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Edited by

I. Gohberg
P. Lancaster
P.N. Shivakumar

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Editors' addresses:

I. Gohberg
School of Mathematical Sciences
Tel Aviv University
69978 Tel Aviv
Israel

P. Lancaster
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta T2N 1NA
Canada

P.N. Shivakumar
Department of Applied Mathematics and
Institute of Industrial Mathematical Sciences
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

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SYSTEM THEORETIC ASPECTS OF COMPLETELY SYMMETRIC SYSTEMS

RAIMUND J. OBER

System theoretic aspects of completely symmetric systems will be discussed both for discrete time and continuous time systems. Realization theoretic results are presented. Necessary and sufficient conditions are given for the boundedness of the observability and reachability operators. The asymptotic, exponential/power stability of a completely symmetric system is characterized through the support of its defining measure. For continuous time systems the boundedness of the system operators is analyzed.

1 Introduction

In this paper we consider completely symmetric systems. Finite dimensional completely symmetric systems or relaxation systems have received a considerable amount of attention (see e.g. [17]). The primary aim of this paper is to examine this class of systems in the infinite dimensional case. In the Russian literature such systems have been investigated in their connection to operator nodes and operator extension problems ([4], [16]). Transfer functions of completely symmetric systems are Stieltjes functions for which there is a rich literature, see e.g. [1] for their role in operator theory and [7] for their role in the theory of differential equations. System theoretic investigations of subclasses of this class of systems can be found for example in [6], [3], [9].

In this paper we present a system theoretic study of this class of systems without additional assumptions such as the boundedness of the input and output operators for continuous time systems. We investigate both discrete time and continuous time systems. Particular emphasis is placed on the analysis of system theoretic properties through properties of the transfer function. The reachability and observability of the systems is characterized through the so-called defining measure of the system. It will be shown that a completely symmetric system has bounded reachability and observability operator if and only if the defining measure is a Carleson measure. The exponential stability of a completely symmetric system is characterized through the support of the defining measure. A realization result is also given.

Discrete time systems are analyzed first in Section 2. Continuous time systems are then investigated in the subsequent section. The bilinear transform that was studied in [10] will be used to translate several of the discrete time results to a continuous time setting.

1.1 Notation

The set of all real numbers is denoted by $\mathcal{R}$ and the set of all complex numbers is denoted by $\mathcal{C}$. If $A \subseteq \mathcal{R}$ then χ_A denotes the characteristic function of the set A, i.e. $\chi_A(\lambda) = 1$ for $\lambda \in A$ and $\chi_A(\lambda) = 0$ for $\lambda \in \mathcal{R} \setminus A$.

All Banach spaces considered in this paper are spaces over the complex field $\mathcal{C}$. Given a Hilbert space our convention is that the scalar product is linear in the first component and anti-linear in the second component. For H_1, H_2 Hilbert spaces, $L(H_1, H_2)$ denotes the space of bounded linear operators $T : H_1 \to H_2$. For an operator T on a Hilbert space the spectrum is denoted by $\sigma(T)$. The point spectrum is denoted by $\sigma_p(T)$. For an operator T the Hilbert space adjoint is denoted by T^*. The open unit disk is denoted by $\mathcal{D}$, i.e. $\mathcal{D} = \{z \in \mathcal{C} \mid |z| < 1\}$. The exterior of the closed unit disk is denoted by $\mathcal{D}_e$, i.e. $\mathcal{D}_e = \{z \in \mathcal{C} \mid |z| > 1\}$. We denote by RHP the open right half plane, i.e. $RHP = \{s \in \mathcal{C} \mid Re(s) > 0\}$.

For a measurable function $F : \Omega \to \mathcal{C}$ we say that the integral $\int_\Omega f d\nu$ exists if $\int_\Omega |f| d\nu < \infty$. For a regular positive Borel measure ν on a subset A of $\mathcal{R}$, the Hilbert space of functions on A that are square integrable with respect to ν is denoted by $L^2(A, \nu)$.

The Hardy space $H^\infty(RHP)$ is the Banach space of functions analytic in RHP and uniformly bounded in RHP with norm $\|f\|_\infty := \sup_{s \in RHP} |f(s)|$, for $f \in H^\infty(RHP)$. The Hardy space $H^2(RHP)$ is the Hilbert space of analytic functions in RHP, such that $\sup_{\substack{x \in \mathcal{R} \\ x>0}} \int_{\mathcal{R}} |f(x+iy)|^2 dy < \infty$, with norm $\|f\|_2 = \left(\sup_{\substack{x \in \mathcal{R} \\ x>0}} \int_{\mathcal{R}} |f(x+iy)|^2 dy\right)^{\frac{1}{2}}$, for $f \in H^2(RHP)$.

2 Discrete time systems

A quadruple (A_d, B_d, C_d, D_d) of operators is called a discrete-time system with input space U, output space Y and state space X with U, Y, X being separable Hilbert spaces if A_d is a contraction on X, $B_d \in L(U, X)$, $C_d \in L(X, Y)$, $D_d \in L(U, Y)$. The system is called admissible if $-1 \notin \sigma_p(A_d)$ and $\lim_{\lambda > 1, \lambda \to 1} C_d(\lambda I + A_d)^{-1} B_d$ exists in the norm topology. We denote by $D_X^{U,Y}$ the set of admissible systems with state space X, input space U and output space Y.

We now define what we mean by completely symmetric discrete time systems. In this paper we only consider single input single output systems, i.e. systems such that U and Y are one-dimensional.

Definition 2.1 *A discrete-time (admissible) single input single output system* (A_d, B_d, C_d, D_d) *is called* completely symmetric *if it coincides with its dual system, i.e. if*

$$A_d = A_d^*, \quad B_d = C_d^*, D_d = D_d^*,$$

and $\pm 1 \notin \sigma_p(A_d)$.

The following proposition gives a characterization of the transfer function of a completely symmetric system.

Proposition 2.1 *Let (A_d, B_d, C_d, D_d) be a completely symmetric discrete time system with transfer function $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$, $z \in \mathcal{D}_e$. Set $G_d^{\perp}(z) := \frac{1}{z}[G_d(\frac{1}{z}) - G_d(\infty)]$, $z \in \mathcal{D}$. Then there exists a unique positive finite Borel measure ν on $[-1,1]$ such that for $z \in \mathcal{D}$,*

$$G_d^{\perp}(z) = \sum_{n=0}^{\infty} a_n z^n = \int_{[-1,1]} \frac{1}{1-tz} d\nu(t),$$

and for $z \in \mathcal{D}_e$,

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + D_d$$

with $a_n := \int_{[-1,1]} t^n d\nu(t)$, $n \geq 0$.
Moreover,

1.
$$\nu(\{-1,1\}) = 0,$$

2.
$$\lim_{n\to\infty} a_n = 0.$$

In particular, if $A_d = \int_{[-1,1]} t dE(t)$, is the spectral decomposition of A_d with Borel σ-algebra Ω then ν is the Borel measure given by

$$\nu(\omega) =< E(\omega)B_d, B_d >, \quad \omega \in \Omega.$$

Proof: We have for $z \in \mathcal{D}$,

$$G_d^{\perp}(z) = \frac{1}{z}C_d(\frac{1}{z}I - A_d)^{-1}B_d = C_d(I - zA_d)^{-1}B_d$$

$$= C_d \sum_{n=0}^{\infty}(zA_d)^n B_d = \sum_{n=0}^{\infty} C_d z^n A_d^n B_d = \sum_{n=0}^{\infty} B_d^* z^n A_d^n B_d$$

$$= \sum_{n=0}^{\infty} B_d^* z^n \int_{[-1,1]} t^n dE(t) B_d = \sum_{n=0}^{\infty} z^n \int_{[-1,1]} t^n d < E(t)B_d, B_d >$$

$$= \sum_{n=0}^{\infty} z^n \int_{[-1,1]} t^n d\nu(t) = \int_{[-1,1]} \sum_{n=0}^{\infty} z^n t^n d\nu(t)$$

$$= \int_{[-1,1]} \frac{1}{1-tz} d\nu(t),$$

where $A_d = \int_{[-1,1]} t dE(t)$, is the spectral decomposition of A_d with Borel σ-algebra Ω and ν is the Borel measure given by $\nu(\omega) =< E(\omega)B_d, B_d >$, $\omega \in \Omega$. This measure is finite and positive since for $\omega \in \Omega$

$$\nu(\omega) =< E(\omega)B_d, B_d >=< E(\omega)B_d, E(\omega)B_d >= \|E(\omega)B_d\|^2 \leq \|B_d\|^2$$

and

$$\nu(\omega) =< E(\omega)B_d, B_d > \geq 0.$$

Assume that there is another positive finite regular Borel measure μ such that

$$a_n = \int_{[-1,1]} t^n d\nu(t) = \int_{[-1,1]} t^n d\mu(t), \quad n \geq 0.$$

Let f be a continuous function on $[-1,1]$. By Weierstrass's theorem for $\epsilon > 0$ there exists a polynomial $p_n(t) = \sum_{k=0}^{n} \lambda_n t^n$ such that $\sup_{t\in[-1,1]} |f(t) - p_n(t)| < \epsilon$. Then since $\int_{[-1,1]} p_n d\mu = \int_{[-1,1]} p_n d\nu$,

$$\begin{aligned} |\int_{[-1,1]} f d\mu - \int_{[-1,1]} f d\nu| &= |\int_{[-1,1]} f d\mu - \int_{[-1,1]} p_n d\mu - (\int_{[-1,1]} f d\nu - \int_{[-1,1]} p_n d\nu)| \\ &\leq \int_{[-1,1]} |f - p_n| d\mu + \int_{[-1,1]} |f - p_n| d\nu \\ &\leq \epsilon\mu([-1,1]) + \epsilon\nu([-1,1]). \end{aligned}$$

Hence $\int_{[-1,1]} f d\mu = \int_{[-1,1]} f d\nu$ for all continuous functions f on $[-1,1]$. Therefore, by the Riesz representation theorem ([13], p. 40) $\mu = \nu$.

