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On Stieltjes functions and Hankel operators [1]

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Abstract

Let G be a Stieltjes function which is analytic in the open right half plane. It is shown that G is in $H^{\infty}(RHP)$ if and only if the Hankel operator H_G on $H^2(RHP)$ with symbol G is nuclear. If G is in $H^{\infty}(RHP)$ it is shown that the non-tangential limit of G at $s = 0$ equals twice the nuclear norm of H_G.

1. Introduction

We will study Stieltjes functions which are analytic in the open right half plane. More precisely, let μ be a positive regular Borel measure on $]-\infty, 0]$ such that $\mu(\{0\}) = 0$ and $\int_{]-\infty,0]}(1/(1-r))\,\mathrm{d}\mu(r) < \infty$, and set

$$G(s) = \int_{]-\infty,0]} \frac{1}{s-r}\,\mathrm{d}\mu(r)$$

for $s \in \mathscr{C} \setminus supp(\mu)$. This function is analytic on $\mathscr{C} \setminus supp(\mu)$ and in particular is analytic in the open right half plane. Such functions appear in many areas of applications and have also been studied extensively from a theoretical point of view. In the rational case, the application of such functions in the area of circuit theory and systems theory has been studied in great detail by Willems [12]. System theoretic aspects and realization theory of these functions have been considered by Fuhrmann in [3] and more recently in [8]. In the Russian literature such functions have been investigated in their connection to operator nodes and operator extension problems [2, 10, 1] and [5] for their role in the theory of differential equations.

Given such a Stieltjes function G we would like to analyze the Hankel operator H_G on $H^2(RHP)$ with symbol G.

Definition 1.1. If G is an analytic function on RHP, then the operator

$$H_G : D(H_G) \to H^2(RHP); f \mapsto P_+ M_G R f$$

where

$Rf(s) = f(-s)$,

M_G multiplication operator by G,

P_+ projection on $H^2(RHP)$

with $D(H_G) = \{f \in H^2(RHP) : f$ rational, GRf has non-tangential limit a.e. on $\mathrm{i}\Re$ that is in $L^2(\mathrm{i}\Re)\}$ is called the *Hankel operator* H_G with symbol G. If H_G extends to a bounded operator on $H^2(RHP)$, this extension is also called the Hankel operator H_G.

It follows by a theorem by Widom [11] concerning positive definite Hankel matrices acting on l^2 that H_G is a bounded operator if and only if

$$\mu([r, 0]) = O(r)$$

as $r \to 0$ [9, 8].

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Here we are interested in the question when G is in $H^\infty(RHP)$. It follows by Nehari's theorem, that if G is in $H^\infty(RHP)$ then H_G is bounded, but the converse is of course not true in general. Actually we will show that $G \in H^\infty(RHP)$ if and only if H_G is nuclear, i.e. if and only if H_G is compact and $\sum_{i=1}^\infty \sigma_i < \infty$, where σ_i is the ith singular value of H_G.

If G is in $H^\infty(RHP)$, we will also show that the non-tangential limit $G(0)$ of G at 0 exists and that we have the explicit formula

$$G(0) = 2\sum_{i=1}^{\infty} \sigma_i.$$

1.1. Notation

The set of all real numbers is denoted by $\mathscr{R}$ and the set of all complex numbers is denoted by $\mathscr{C}$. All Banach spaces considered in this paper are spaces over the complex field $\mathscr{C}$. We denote by RHP the open right half plane, i.e. $RHP = \{s \in \mathscr{C} \mid \mathrm{Re}(s) > 0\}$. The Hardy space $H^\infty(RHP)$ is the Banach space of functions analytic in RHP and uniformly bounded in RHP with norm $\|f\|_\infty := \sup_{s\in RHP} |f(s)|$, for $f \in H^\infty(RHP)$. The Hardy space $H^2(RHP)$ is the Hilbert space of analytic functions in RHP, such that $\sup_{x\in\mathscr{R}, x>0} \int_{\mathscr{R}} |f(x+\mathrm{i}y)|^2\,\mathrm{d}y < \infty$, with norm $\|f\|_2 = \left(\sup_{x\in\mathscr{R}, x>0} \int_{\mathscr{R}} |f(x+\mathrm{i}y)|^2\,\mathrm{d}y\right)^{1/2}$, for $f \in H^2(RHP)$.

2. Main result

We can now proceed to the proof of the main result. We will use Howland's approach [4, 9] to the characterization of nuclear positive Hankel matrices.

Theorem 2.1. *Let μ be a positive regular Borel measure on $]-\infty,0]$ such that $\mu(\{0\}) = 0$ and $\int_{]-\infty,0]} (1/(1-r))\,\mathrm{d}\mu(r) < \infty$. Let*

$$G(s) = \int_{]-\infty,0]} \frac{1}{s-r}\,\mathrm{d}\mu(r), \quad s \in \mathscr{C}\backslash supp(\mu).$$

Then

(1) *the following statements are equivalent:*
 (a) *$G(0) := \lim_{s\in\mathscr{R}, s\to 0+} G(s)$ exists and is finite.*
 (b) *The Hankel operator H_G is nuclear, i.e. H_G is compact and $\sum_{i=1}^\infty \sigma_i < \infty$, where σ_i is the ith singular value of H_G, i.e. the square root of the ith eigenvalue of $H_G^* H_G$ ordered according to magnitude.*
 (c) *$G \in H^\infty(RHP)$.*
 (d) *$\int_{]-\infty,0]} (1/-r)\,\mathrm{d}\mu(r)$ exists.*

