balanced realizations may provide a good canonical forms for system identification has for the first time been suggested by Maciejowski ([16]).

The parametrization results for $S_n^{p,m}$, $L_n^{p,m}$, $B_n^{p,m}$ and P_n^m allow an analysis of the number of connected components of the associated manifolds of linear systems ([24], [25]). A disadvantage of the balanced parametrization is, however, that it does not induce an atlas for the manifold of systems. This property of the canonical form is not ideal for the implementation in system identification algorithms where overlapping charts are of importance ([10]). That the canonical forms can be changed to lead to an overlapping parametrization was shown in ([12]).

The importance of Lyapunov balanced realizations in the theory of Hankel operators and H^∞ control is due to the interpretation of the Lyapunov singular values as the singular values of the Hankel operator whose kernel is given by the impulse response of the system. Let $(A, B, C, D) \in S_n^{p,m}$ and set $H(t) := Ce^{tA}B$, $t \geq 0$. Then the *Hankel operator* with kernel H is defined by

$$\mathcal{H}: \quad L^2([0,\infty]) \to L^2([0,\infty])$$
$$u \mapsto (\mathcal{H}(u))(\cdot) = \int_0^\infty H(t + \cdot)u(t)dt$$

This operator has rank n and we have that the singular values of $\mathcal{H}$ are given by

$$\sigma((\mathcal{H}^*\mathcal{H})^{\frac{1}{2}}) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\},$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the Lyapunov singular values of (A, B, C, D). Using Theorem 3.2 it is therefore quite straightforward to construct finite rank Hankel operators with prescribed singular values. This system theoretic approach was used in ([22], [23], [32]) to solve the inverse spectral problem for Hankel operators.

References

1. Caines P.E., *Linear Stochastic Systems.* John Wiley, New York, 1988.
2. Desai U.B., and D. Pal, "A transformation approach to stochastic model reduction", *IEEE Transactions on Automatic Control*, 29:1097–1100, 1984.
3. Enns D., "Model reduction for control system design", *PhD thesis*, Department of Aeronautics and Astronautics, Stanford University, 1984.
4. Francis B., *A course in H^∞ control theory.* Lecture Notes in Control and Information Sciences, volume 88. Springer, 1987.
5. Fuhrmann P.A., and R. J. Ober, "A functional approach to LQG balancing", *International Journal of Control*, 57:627–741, 1993.
6. Genin Y.P., P. V. Dooren, and T. Kailath, "On Σ-lossless transfer functions and related questions", *Linear Algebra and its Applications*, 50:251–275, 1983.
7. Glover K., "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds", *International Journal of Control*, 39:1115–1193, 1984.
8. Guidorzi R.P., "Invariants and canonical forms for systems: structural and parametric identification", *Automatica*, 17:117–133, 1981.
9. Hannan E.J., and M. Deistler, *The statistical theory of linear systems.* John Wiley, New York, 1988.
10. Hanzon B., "Recursive identification and spaces of linear dynamical systems", CWI Tracts 63, 64, CWI Amsterdam, 1989.
11. Hanzon B., "A new balanced canonical form for stable multivariable systems", *Research Memorandum*, Vakgroep Econometrie 1993-13, Free University Amsterdam, 1993.
12. Hanzon B., and R. J. Ober, "Overlapping block-balanced canonical forms and parametrizations: the SISO case", *SIAM Journal on Control and Optimization.* To appear.
13. Hinrichsen D., "Canonical forms and parametrization problems in linear system theory", in P. Cook, editor, *Fourth IMA International Conference on Control Theory.* Academic Press, New York, 1986.
14. Jonckheere E.A., and L. M. Silverman, "A new set of invariants for linear systems–application to reduced order compensator design", *IEEE Transactions on Automatic Control*, 28:953–964, 1983.
15. Kabamba P.T., "Balanced forms: canonicity and parametrization", *IEEE Transactions on Automatic Control*, 30:1106–1109, 1985.
16. Maciejowski J.M., "Balanced realizations in system identification", in *IFAC Identification and System Parameter Estimation, York, U.K.*, pages 1823–1827. Pergamon Press, Oxford, 1985.
17. McGinnie B.P., J. M. Maciejowski, and R. J. Ober, "Balanced realizations in system identification", Monograph, in preparation.
18. McGinnie B.P., R. J. Ober, and J. M. Maciejowski, "Balanced parametrizations in time series identification", in *Proceedings Conference on Decision on Control*, 1990.
19. Moore B.C., "Principal component analysis in linear systems: controllability, observability and model reduction", *IEEE Transactions on Automatic Control*, 26:17–32, 1981.
20. Mullis C.T., and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters", *IEEE Transactions on Circuits and Systems*, CAS 23:551–562, 1976.
21. Ober R.J., "Balanced realizations: canonical forms, parametrizations, model reduction", *International Journal of Control*, 46:643–670, 1987.

22. Ober R.J., "A note on a system theoretic approach to a conjecture by Peller-Khrushchev", *Systems and Control Letters*, 8:303–306, 1987.
23. Ober R.J., "A note on a system theoretic approach to a conjecture by Peller-Khrushchev: the general case", *IMA Journal of Mathematical Control and Information*, 4:263–279, 1987.
24. Ober R.J., "Topology of the set of asymptotically stable systems", *International Journal of Control*, 46:263–280, 1987.
25. Ober R.J., "Connectivity properties of classes of linear systems", *International Journal of Control*, 50:2049–2073, 1989.
26. Ober R.J., "Balanced parametrization of classes of linear systems", *SIAM Journal on Control and Optimization*, 29:1251–1287, 1991.
27. Ober R.J., and P. A. Fuhrmann, "Diffeomorphisms between sets of linear systems", in P. V. Dooren and B. Wyman, editors, *Linear Algebra for Control Theory*, IMA Volumes in Mathematics and Its Applications, volume 62, pages 117–157. Springer, 1994.
28. Ober R.J., and D. McFarlane, "Balanced canonical forms for minimal systems: a normalized coprime factorization approach", *Linear Algebra and its Applications*, 122-124:23–64, 1989.
29. Opdenacker P.C., and E. A. Jonckheere. "A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds", *IEEE Transactions on Circuits and Systems*, 35:184–189, 1988.
30. Pernebo L., and L. M. Silverman, "Model reduction via balanced state space representations", *IEEE Transactions on Automatic Control*, 37:382–387, 1982.
31. Rissanen J., "Basics of invariants and canonical forms for linear dynamic systems", *Automatica*, 10:175–182, 1974.
32. Treil S.R., "The inverse spectral problem for the modulus of a Hankel operator and balanced realizations", *Leningrad Mathematical Journal*, 2:353–375, 1991.
33. Verriest E.I., "Low sensitivity design and optimal order reduction for the LQG-problem", in *Proceedings 24th Midwest Symposium on Circuits and Systems Albuquerque, New Mexico*, June 1981.
34. Willems J.C., "Least squares stationary optimal control and algebraic Riccati equation", *IEEE Transactions on Automatic Control*, 16:621–634, 1971.