1.) Note that since by assumption $\pm 1 \notin \sigma_p(A_d)$, it follows ([14], Theorem 12.29) that $E(\{-1,+1\}) = 0$. Hence $\nu(\{-1,+1\}) =< E(\{-1,+1\} B_d, B_d >= 0$.

2.) Clearly, $|t^n| < 1$ and $\lim_{n\to\infty} t^n = 0$ for $t \in]-1,1[$. As $\nu(\{-1,1\}) = 0$ we have for $n \geq 0$ that

$$a_n = \int_{[-1,1]} t^n d\nu(t) = \int_{[-1,1]} t^n d\nu(t) = \int_{]-1,1[} t^n d\nu(t).$$

Since the measure ν is finite we have by the Lebesgue dominated convergence theorem that

$$a_n = \int_{]-1,1[} t^n d\nu(t) \to 0$$

as $n \to \infty$.

□

Given a completely symmetric discrete-time system or its transfer function we call the measure ν constructed in the previous Proposition the *defining measure* of the system or transfer function.

We now show that functions with the above integral representation are analytic outside the support $supp(\nu)$ of the measure ν, where $supp(\nu)$ is the complement of the largest open set A with $\nu(A) = 0$.

Lemma 2.1 *Let ν be a finite positive regular Borel measure on $[-1,1]$ such that $\nu(\{-1,+1\}) = 0$. Then the function f given by*

$$z \mapsto f(z) := \int_{[-1,1]} \frac{1}{z-t} d\nu(t)$$

is analytic on $C \setminus supp(\nu)$.

A consequence of this Lemma is that if $G_d(z)$, $z \in \mathcal{D}_e$, is the transfer function of a completely symmetric system (A_d, B_d, C_d, D_d), G_d can be extended analytically to $C \setminus supp(\nu)$ where ν is the defining measure of the system. The continuation has the same integral representation

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + D_d,$$

$z \in C \setminus supp(\nu)$.

2.1 Stability

A discrete-time system (A_d, B_d, C_d, D_d) is called *asymptotically stable* if $\lim_{n\to\infty} A_d^n x = 0$ for $x \in X$ and *power stable* if there exists $0 \le r < 1$ and $0 \le M < \infty$ such that $\|A_d^n\| \le Mr^n$ for $n = 0, 1, 2, \ldots$.

Lemma 2.2 *Let (A_d, B_d, C_d, D_d) be a completely symmetric discrete-time system. Then*

1. *(A_d, B_d, C_d, D_d) is asymptotically stable.*
2. *(A_d, B_d, C_d, D_d) is power stable if and only if $\sigma(A_d) \subseteq [-\alpha, \alpha]$ for some $0 \le \alpha < 1$.*

Proof: Let $A_d = \int_{[-1,1]} \lambda dE(\lambda)$ be the spectral representation of A_d. Note that as $\pm 1 \notin \sigma_p(A_d)$, we have that $E(\{-1, 1\}) = 0$. Let x be a vector in the state space X. Then for $n \ge 0$,

$$\|A_d^n x\|^2 = \int_{[-1,1]} |\lambda^n|^2 dE_{x,x}(\lambda) = \int_{]-1,1[} |\lambda^{2n}| dE_{x,x}(\lambda),$$

as $E(\{-1, 1\}) = 0$. Since $|\lambda^{2n}| \to 0$, as $n \to \infty$, it follows by the Lebesgue dominated convergence theorem that $\|A_d^n x\|^2 \to 0$. Hence the system is asymptotically stable.

It follows from the formula for the spectral radius, i.e. $\rho(A_d) = \inf_{n\ge1} \|A_d^n\|^{1/n}$ that the system is power stable if and only if the spectral radius is strictly less than 1. This is of course the case here if and only if $\sigma(A_d) \subseteq [-\alpha, \alpha]$ for some $0 \le \alpha < 1$. □

2.2 Observability and reachability

For a discrete-time system (A_d, B_d, C_d, D_d) the *observability operator* $\mathcal{O}_d$ is defined by $\mathcal{O}_d : D(\mathcal{O}_d) \to l_Y^2$; $x \mapsto (C_d A_d^n x)_{n\ge0}$, where $D(\mathcal{O}_d) = \{x \in X \mid (C_d A_d^n x)_{n\ge0} \in l_Y^2\}$. The system is said to have bounded observability operator if $D(\mathcal{O}_d) = X$ in which case $\mathcal{O}_d$ is a bounded operator. The system is called *observable* if it has a bounded observability operator with zero kernel. The *reachability operator* $\mathcal{R}_d$ of the system is defined by $\mathcal{R}_d : D(\mathcal{R}_d) \to X$; $\mathcal{R}_d((u_i)_{1\le i\le k}) = \sum_{i=0}^{\infty} A^i B u_i$, where $D(\mathcal{R}_d)$ is the set of finite sequences in l_U^2. The system is said to have bounded reachability operator if $\mathcal{R}_d$ extends to a bounded operator on l_U^2. If such an extension exists, the extension will also be called the reachability operator and will be also be denoted by $\mathcal{R}_d$. The system is called *reachable* if it has a bounded reachability operator with dense range.

For power stable systems it is easily seen that they have bounded observability respectively reachability operators.

Lemma 2.3 *A power stable completely symmetric discrete-time system has bounded reachability and observability operator.*

Proof: The result is easily verified. □

In order to give a characterization of the boundedness of the observability and reachability operators for general completely symmetric systems we need to introduce the notion of a Hankel operator. Let $h_n \in C$ for $n = 0, 1, \ldots$ and consider the operator $H : l_U^2 \to l_Y^2$ given by the matrix $H = (h_{n+m})_{n,m\ge0}$.

Lemma 2.4 *Let (A_d, B_d, C_d, D_d) be a completely symmetric discrete-time system. Then $D(\mathcal{O}_d)$ is dense in X, $\mathcal{O}_d = \mathcal{R}_d^*$ and the following statements are equivalent.*

1. *The system has bounded reachability operator.*
2. *The system has bounded observability operator.*
3. *The Hankel operator H is bounded where H is given by the matrix $H = (C_d A_d^{i+j} B_d)_{i,j\geq 0}$.*

Moreover, the system is observable if and only if it is reachable.

Proof: The proof follows from duality arguments and the fact that $H = \mathcal{O}\mathcal{R}$. □

The following theorem is now an immediate consequence of Widom's theorem that characterizes the boundedness of positive Hankel operators (see [11]).

Theorem 2.1 *Let (A_d, B_d, C_d, D_d) be a completely symmetric discrete-time system with defining measure ν. The system has bounded reachability respectively observability operator if and only if ν is a Carleson measure on $[-1,1]$ which is the case if and only if*

$$\nu([\alpha,1]) + \nu([-1,-\alpha]) = O(1-\alpha)$$

as $\alpha \to 1$.

Proof: The theorem follows by combining the previous Lemma with Widom's theorem ([11], Theorem 1.6). □

We can now address the problem of the observability and reachability of completely symmetric discrete time systems.

Theorem 2.2 *Let (A_d, B_d, C_d, D_d) be a completely symmetric discrete-time system with bounded reachability and observability operator. Let $A_d = \int_{[-1,1]} \lambda dE(\lambda)$, be the spectral decomposition of A_d with spectral decomposition E defined on the Borel σ-algebra Ω on $[-1,1]$. Then the system is reachable/observable if and only if*

$$\cap_{\omega\in\Omega} \ker(C_d E(\omega)) = \{0\}.$$

Proof: We show that $\cap_{\omega\in\Omega} \ker(C_d E(\omega)) = ker(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the observability operator of the system. Let $x \in ker(\mathcal{O}_d)$, then for each $n \geq 0$, $y \in Y$,

$$0 =< y, C_d A_d^n x >=< C_d^* y, A_d^n x >=< C_d^* y, \int_{[-1,1]} \lambda^n dE(\lambda) x >= \int_{[-1,1]} \lambda^n dE_{x,C_d^* y}(\lambda).$$

By Weierstrass's theorem and the Riesz representation theorem ([13], Theorem 6.19) this implies that the complex Borel measure $\omega \mapsto E_{x,C_d^* y}(\omega) =< C_d^* y, E(\omega)x >$ on $[-1,1]$ is the zero measure, i.e. $< y, C_d E(\omega) x >= 0$ for all $\omega \in \Omega$ and therefore $x \in \ker(C_d E(\omega))$ for all $\omega \in \Omega$. Hence $x \in \cap_{\omega\in\Omega} \ker(C_d E(\omega))$.