(2) *If one of the conditions of (1) holds, then*

$$G(0) = \int_{]-\infty,0]} \frac{1}{-r}\,\mathrm{d}\mu(r) = 2\sum_{n=1}^{\infty} \sigma_n = \|G\|_\infty.$$

Proof. (1) By the results in [8] (see also [9] for a similar but different construction), it follows that the Hankel operator H_G is unitarily equivalent to the Hankel matrix $H = (h_{i+j})_{0\leqslant i,j<\infty}$ acting on l^2 with $h_i = \int_{[-1,1]} t^n\,\mathrm{d}\nu(t)$, $n \geqslant 0$, where ν is the postive finite Borel measure on $[-1,1]$ given by

$$\nu(A) := \int_{\rho(A)} \frac{2}{(1-r)^2}\,\mathrm{d}\mu(r),$$

for A a Borel set in $[-1,1]$ and

$$\rho : [-1,1] \to [-\infty, 0];\ r \mapsto \frac{t-1}{t+1},$$

where we take $\rho(-1) = -\infty$. The Hankel matrix H is nuclear if and only if the diagonal of H is summable [9]. This is the case if and only if

$$\sum_{n=0}^{\infty} h_{2n} = \sum_{n=0}^{\infty} \int_{[-1,1]} t^{2n}\,\mathrm{d}\nu(t) < \infty.$$

By Lebesgue's monotone convergence theorem

$$\sum_{n=0}^{\infty} \int_{[-1,1]} t^{2n}\,\mathrm{d}\nu(t) = \int_{[-1,1]} \sum_{n=0}^{\infty} t^{2n}\,\mathrm{d}\nu(t) = \int_{[-1,1]} \frac{1}{1-t^2}\,\mathrm{d}\nu(t).$$

Hence H and therefore H_G is nuclear if and only if $\int_{[-1,1]} (1/(1-t^2))\,\mathrm{d}\nu(t) < \infty$. Since

$$\int_{[-1,1]} \frac{1}{1-t^2}\,\mathrm{d}\nu(t) = \int_{]-\infty,0]} \frac{1}{1-((1+r)/(1-r))^2} \frac{2}{(1-r)^2}\,\mathrm{d}\mu(r) = \int_{]-\infty,0]} \frac{1}{-2r}\,\mathrm{d}\mu(r),$$

H_G is nuclear if and only if $\int_{]-\infty,0]} (1/-2r)\,\mathrm{d}\mu(r) < \infty$, i.e. (b) is equivalent to (d).

This also implies that if H_G is nuclear then

$$\sum_{n=1}^{\infty} \sigma_n = \int_{]-\infty,0]} \frac{1}{-2r}\,\mathrm{d}\mu(r).$$

(d) implies (a): For $s > 0$, $r < 0$, $|1/(s-r)| \leqslant 1/-r$. Since $\int_{]-\infty,0]} (1/-r)\,\mathrm{d}\mu(r) < \infty$, by Lebesgue's dominated convergence theorem

$$\lim_{\substack{s\in\mathscr{R}\\ s\to 0-}} G(s) = \lim_{\substack{s\in\mathscr{R}\\ s\to 0-}} \int_{]-\infty,0]} \frac{1}{s-r}\,\mathrm{d}\mu(r) = \int_{]-\infty,0]} \frac{1}{-r}\,\mathrm{d}\mu(r).$$

(a) implies (d): Since for $s > 0$ and $r < 0$, $1/(s-r) > 0$, we have by Fatou's Lemma

$$\int_{]-\infty,0]} \frac{1}{-r}\,\mathrm{d}\mu(r) = \int_{]-\infty,0]} \liminf_{\substack{s\in\mathscr{R}\\ s\to 0+}} \frac{1}{-r}\,\mathrm{d}\mu(r) \leqslant \liminf_{\substack{s\in\mathscr{R}\\ s\to 0+}} \int_{]-\infty,0]} \frac{1}{s-r}\,\mathrm{d}\mu(r) = \lim_{\substack{s\in\mathscr{R}\\ s\to 0-}} G(s) =: G(0) < \infty.$$

Hence (d)

(d) implies (c): For $s \in RHP$ and $r \in \mathscr{R}$, $r < 0$ we have that $|1/(s-r)| \leqslant |1/r|$. Hence

$$|G(s)| \leqslant \int_{]-\infty,0]} \frac{1}{|s-r|}\,\mathrm{d}\mu(r) \leqslant \int_{]-\infty,0]} \frac{1}{-r}\,\mathrm{d}\mu(r) < \infty.$$

Therefore $G \in H^\infty(RHP)$.

(c) implies (d): For $s \in RHP$, $r \in \mathscr{R}$, $r < 0$, $1/(s-r) > 0$. Hence by Fatou's Lemma

$$\int_{]-\infty,0]} \frac{1}{-r}\,\mathrm{d}\mu(r) = \int_{]-\infty,0]} \liminf_{\substack{s\in\mathscr{R}\\ s\to 0+}} \frac{1}{s-r}\,\mathrm{d}\mu(r) \leqslant \liminf_{\substack{s\in\mathscr{R}\\ s\to 0+}} \int_{]-\infty,0]} \frac{1}{s-r}\,\mathrm{d}\mu(r) = \lim_{\substack{s\in\mathscr{R}\\ s\to 0+}} G(s) \leqslant \|G\|_\infty.$$

(2) Follows by the identities established in (1).

Note that part (2) of the theorem generalizes a formula for rational functions [6, 7] to the general situation discussed here.

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