Let now $x \in \cap_{\omega\in\Omega} \ker(C_d E(\omega))$. Then for $y \in Y$, $\omega \in \Omega$,

$$E_{x,C_d^* y}(\omega) = < C_d^* y, E(\omega)x > = < y, C_d E(\omega)x > = 0.$$

Hence $E_{x,C_d^* y}(\omega) = 0$ for all $\omega \in \Omega$ and $n \geq 0$

$$0 = \int_{[-1,1]} \lambda^n dE_{x,C_d^* y}(\lambda) = < C_d^* y, \int_{[-1,1]} \lambda^n dE(\lambda)x > = < C_d^* y, A_d^n x > = < y, C_d A_d^n x > .$$

This implies that $C_d A_d^n x = 0$, $n \geq 0$, and therefore $\mathcal{O}_d x = 0$, i.e. $x \in Ker(\mathcal{O}_d)$. □

2.3 Realization theory

In Proposition 2.1 we showed that the transfer function of a discrete time completely symmetric system has a particular integral representation that is determined by the defining measure ν. The defining measure was shown to be a positive finite Borel measure on $[-1, 1]$ such that $\nu(\{-1, 1\}) = 0$. In the following realization result we are going to show that the converse is also true. Given a positive finite Borel measure ν on $[-1, 1]$ such that $\nu(\{-1, 1\}) = 0$ we establish the existence of a completely symmetric discrete time system whose defining measure is ν.

Theorem 2.3 *Let ν be a positive finite Borel measure on $[-1, 1]$, such that $\nu(\{-1, 1\}) = 0$. Let $c \in C$ and let*

$$G_d(z) := \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + c$$

for $z \in C \setminus supp(\nu)$. Let $X = L^2([-1, 1], \nu)$ and define

$$B_d : C \to X, \qquad u \mapsto \chi_{[-1,1]} u;$$

$$A_d : X \to X, \qquad x \mapsto Mx;$$

$$C_d := B_d^*,$$

$$D_d := c,$$

where $(Mx)(t) = tx(t)$, $t \in [-1, 1]$.

Then

1. *(A_d, B_d, C_d, D_d) is a completely symmetric discrete time system whose transfer function is G_d.*

2. *The system (A_d, B_d, C_d, D_d) has bounded reachability or observability operators if and only if ν is a Carleson measure.*

3. *If ν is a Carleson measure then the system is observable and reachable.*

Proof: 1.) Clearly B_d and A_d are bounded operators and A_d is self-adjoint. Since $\sigma(A_d) \subseteq [-1,1]$ and A_d is self-adjoint, A_d is a contraction. As $\nu(\{-1,1\}) = 0$, we have that $\sigma_p(A_d) \subseteq]-1,1[$. Hence the system has a transfer function G_d^1 which is analytic on $\mathcal{D}_e$, where for $z \in \mathcal{D}_e$, and $u, y \in \mathcal{C}$,

$$< y, G_d^1(z)u >=< y, (C_d(zI - A_d)^{-1}B_d + D_d)u >=< B_d y, (zI - A_d)^{-1}B_d u > + < y, D_d u >$$

$$=< \chi_{[-1,1]}y, (zI - M)^{-1}\chi_{[-1,1]}u > + yc\bar{u} = y\int_{[-1,1]} \frac{1}{z-t} d\nu(t)\bar{u} + yc\bar{u} = yG_d(z)\bar{u}.$$

Hence the system is a realization of G_d. Clearly the system is completely symmetric.
2.) This is a consequence of Theorem 2.1 .
3.) Let ν be a Carleson measure. The system is reachable and therefore also observable if *range*$(\mathcal{R})$ is dense in $L^2(\nu)$. Let $u = (u_0, u_1, \ldots, u_n, 0, 0, \ldots)$ then for $t \in [-1,1]$,

$$(\mathcal{R}u)(t) = (\sum_{i=0}^{n} A^i B u_i)(t) = (\sum_{i=0}^{n} M^i \chi_{[-1,1]} u_i)(t) = \sum_{i=0}^{n} t^i u_i.$$

Hence *range*$(\mathcal{R})$ is dense in $L^2(\nu)$ if the polynomial functions $(t^i)_{i\geq 0}$ span $L^2(\nu)$. But this is the case by ([13], p.69) and Weierstrass's theorem. That the system is observable follows by duality. □

An observable and reachable discrete-time system (A_d, B_d, C_d, D_d) with reachability operator $\mathcal{R}$ and observability operator $\mathcal{O}$ is called *par-balanced* if $\mathcal{O}^*\mathcal{O} = \mathcal{R}\mathcal{R}^*$. The duality properties of a completely symmetric observable and reachable system imply that such a system is par-balanced.

The following proposition is due to N. Young ([18]) and shows that a par-balanced realization is unique up to a unitary state-space transformation.

Lemma 2.5 *Let (A_d, B_d, C_d, D_d) be a reachable and observable par-balanced realization of a transfer function G. Then all reachable and observable par-balanced realizations of the transfer function G are given by $(UA_dU^*, UB_d, C_dU^*, D_d)$, where U is unitary.*

Hence we have the following Lemma.

Lemma 2.6 *Let G_d be the transfer function of a completely symmetric discrete time system. Then (A_d, B_d, C_d, D_d) is a completely symmetric realization of G_d if and only if (A_d, B_d, C_d, D_d) is a par-balanced realization of G_d.*

In the following Lemma the spectral minimality of a completely symmetric system is established.

Corollary 2.1 *Let ν be a positive finite measure on $[-1,1]$ such that $\nu(\{-1,1\}) = 0$ and assume that ν is a Carleson measure. Let*

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t),$$

for $z \notin supp(\nu)$. *If* (A_d, B_d, C_d, D_d) *is a par-balanced realization of* G_d, *then*

$$\sigma(G_d) = \sigma(A_d) = supp(\nu),$$

where $\sigma(G_d)$ *denotes the set of singularities of* G_d, *i.e. those points in the complex plane at which* G_d *has no analytic extension. Moreover, the spectrum of* A_d *has only simple multiplicity.*

Proof: The realization of Theorem 2.3 is par-balanced. Since by Lemma 2.5 all par-balanced realizations are related to this realization by a unitary transformation, we can assume without loss of generality that (A_d, B_d, C_d, D_d) is the realization of Theorem 2.3. This realization is reachable, observable and A_d is self-adjoint. Therefore, it is spectrally minimal (see [3],[5]), i.e. $\sigma(G_d) = \sigma(A_d)$ and by ([12], p.229), $\sigma(A_d) = supp(\nu)$. Moreover, by ([12], p.232), A_d only has simple spectrum. □

In the following corollary the stability question is addressed again.

Corollary 2.2 *Let* ν *be a positive finite measure on* $[-1,1]$ *such that* $\nu(\{-1,1\}) = 0$ *and assume that* ν *is a Carleson measure. Let*

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t),$$

for $z \notin supp(\nu)$. *If* (A_d, B_d, C_d, D_d) *is a par-balanced realization of* G_d, *then the system is asymptotically stable. It is power stable if and only if*

$$supp(\nu) \subseteq [-\alpha, \alpha]$$

for some $0 \leq \alpha < 1$.

Proof: This follows immediately from the previous corollary and Lemma 2.2. □

3 Continuous-time systems

In this section we will consider continuous time completely symmetric systems. To study these systems in appropriate generality we need to deal with systems with unbounded operators. Such systems are now defined.

If A is the generator of a strongly continuous semigroup of contractions on the Hilbert space X then $D(A)$ is a Hilbert space with inner product induced by the graph norm $\|x\|_A^2 := \|x\|^2 + \|Ax\|_X^2$, $x \in D(A)$. Denote by $D(A)^{(')}$ the Hilbert space of antilinear functionals on $(D(A), \|\cdot\|_A)$ with norm $\|f\|' := \sup_{\|x\|_A \leq 1} |f(x)|$, $f \in D(A)^{(')}$. We then have the rigged structure

$$D(A) \subseteq X \subseteq D(A)^{(')}.$$

For the adjoint $(A^*, D(A^*))$ we have similarly $D(A^*) \subseteq X \subseteq D(A^*)^{(')}$. We can now define admissible continuous-time systems (see [10]).

Definition 3.1 *A quadruple of operators (A_c, B_c, C_c, D_c) is called an* admissible *continuous time system with state space X, input space U and output space Y, where X, U, Y are separable Hilbert spaces, if*

1. *$(A_c, D(A_c))$ is the generator of a strongly continuous semigroup of contractions on X.*
2. *$B_c : U \to (D(A_c^*)^{(\prime)}, \|\cdot\|')$ is a bounded linear operator.*
3. *$C_c : D(C_c) \to Y$ is linear with $D(C_c) = D(A_c) + (I - A_c)^{-1}B_cU$ and $C_{c|D(A_c)} : (D(A_c), \|\cdot\|_{A_c}) \to Y$ is bounded.*
4. *$C_c(I - A_c)^{-1}B_c \in L(U, Y)$.*
5. *A_c, B_c, C_c are such that $\lim_{\substack{s\in\mathbb{R} \\ s\to+\infty}} C_c(sI - A_c)^{-1}B_c = 0$ in the norm topology.*

We write $C_X^{U,Y}$ for the set of admissible continuous time systems with input space U, output space Y and state space X.

In order to define what we mean by a completely symmetric continuous time system we need to recall the definition of the dual of an admissible continuous time system (see [10]).

Definition 3.2 *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$. Then the* dual system *$(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ of (A_c, B_c, C_c, D_c) is given by*

1. *$(\tilde{A}_c, D(\tilde{A}_c)) := (A_c^*, D(A_c^*))$, the adjoint operator of $(A_c, D(A_c))$.*
2. *$\tilde{B}_c : Y \to D(A_c)^{(\prime)}$; $y \mapsto \tilde{B}_c(y)[\cdot] :=< y, C_c(\cdot) >$.*
3. *$\tilde{C}_c : D(\tilde{C}_c) \to U$, $D(\tilde{C}_c) = D(\tilde{A}_c) + (I - \tilde{A}_c)^{-1}\tilde{B}_cY$, where $\tilde{C}_cx_0$ is defined by*

$$< u, \tilde{C}_cx_0 >= B_c(u)[x_0]$$

for $x_0 \in D(A_c^*)$, $u \in U$, and

$$< \tilde{C}_cx_0, u >=< y_0, C_c(I - A_c)^{-1}B_cu >$$

for $x_0 = (I - \tilde{A}_c)^{-1}\tilde{B}_cy_0$, $y_0 \in Y$, $u \in U$.

4. *$\tilde{D}_c := D_c^* : Y \to U$.*

The dual system of an admissible system is admissible. If the continuous time transfer function $G(s) : RHP \to L(U, Y)$ has an admissible realization (A_c, B_c, C_c, D_c), then the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a realization of the transfer function $\tilde{G}(s) := (G(\bar{s}))^*$, $s \in RHP$.

We now define a completely symmetric continuous-time system. As in the discrete-time case we restrict ourselves to systems with one dimensional input and output spaces.

Definition 3.3 *An admissible system (A_c, B_c, C_c, D_c) with one dimensional input and output space is called* completely symmetric *if*

$$A_c = \tilde{A}_c, \quad B_c = \tilde{C}_c, \quad D_c = \tilde{D}_c$$

and $0 \notin \sigma_p(A_c)$, where $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is the dual system.

Our method of analysis of continuous time completely symmetric systems is mainly based on relating these systems to discrete time completely symmetric systems. This will be done by the bilinear transform between continuous time and discrete time admissible systems. For a discussion of the background of this technique and the particular formulation which we will need see [10].

In the following theorem (see [10]) we introduce the map $T : D_X^{U,Y} \to C_X^{U,Y}$ that transforms discrete time systems to continuous time systems.

Theorem 3.1 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, then $T((A_d, B_d, C_d, D_d)) =: (A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, where the operators A_c, B_c, C_c, D_c are defined as follows:*

1. $A_c := (I+A_d)^{-1}(A_d-I) = (A_d-I)(I+A_d)^{-1}$, $D(A_c) := D((I+A_d)^{-1})$. *This operator generates a strongly continuous semigroup of contractions on X.*

2. *The operator B_c is given by:*
$$B_c := \sqrt{2}(I + A_d)^{-1}B_d : U \to D(A_c^*)^{(')};$$
$$u \mapsto \sqrt{2}(I + A_d)^{-1}B_d(u)[x] :=< B_d(u), (I + A_d^*)^{-1}(x) >_X .$$

3. *The operator C_c is given by:*
$$C_c : D(C_c) \to Y; \qquad x \mapsto \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} \sqrt{2}C_d(\lambda I + A_d)^{-1}x,$$
where $D(C_c) = D(A_c)+(I-A_c)^{-1}B_cU$. On $D(A_c)$ we have $C_{c|D(A_c)} = \sqrt{2}C_d(I+A_d)^{-1}$.

4. $D_c := D_d - \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} C_d(\lambda I + A_d)^{-1}B_d$.

Moreover, let the admissible discrete time system (A_d, B_d, C_d, D_d) be a realization of the transfer function
$$G_d(z) : \mathcal{D}_e \to L(U,Y),$$
i.e. $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$ for $z \in \mathcal{D}_e$. Then
$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d))$$
is an admissible continuous time realization of the transfer function
$$G_c(s) := G_d\left(\frac{1+s}{1-s}\right) : RHP \to L(U,Y),$$
$s \in RHP$.

The inverse map is considered in the next theorem ([10]).

Theorem 3.2 *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, then $T^{-1}((A_c, B_c, C_c, D_c)) := (A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, where the operators A_d, B_d, C_d, D_d are defined as*

1. $A_d := (I + A_c)(I - A_c)^{-1}$, *and for $x \in D(A_c)$ we have that $A_dx = (I - A_c)^{-1}(I + A_c)x$.*

2. $B_d := \sqrt{2}(I - A_c)^{-1}B_c$.

3. $C_d := \sqrt{2}C_c(I - A_c)^{-1}$.

4. $D_d := C_c(I - A_c)^{-1}B_c + D_c$.

Moreover, let the admissible continuous time system (A_c, B_c, C_c, D_c) *be a realization of the transfer function* $G_c : RHP \to L(U,Y)$, *i.e.* $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$, $s \in RHP$. *Then*

$$(A_d, B_d, C_d, D_d) = T^{-1}((A_c, B_c, C_c, D_c))$$

is an admissible discrete time realization of the transfer function

$$G_d(z) := G_c\left(\frac{z-1}{z+1}\right), \qquad z \in \mathcal{D}_e.$$

The following Lemma shows that T maps completely symmetric discrete-time systems to completely symmetric continuous-time systems.

Lemma 3.1 *Let* (A_d, B_d, C_d, D_d) *be an admissible discrete-time system and let* $(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$. *Then* (A_c, B_c, C_c, D_c) *is completely symmetric if and only if* (A_d, B_d, C_d, D_d) *is completely symmetric.*

Proof: This follows immediately from the fact that the map T maps the dual system of (A_d, B_d, C_d, D_d) to the dual system of (A_c, B_c, C_c, D_c) (see [10]). Moreover, $+1 \notin \sigma_p(A_d)$ if and only if $0 \notin \sigma_p(A_c)$. Note that $-1 \notin \sigma_p(A_d)$ by the definition of admissibility. □

In order to be able to define the bilinear transform for a discrete-time completely symmetric system (A_d, B_d, C_d, D_d) the following admissibility condition (Section 2). has to be satisfied. It is required that the limit $\lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1}B_d$ exists. If

$$G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + D_d,$$

$z \notin C \setminus supp(\nu)$, is the transfer function of the system this is equivalent to requiring that

$$\lim_{\substack{\lambda > 1 \\ \lambda \to 1}} C_d(\lambda I + A_d)^{-1}B_d = \lim_{\substack{\lambda > 1 \\ \lambda \to 1}} \int_{[-1,1]} \frac{1}{\lambda + t} d\nu(t)$$

exists.

The following Lemma gives a necessary and sufficient condition for a discrete time completely symmetric system to be admissible.

Lemma 3.2 *Let* (A_d, B_d, C_d, D_d) *be a completely symmetric discrete-time system with transfer function*

$$G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + D_d,$$

$z \notin C \setminus supp(\nu)$, where ν is the defining measure. Then the system is admissible, i.e. $\lim_{\substack{\lambda>1\\ \lambda\to 1}} C_d(\lambda I + A_d)^{-1}B_d$ *exists if and only if the integral*

$$\int_{[-1,1]} \frac{1}{1+t} d\nu(t)$$

exists. Moreover, if $\int_{[-1,1]} \frac{1}{1+t} d\nu(t)$ *exists then*

$$\lim_{\substack{\lambda>1\\ \lambda\to 1}} C_d(\lambda I + A_d)^{-1}B_d = \int_{[-1,1]} \frac{1}{1+t} d\nu(t).$$

Proof: Let for $\lambda \geq 1$, $t \in [-1,1]$

$$h_\lambda(t) = \frac{1}{\lambda + t}$$

Clearly $h_\lambda(t) > 0$ for $t \in [-1,1]$. Let $\lambda_1 > \lambda_2 > 1$. Then for $t \in [-1,1]$

$$h_{\lambda_1}(t) - h_{\lambda_2}(t) = \frac{1}{\lambda_1 + t} - \frac{1}{\lambda_2 + t} = \frac{\lambda_2 - \lambda_1}{(\lambda_1 + t)(\lambda_2 + t)} < 0.$$

Hence as $\lambda \to 1$, $\lambda > 1$, h_λ monotonically increases to h_1. Assume that

$$\int_{[-1,1]} \frac{1}{1+t} d\nu(t) = \int_{[-1,1]} h_1(t) d\nu(t)$$

exists. Then by Lebesgue's monotone convergence theorem

$$\lim_{\substack{\lambda>1\\ \lambda\to 1}} C_d(\lambda I + A_d)^{-1}B_d = \lim_{\substack{\lambda>1\\ \lambda\to 1}} \int_{[-1,1]} h_\lambda(t) d\nu(t) = \int_{[-1,1]} h_1(t) d\nu(t)$$

and the system is admissible.

Now assume that the system is admissible, i.e. $\lim_{\substack{\lambda>1\\ \lambda\to 1}} C_d(\lambda I + A_d)^{-1}B_d$ exists and is finite, then by Fatou's Lemma

$$0 \leq \int_{[-1,1]} \frac{1}{1+t} d\nu(t) = \int_{[-1,1]} h_1(t) d\nu(t) = \int_{[-1,1]} \liminf_{\substack{\lambda>1\\ \lambda\to 1}} h_\lambda(t) d\nu(t)$$

$$\leq \liminf_{\substack{\lambda>1\\ \lambda\to 1}} \int_{[-1,1]} h_\lambda(t) d\nu(t) = \lim_{\substack{\lambda>1\\ \lambda\to 1}} C_d(\lambda I + A_d)^{-1}B_d.$$

Therefore $\int_{[-1,1]} \frac{1}{1+t} d\nu(t)$ exists. This completes the proof. □

In the context of the boundedness condition of Theorem 2.1 the following result is of interest for admissible discrete time systems.

Lemma 3.3 *Let ν be a positive finite Borel measure on $[-1,1]$ such that $\nu(\{-1\}) = 0$ and such that* $\int_{[-1,1]} \frac{1}{1+t} d\nu(t)$ *exists. Then*

$$\nu([-1,-\alpha]) = O(1-\alpha)$$

as $\alpha \to 1$.

Proof: We have that

$$\int_{[-1,1]} \frac{1}{1+t} d\nu(t) \geq \int_{[-1,-\alpha]} \frac{1}{1+t} d\nu(t) \geq \frac{1}{1-\alpha} \nu([-1,-\alpha])$$

which implies the claim. □

In the following Lemma many of the technical details are worked out that are necessary to translate the results on the transfer functions of discrete time completely symmetric systems to the continuous time case.

Lemma 3.4 *Let ν be a finite positive Borel measure on $[-1,1]$, such that $\nu(\{-1,1\}) = 0$. Let*

$$\rho : [-1,1] \to [-\infty, 0]; \quad t \mapsto \frac{t-1}{t+1},$$

where we take $\rho(-1) = -\infty$. Then

$$\mu(A) := \int_A \frac{1}{2}(1-r)^2 d(\nu\rho^{-1})(r) = \int_{\rho^{-1}(A)} \frac{2}{(1+t)^2} d\nu(t)$$

for all Borel sets A in $[-\infty, 0]$, defines a, not necessarily finite, positive regular Borel measure on $[-\infty, 0]$, such that $\mu(\{-\infty, 0\}) = 0$. We therefore consider μ as a positive regular Borel measure on $]-\infty, 0]$.

Moreover,

1.
$$\nu(A) = \int_{\rho(A)} d(\nu\rho^{-1})(r) = \int_{\rho(A)} \frac{2}{(1-r)^2} d\mu(r),$$

for all Borel sets A in $[-1,1]$.

2. *for f a measurable function on $[-1,1]$, $\int_{[-1,1]} f(t)d\nu(t)$ exists if and only if $\int_{]\infty,0]}(f \circ \rho^{-1})(r)\frac{2}{(1-r)^2}d\mu(r)$ exists. If one of the integral exists, both integrals exist.*

   For g a measurable function on $]-\infty,0]$, $\int_{]-\infty,0]} g(r)d\mu(r)$ exists if and only if $\int_{[-1,1]}(g \circ \rho)(t)\frac{2}{(1+t)^2}d\nu(t)$ exists. If one of the integrals exists, both integrals are equal.

3. *The map $V : L^2([-1,1],\nu) \to L^2(]\infty,0],\mu)$ with*

$$(V(f))(r) = \left(\frac{\sqrt{2}}{1-r} f\left(\frac{1+r}{1-r}\right)\right), \qquad -\infty < r \leq 0,$$

is unitary with inverse $V^{-1} : L^2(]-\infty,0],\mu) \to L^2([-1,1],\nu)$, where

$$(V^{-1}(g))(t) = \left(\frac{\sqrt{2}}{1+t} g\left(\frac{t-1}{t+1}\right)\right), \qquad -1 < t \leq 1,$$

and $(V^{-1}(g))(-1)$ arbitrary.

$$supp(\mu) = \rho(supp(\nu) \setminus \{-1\}),$$
$$supp(\nu) \setminus \{-1\} = \rho^{-1}(supp(\mu)).$$

5. *μ is such that $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r)$ exists if and only if ν is such that $\int_{[-1,1]} \frac{1}{1+t} d\nu(t)$ exists. If one of the integrals exists then both are equal.*

6. *if μ is such that $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r) < \infty$ then $\mu([r,0]) = O(r)$ as $r \to 0-$ if and only if $\nu([-1,-\alpha]) + \nu([\alpha,1]) = O(1-\alpha)$ as $\alpha \to 1$.*

7. *if μ is such that $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r)$ exists then*
$$G_c(s) = \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r)$$
is an analytic function on $C \setminus supp(\mu) = C \setminus \rho^{-1}(supp(\nu) \setminus \{-1\})$.
Moreover, if
$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + \int_{[-1,1]} \frac{1}{1+t} d\nu(t)$$
for $z \in C \setminus supp(\nu)$ then
$$G_c(s) = G_d\left(\frac{1+s}{1-s}\right),$$
for $s \in C \setminus supp(\mu)$. Also
$$\lim_{\substack{s \to +\infty \\ s \in R}} \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) = 0.$$

Proof: Let $\tilde{\mu}$ be the finite positive regular Borel measure on $[-\infty, 0]$ defined by
$$\tilde{\mu}(A) := (\nu\rho^{-1})(A) := \nu(\rho^{-1}(A))$$
for each Borel set A in $[-\infty, 0]$. Note (see e.g. [2], Theorem 6.12, p. 213) that for each measurable function f on $[-1,1]$ we have that $\int_{[-1,1]} f(t) d\nu(t)$ exists if and only if $\int_{[-\infty,0]} (f \circ \rho^{-1})(r) d(\nu\rho^{-1})(r)$ exists. If one of the integrals exists both are equal. For the measure μ defined by
$$\mu(A) := \int_A \frac{1}{2}(1-r)^2 d(\nu\rho^{-1})(r) = \int_A \frac{1}{2}(1-r)^2 d\tilde{\mu}(r) = \int_{\rho^{-1}(A)} \frac{2}{(1+t)^2} d\nu(t)$$
we therefore have
$$\nu(A) = \int_{\rho(A)} d(\nu\rho^{-1})(r) = \int_{\rho(A)} \frac{2}{(1-r)^2} d\mu(r)$$
for each Borel set A in $[-1,1]$. We have used that if $r = \frac{t-1}{t+1}$, for $t \in [-1,1]$, then $t = \frac{1+r}{1-r}$ and $\frac{2}{(1+t)^2} = \frac{1}{2}(1-r)^2$. We have that
$$\mu(\{-\infty, 0\}) = \int_{\nu(\{-1,1\})} \frac{2}{(1+t)^2} d\nu(t) = 0,$$

since $\nu(\{-1,1\})=0$. Hence we can consider ν as a positive Borel measure on $]-\infty,0]$. We have also shown 1.).
2.) Follows immediately from the proof of 1.).
3.) Follows immediately from 2.).
4.) This is verified easily.
5.) We have

$$\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)=\int_{]-\infty,0]}\frac{1}{2}(1-r)d(\nu\rho^{-1})(r)=\int_{[-1,1]}\frac{1}{2}(1-\frac{t-1}{t+1})d\nu(t)$$
$$=\int_{[-1,1]}\frac{1}{1+t}d\nu(t),$$

which implies the claim.
6.) Note that by Lemma 3.3 $\nu([-1,-\alpha])=O(1-\alpha)$ as $\alpha\to 1$ since $\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)=\int_{[-1,1]}\frac{1}{1+t}d\nu(t)$ exists. For $0<\alpha<1$, we have that $-1<\rho(\alpha)<0$. Then

$$\nu([\alpha,1])=\int_{\rho([\alpha,1])}\frac{2}{(1-r)^2}d\mu(r)=\int_{[\rho(\alpha),0]}\frac{2}{(1-r)^2}d\mu(r).$$

This identity implies that

$$\frac{1}{2}\mu([\rho(\alpha),0])\leq\frac{2}{(1-\rho(\alpha))^2}\mu([\rho(\alpha),0])\leq\int_{[\rho(\alpha),0]}\frac{2}{(1-r)^2}d\mu(r)$$
$$=\nu([\alpha,1])\leq\frac{2}{(1-0)^2}\mu([\rho(\alpha,0])=2\mu([\rho(\alpha),0]).$$

Since $0\leq-\rho(\alpha)\leq\frac{-2\rho(\alpha)}{1-\rho(\alpha)}=1-\alpha\leq-2\rho(\alpha)$ we therefore have

$$\frac{\mu([\rho(\alpha),0])}{-4\rho(\alpha)}=\frac{1}{2}\frac{\mu([\rho(\alpha),0])}{-2\rho(\alpha)}\leq\frac{\nu([\alpha,1])}{1-\alpha}\leq 2\mu([\rho(\alpha),0])\leq-\rho(\alpha).$$

As $\alpha\to 1$ if and only if $\rho(\alpha)\to 0-$, these inequalities imply that $\mu([r,0])=O(r)$ as $r\to 0-$ if and only if $\nu([\alpha,1[)=O(1-\alpha)$ as $\alpha\to 1$.
7.) Assume that the measure μ is such that $\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)$ exists. Then by 5.) $\int_{[-1,1]}\frac{1}{1+t}d\nu(t)$ exists. Hence by Lemma 2.1

$$G_d(z)=\int_{[-1,1]}\frac{1}{z-t}d\nu(t)+\int_{[-1,1]}\frac{1}{1+t}d\nu(t)$$

defines an analytic function on $C\setminus supp(\nu)$. For $s\in C\setminus supp(\mu)$ let

$$G_c(s)=G_d\left(\frac{1+s}{1-s}\right)=\int_{[-1,1]}\frac{1}{\frac{1+s}{1-s}-t}d\nu(t)+\int_{[-1,1]}\frac{1}{1+t}d\nu(t)$$
$$=\int_{[-1,1]}\frac{1}{s-\frac{t-1}{t+1}}\frac{2}{(1+t)^2}d\nu(t)$$
$$=\int_{]-\infty,0]}\frac{1}{s-r}\frac{1}{2}(1-r)^2d(\nu\rho^{-1})(r)$$

$$= \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r).$$

Note that for $s \in \Re$, $s > 0$, the function

$$f_s : \ ]-\infty,0] \to \Re; \quad r \mapsto \frac{1}{s-r}$$

is positive. Also for $s > 1$, $f_s \leq f_1$ and by assumption $f_1 = |f_1|$ is integrable. By Lebesgue's dominated convergence theorem we have as $\lim_{s\to\infty} f_s = 0$ that

$$\lim_{\substack{s\to\infty \\ s\in\Re}} \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) = \lim_{s\to\infty} \int_{]-\infty,0]} f_s(r) d\mu(r) = \int_{]-\infty,0]} \lim_{s\to\infty} f_s(r) d\mu(r)$$

$$= \int_{]-\infty,0]} 0 d\mu(r) = 0.$$

□

Remark 3.1 *In [11] a transform technique similar but not identical to the one in the previous Lemma was used to analyze unitarily equivalent Hankel operators.*

We now show that the transfer function of a completely symmetric continuous time system has an integral representation similar to discrete time completely symmetric systems.

Proposition 3.1 *Let (A_c, B_c, C_c, D_c) be a completely symmetric continuous time system with transfer function $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$, $s \in RHP$. Then there exists a unique positive regular Borel measure μ on $]-\infty, 0]$ such that*

$$G_c(s) = \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) + D_c, \qquad s \in RHP.$$

Moreover,

1. *G_c can be extended analytically to $C \setminus supp(\mu)$ where the extension is given by*

$$G_c(s) = \int_{]-\infty,0]} \frac{1}{s-t} d\mu(t) + D_c, \qquad s \in C \setminus supp(\mu).$$

2. *the integrals*

$$\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r)$$

and

$$\int_{]-\infty,0]} \frac{1}{(1-r)^2} d\mu(r)$$

exist.

Proof: Let $(A_d, B_d, C_d, D_d) = T^{-1}((A_c, B_c, C_c, D_c))$ be the corresponding discrete time admissible system. Since the bilinear transform preserves duality, the discrete time system is an admissible completely symmetric system. Let G_d be the transfer function of (A_d, B_d, C_d, D_d). By Proposition 2.1 there exists a unique positive finite Borel measure ν such that

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + D_d, \qquad z \in \mathcal{D}_e.$$

Let G_c be the transfer function of the continuous time system. Then by Theorem 3.1

$$G_c(s) = G_d\left(\frac{1+s}{1-s}\right), \qquad s \in RHP.$$

Let μ be the positive Borel measure on $]-\infty, 0]$ constructed in Lemma 3.4. Since the discrete time system is admissible we have by Lemma 3.4 that $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r) = \int_{[-1,1]} \frac{1}{1+t} d\nu(t)$ exist. Using Lemma 3.4 part 7, for $s \in RHP$,

$$G_c(s) = \int_{[-1,1]} \frac{1}{\frac{1+s}{1-s} - t} d\nu(t) + D_d = \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) + D_d - \int_{[-1,1]} \frac{1}{1+t} d\nu(t)$$

Since $\lim_{\substack{s\in\mathcal{R} \\ s\to\infty}} \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) = 0$ we have that $D_c = D_d - \int_{[-1,1]} \frac{1}{1+t} d\nu(t)$. Hence

$$G_c(s) = \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r) + D_c.$$

Also by Lemma 3.4 part 7 G_c is analytic on $\mathcal{C} \setminus supp(\mu)$. The uniqueness of μ follows from the fact that G_c and G_d are bilinearly related and that ν is unique. This shows 1.)
2.) That $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r)$ exists has already been established. To complete the proof note that

$$\int_{]-\infty,0]} \frac{1}{(1-r)^2} d\mu(r) = \int_{]-\infty,0]} \frac{1}{2} \frac{1}{(1-r)^2} (1-r)^2 d(\nu\rho^{-1})(r) = \int_{[-1,1]} \frac{1}{2} d\nu(t) < \infty.$$

□

As in the discrete-time case we refer to the measure μ as the *defining measure* of the continuous time completely symmetric system or its transfer function.

3.1 Stability

A continuous-time system (A_c, B_c, C_c, D_c) is *asymptotically stable* if $\lim_{t\to\infty} e^{tA_c}x = 0$ for all $x \in X$ and *exponentially stable* if there exists $0 \leq M < \infty$ and $\omega < 0$ such that $\|e^{tA_c}\| \leq Me^{\omega t}$ for all $t \geq 0$.

Proposition 3.2 *Let (A_c, B_c, C_c, D_c) be a completely symmetric continuous-time system. Then*

1. *(A_c, B_c, C_c, D_c) is asymptotically stable.*

2. (A_c, B_c, C_c, D_c) *is exponentially stable if and only if* $\sigma(A_c) \subseteq]-\infty, \beta]$ *for some* $\beta < 0$.

Proof: 1.) The asymptotic stability follows from the discrete-time result by applying the fact ([15]) that a semigroup is asymptotically stable if and only if the co-generator is also asymptotically stable.
2.) Let $(e^{tA_c})_{t\geq 0}$ be the semigroup of contractions with generator A_c. Let $A_c = \int_{]-\infty,0]} \lambda dE_c(\lambda)$ be the spectral decomposition of A_c. Then by the functional calculus for unbounded selfadjoint operators ([14]), for $t \geq 0$

$$e^{tA_c} = \int_{]-\infty,0]} e^{t\lambda} dE_c(\lambda).$$

It follows by ([8], Proposition A-III, 2.1) that the semigroup is exponentially stable if and only if $r(e^{A_c}) < 1$, where $r(T)$ is the spectral radius of the operator T. But by the spectral mapping theorem for selfadjoint operators

$$\sigma(e^{A_c}) = \overline{e^{\sigma(A_c)}}.$$

This implies that the semigroup is exponentially stable if and only if $\sigma(A_c) \subseteq]-\infty, \beta]$ for some $\beta < 0$. □

3.2 Observability and Reachability

The definition of observability and reachability of admissible continuous time systems is now given.

Definition 3.4 *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$*, then the operator*

$$\mathcal{O}_c : D(\mathcal{O}_c) \to L^2_Y([0,\infty[); x \mapsto (C_c e^{tA_c} x)_{t\geq 0}$$

is called the observability operator *of the system* (A_c, B_c, C_c, D_c)*, where* $D(\mathcal{O}_c) =$

$$\{x \in X \mid C_c e^{tA_c} x \text{ exists for almost all } t \in [0,\infty[\text{ and } (C_c e^{tA_c} x)_{t\geq 0} \in L^2_Y([0,\infty[)\}.$$

We say that (A_c, B_c, C_c, D_c) *has a bounded observability operator if* $D(A_c) \subseteq D(\mathcal{O}_c)$ *and* $\mathcal{O}_c$ *extends to a bounded operator on* X*. This extension will also be denoted by* $\mathcal{O}_c$.

If (A_c, B_c, C_c, D_c) *has a bounded observability operator* $\mathcal{O}_c$ *such that* $ker(\mathcal{O}_c) = \{0\}$*, then the system* (A_c, B_c, C_c, D_c) *is called* observable.

Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *be the dual of* (A_c, B_c, C_c, D_c)*. If the observability operator* $\tilde{\mathcal{O}}_c$ *of* $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *is a bounded operator on* X*, the adjoint of* $\tilde{\mathcal{O}}_c$ *is called the* reachability operator, *denoted* $\mathcal{R}_c$ *of* (A_c, B_c, C_c, D_c)*, i.e.* $\mathcal{R}_c = \tilde{\mathcal{O}}_c^*$*. If* $\mathcal{R}_c$ *exists and* $range(\mathcal{R}_c)$ *is dense in* X*, the system* (A_c, B_c, C_c, D_c) *is said be* reachable.

We need to define Hankel operators for continuous time transfer functions.

Definition 3.5 *If G_c is a $L(U,Y)$ valued function analytic on RHP, then the operator*

$$H_{G_c,RHP} : D(H_{G_c,RHP}) \to H^2_Y(RHP); f \mapsto P_+ M_{G_c} Rf$$

where

$$Rf(s) = f(-s)$$

$$M_{G_c} \quad \text{multiplication operator by } G_c$$

$$P_+ \quad \text{projection on } H^2_Y(RHP)$$

with $D(H_{G_c,RHP}) = \{f \in H^2_U(RHP) : \; f \text{ rational}, G_c Rf \text{ has non-tangential limit a.e. on } i\Re \text{ that is in } L^2_Y(i\Re)\}$ *is called the* Hankel operator $H_{G_c,RHP}$ *with symbol G_c. If $H_{G_c,RHP}$ extends to a bounded operator on $H^2_U(RHP)$, this extension is also called the Hankel operator $H_{G_c,RHP}$.*

If it is clear from the context that the Hankel operator is defined with respect to RHP we will drop the subscript RHP and write H_G instead of $H_{G,RHP}$.

Lemma 3.5 *Let (A_c, B_c, C_c, D_c) be a completely symmetric continuous time system and let G_c be its transfer function. If $\mathcal{O}_c$ is the observability operator and $\mathcal{R}_c$ the reachability operator, then $D(\mathcal{O}_c)$ is dense in X, $\mathcal{O}_c = \mathcal{R}_c^*$ and the following statements are equivalent,*

1. *the system has bounded reachability operator.*
2. *the system has bounded observability operator.*
3. *the Hankel operator $H_{G_c,RHP}$ is bounded.*

Moreover, the system is observable if and only if it is reachable.

Proof: The proof follows from the discrete time result and the fact (Theorem 7.7 in [10]) that under the bilinear transform the discrete time observability (reachability/Hankel) operator and the continuous time observability (reachability/Hankel) operator are unitarily equivalent. □

We can now characterize the boundedness of the observability/reachability operator of a continuous time completely symmetric system.

Theorem 3.3 *Let (A_c, B_c, C_c, D_c) be a completely symmetric continuous time system with defining measure μ. The system has bounded reachability/observability operator if and only if*

$$\mu([r,0]) = O(r)$$

as $r \to 0-$.

Proof: Let $(A_d, B_d, C_d, D_d) := T^{-1}((A_c, B_c, C_c, D_c))$ be the corresponding discrete time system with defining measure ν. By Proposition 3.1 $\int_{]-\infty,0]} \frac{1}{1-r} d\mu(r)$ exists. Hence by Lemma 3.4 part 6; $\nu([-1,-\alpha]) + \nu([\alpha,1]) = O(1-\alpha)$ as $\alpha \to 1$ if and only if $\mu([r,0]) = O(r)$ as $r \to 0-$. Since by Theorem 7.7 in [10] the observability operator of the continuous time and discrete time system are unitarily equivalent, the result now follows from the discrete time result (Theorem 2.1). □

We now establish a reachability/observability criterion for continuous time completely symmetric systems.

Corollary 3.1 *Let (A_c, B_c, C_c, D_c) be a completely symmetric continuous-time system with bounded reachability and observability operator. Let $A_c = \int_{-\infty}^{0} \lambda dE(\lambda)$ be the spectral decomposition of A_c, with spectral family E defined on the Borel σ-algebra on $]-\infty, 0]$. Then the system is observable/reachable if and only if*

$$\cap_{\omega \in \Omega} \{x \in D(A_c) \mid x \in ker(C_c E(\omega))\} = \{0\}.$$

Proof: Let $(A_d, B_d, C_d, D_d) := T^{-1}((A_c, B_c, C_c, D_c))$ be the corresponding discrete time system. The proof is based on the fact that (A_d, B_d, C_d, D_d) is observable if and only if (A_c, B_c, C_c, D_c) is observable ([10]). Also since A_d and A_c are related by a Cayley transformation, E is also the spectral family associated with the spectral decomposition of A_d. Note that by ([14], p.365)

$$E(\omega)(I + A_d)^{-1} \subseteq (I + A_d)^{-1} E(\omega)$$

for $\omega \in \Omega$. We first show that $\cap_{\omega \in \Omega} \{x \in D(A_c) \mid x \in ker(C_c E(\omega))\} \neq \{0\}$ implies that $\cap_{\omega \in \Omega} ker C_d E(\omega) \neq \{0\}$. Let $\omega \in \Omega$ and $x \in D(A_c) = D((I + A_d)^{-1})$ such that $C_c E(\omega) x = 0$. Let $y = (I + A_d)^{-1} x$. Then by the above

$$E(\omega)(I + A_d)^{-1} x = (I + A_d)^{-1} E(\omega) x$$

and therefore $E(\omega)x \in D((I + A_d)^{-1})$. Therefore

$$C_d E(\omega) y = C_d E(\omega)(I + A_d)^{-1} x = C_d (I + A_d)^{-1} E(\omega) x = \frac{1}{\sqrt{2}} C_c E(\omega) x = 0.$$

Hence $\cap_{\omega \in \Omega} ker C_d E(\omega) \neq \{0\}$. We now show the other implication. Let $\omega \in \Omega$ and $x \neq 0$ such that $C_d E(\omega) x = 0$. Set $y = (I + A_d)x$. Note that $y \neq 0$ since $-1 \notin \sigma_p(A_d)$. Then $E(\omega)x = E(\omega)(I + A_d)^{-1} y = (I + A_d)^{-1} E(\omega) y$ and therefore $E(\omega) y \in D((I + A_d)^{-1}) = D(A_c)$. Also

$$0 = C_d E(\omega) x = C_d E(\omega)(I + A_d)^{-1} y = C_d (I + A_d)^{-1} E(\omega) y = \frac{1}{\sqrt{2}} C_c E(\omega) y.$$

Hence $y \in \cap_{\omega \in \Omega} \{x \in D(A_c) \mid C_c E(\omega) x = 0\}$. □

3.3 Realization theory

To most efficiently study realization theory for continuous time completely symmetric systems we first apply the bilinear transform to the discrete time realization of Theorem 2.3

Lemma 3.6 *Let (A_d, B_d, C_d, D_d) be the discrete time realization of Theorem 2.3 of the transfer function*

$$G_d(z) = \int_{[-1,1]} \frac{1}{z-t} d\nu(t) + c, \qquad z \in C \setminus supp(\nu),$$

where ν is a positive finite Borel measure on $[-1,1]$ such that $\int_{[-1,1]} \frac{1}{1+t} d\nu(t) < \infty$ and c is a constant. Let

$$(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$$

be the corresponding continuous time system. Then the state space of the continuous time system is $X = L^2([-1,1],\nu)$. The operators of the continuous time system are given by

1.

$$A_c : D(A_c) \to X;$$

where

$$(A_c x)(t) = \begin{cases} \left(\frac{t-1}{t+1}\right) x(t), & -1 < t \leq 1 \\ 0, & t = -1 \end{cases}$$

and

$$D(A_c) = \{x \in L^2([-1,1],\nu) \mid \int_{[-1,1]} \frac{|x(t)|^2}{(1+t)^2} d\nu(t) < \infty\}.$$

2.

$$B_c : U \to (D(A_c))^{(')},$$

where for $u \in U$

$$B_c(u) : D(A_c) \to C; \quad x \mapsto \sqrt{2} \int_{[-1,1]} \frac{1}{1+t} \overline{x(t)} d\nu(t) u.$$

3. For $x \in D(A_c)$,

$$C_c x = \sqrt{2} \int_{[-1,1]} \frac{1}{1+t} x(t) d\nu(t).$$

If $x \in (I - A_c)^{-1} B_c u$, then $x = \chi_{[-1,1]} u$ for some $u \in U$, and

$$C_c x = \int_{[-1,1]} \frac{1}{1+t} d\nu(t) u.$$

4.

$$D_c = c - \int_{[-1,1]} \frac{1}{1+t} d\nu(t).$$

Proof: We use Theorem 2.3.
1.) We know that $D(A_c) = D((I+A_d)^{-1})$. But

$$D((I+A_d)^{-1}) = \{x \in L^2([-1,1],\nu) \mid \left(\frac{1}{1+t}x(t)\right)_{-1\le t\le 1} \in L^2([-1,1],\nu)\}.$$

For $x \in D(A_c)$

$$A_c x = (A_d - I)(I+A_d)^{-1}x = \left(\left(\frac{t-1}{t+1}\right)x(t)\right)_{-1\le t\le 1}.$$

2.) Since A_c is self adjoint, $(D(A_c^*))^{(')} = (D(A_c))^{(')}$. For $u \in U$, $x \in D(A_c)$

$$B_c(u)[x] = \sqrt{2} < B_d u, (I+A_d^*)^{-1}x >= \sqrt{2}\int_{[-1,1]} \chi_{[-1,1]}(t)u\frac{1}{1+t}\overline{x(t)}d\nu(t)$$

$$= \sqrt{2}\int_{[-1,1]} \frac{1}{1+t}\overline{x(t)}d\nu(t)u.$$

3.) For $x \in D(A_c)$,

$$C_c x = \sqrt{2}C_d(I+A_d)^{-1}x.$$

For $y \in Y$,

$$< C_c x, y >_Y=< \sqrt{2}C_d(I+A_d)^{-1}x, y >_Y= \sqrt{2} < (I+A_d)^{-1}x, C_d^* y >_Y$$

$$= \sqrt{2}\int_{[-1,1]} \frac{1}{1+t}x(t)\chi_{[-1,1]}(t)\bar{y}d\nu(t)$$

$$= \sqrt{2}\int_{[-1,1]} \frac{1}{1+t}x(t)d\nu(t)\bar{y}.$$

Hence

$$C_c x = \sqrt{2}\int_{[-1,1]} \frac{1}{1+t}x(t)d\nu(t).$$

If $x \in (I - A_c)^{-1}B_c U$, then $x = \frac{1}{\sqrt{2}}B_d u = \frac{1}{\sqrt{2}}\chi_{[-1,1]}u$ for some $u \in U$ (see p.448 in [10]). Hence

$$C_c x = \frac{1}{\sqrt{2}}C_c B_d u = \lim_{\substack{\lambda\to 1\\ \lambda>1}} C_d(\lambda I + A_d)^{-1}B_d u = \int_{[-1,1]} \frac{1}{1+t}d\nu(t)u$$

by the admissibility of (A_d, B_d, C_d, D_d) and Lemma 3.2.
4.) The expression for D_c follows since

$$D_c = D_d - \lim_{\substack{\lambda\to 1\\ \lambda>1}} C_d(\lambda I + A_d)^{-1}B_d = c - \int_{[-1,1]} \frac{1}{1+t}d\nu(t).$$

□

We are now in a position to prove the realization theorem for continuous time completely symmetric systems.

Theorem 3.4 *Let μ be a positive regular Borel measure on $]-\infty,0]$ such that $\mu(\{0\})=0$ and $\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)<\infty$ and let*

$$G_c(s)=\int_{]-\infty,0]}\frac{1}{s-r}d\mu(r)$$

for $s\in C\setminus supp(\mu)$. Let $X=L^2(]-\infty,0],\mu)$. Define

1. $$A_c: D(A_c)\to X;\quad x\mapsto (rx(r))_{-\infty<r\leq 0},$$
   with $D(A_c)=\{x\in L^2(]-\infty,0],\mu)\mid \int_{]-\infty,0]}|x(r)r|^2d\mu(r)<\infty\}$.

2. $$B_c: U\to D(A_c)^{(')};$$
   $$u\mapsto\left(x\mapsto\int_{]-\infty,0]}\overline{x(r)}d\mu(r)u\right).$$

3. *For $x\in D(A_c)$,*
   $$C_cx=\int_{]-\infty,0]}x(r)d\mu(r).$$
   For $x\in(I-A_c)^{-1}B_cU$ we have $x(r)=\frac{\sqrt{2}}{1-r}\chi_{]-\infty,0]}(r)u$, $\infty<r\leq 0$, for some $u\in U$, then set
   $$C_cx=\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)u.$$

4. $D_c=0$.

Then

1. *the system (A_c,B_c,C_c,D_c) is an admissible completely symmetric system with transfer function G_c.*
2. *the system (A_c,B_c,C_c,D_c) has bounded reachability and observability operator if and only if $\mu([r,0])=O(r)$ as $r\to 0-$.*
3. *if the system (A_c,B_c,C_c,D_c) has bounded reachability and observability operator then the system is reachable and observable.*

Proof: Note that by Lemma 3.4 G_c is an analytic function on $C\setminus supp(\mu)$ such that $\lim_{\substack{s\to\infty\\ s\in\Re}}G_c(s)=0$. Let ν be the measure on $[-1,1]$ as in Lemma 3.4 then by Lemma 3.4part 4

$$G_c(s)=G_d\left(\frac{1+s}{1-s}\right)$$

for $s\in C\setminus supp(\mu)$, where

$$G_d(z)=\int_{[-1,1]}\frac{1}{z-t}d\nu(t)+\int_{[-1,1]}\frac{1}{1+t}d\nu(t),$$

$z \in C \setminus supp(\nu)$. Let (A_d, B_d, C_d, D_d) be the realization of G_d given in Theorem 2.3. Then by Lemma 3.6 and Theorem 3.1

$$(A_c', B_c', C_c', D_c') := T((A_d, B_d, C_d, D_d))$$

is an admissible completely symmetric continuous time realization of G_c with state space $L^2([-1,1],\nu)$. To obtain a realization with state space $L^2(]-\infty,0],\mu)$ we perform a state space transformation with the unitary operator V of Lemma 3.4part 3. We will show that the resulting system (A_c, B_c, C_c, D_c) is as defined in the statement of the Theorem.

By Lemma 3.6

$$D(A_c') = \{x \in L^2([-1,1],\nu) \mid \int_{[-1,1]} \frac{|x(t)|^2}{(1+t)^2} d\nu(t) < \infty\}.$$

Let $x \in D(A_c')$ then by Lemma 3.4

$$\int_{[-1,1]} \frac{|x(t)|^2}{(1+t)^2} d\nu(t) = \int_{]-\infty,0]} \frac{\left|x\left(\frac{1+r}{1-r}\right)\right|}{\left(1+\frac{1+r}{1-r}\right)^2} \frac{2}{(1-r)^2} d\mu(r)$$

$$= \int_{]-\infty,0]} \left|\frac{\sqrt{2}}{1-r} x\left(\frac{1+r}{1-r}\right)\right|^2 \frac{(1-r)^2}{2} d\mu(r) = \int_{]-\infty,0]} |(V(x))(r)|^2 \frac{(1-r)^2}{2} d\mu(r).$$

Hence

$$V(D(A_c')) = \{g \in L^2(]-\infty,0],\mu) \mid \int_{]-\infty,0]} \left|g(r)\frac{(1-r)^2}{\sqrt{2}}\right|^2 d\mu(r) < \infty\}$$
$$= \{g \in L^2(]-\infty,0],\mu) \mid \int_{]-\infty,0]} |g(r)r|^2 d\mu(r) < \infty\}.$$

To determine A_c let $g \in V(D(A_c))$. Then

$$(V^{-1}g)(t) = \frac{\sqrt{2}}{1+t} g\left(\frac{t-1}{t+1}\right)$$

and

$$(A_c V^{-1} g)(t) = \sqrt{2}\frac{t-1}{(t+1)^2} g\left(\frac{t-1}{t+1}\right),$$

$-1 < t \leq 1$. Hence

$$(VA_cV^{-1}g)(r) = \frac{\sqrt{2}}{1-r}\sqrt{2}\frac{\frac{1+r}{1-r}-1}{\left(\frac{1+r}{1-r}+1\right)^2} g\left(\frac{\frac{1+r}{1-r}-1}{\frac{1+r}{1-r}+1}\right) = rg(r)$$

for $-\infty < r \leq 0$.

To determine B_c let $u \in U$, $x \in D(A_c')$, then

$$B_c'(u)[x] = \sqrt{2}\int_{[-1,1]} \frac{1}{1+t}\overline{x(t)} d\nu(t) u = \sqrt{2}\int_{]-\infty,0]} \frac{1}{1+\frac{1+r}{1-r}} \overline{x\left(\frac{1+r}{1-r}\right)} \frac{2}{(1-r)^2} d\mu(r) u$$

$$= \int_{]-\infty,0]} \overline{\frac{\sqrt{2}}{1-r} x\left(\frac{1+r}{1-r}\right)} d\mu(r) u$$
$$= \int_{]-\infty,0]} \overline{(Vx)(r)} d\mu(r) u.$$

Hence for $g \in D(A_c)$, $u \in U$,

$$B_c(u)[g] = \int_{]-\infty,0]} \overline{g(r)} d\mu(r) u.$$

To determine C_c recall that for $x \in D(A_c')$

$$C_c' x = \sqrt{2} \int_{[-1,1]} \frac{1}{1+t} x(t) d\nu(t) = \sqrt{2} \int_{]-\infty,0]} \frac{1}{1+\frac{1+r}{1-r}} x\left(\frac{1+r}{1-r}\right) \frac{2}{(1-r)^2} d\mu(r)$$
$$= \int_{]-\infty,0]} \frac{\sqrt{2}}{1-r} x\left(\frac{1+r}{1-r}\right) d\mu(r)$$
$$= \int_{]-\infty,0]} (Vx)(r) d\mu(r).$$

Hence for $g \in D(A_c)$,

$$C_c g = \int_{]-\infty,0]} g(r) d\mu(r).$$

If $x \in (I - A_c')^{-1} B_c' U$, then $x = \chi_{]-\infty,0]} u$ for some $u \in U$. Therefore

$$C_c' x = \int_{[-1,1]} \frac{1}{1+t} d\nu(t) u = \int_{]-\infty,0]} \frac{1}{1+\frac{1+r}{1-r}} \frac{2}{(1-r)^2} d\mu(r) u$$
$$= \int_{]-\infty,0]} \frac{1}{1-r} d\mu(r) u = \int_{]-\infty,0]} \frac{1}{\sqrt{2}} (Vx)(r) d\mu(r)$$
$$= C_c V(x).$$

Since by Lemma 3.4

$$\lim_{\substack{s \to \infty \\ s \in \mathbb{R}}} G_c(s) = 0,$$

we have that $D_c = 0$.

Since (A_c', B_c', C_c', D_c') is a completely symmetric admissible realization of G_c. Since V is unitary (A_c, B_c, C_c, D_c) is a completely symmetric admissible system whose transfer function is G_c. This shows 1.)

2.) Follows from Theorem 3.3.

3.) The bilinear transform and unitary map V preserve the boundedness of the observability and reachability operators. They also preserve observability and reachability of the system ([10]). Since under the assumption (A_d, B_d, C_d, D_d) is reachable and observable this implies the reachability and observability of (A_c, B_c, C_c, D_c). □

An admissible observable and reachable continuous time system (A_c, B_c, C_c, D_c) is called par-balanced if $\mathcal{O}_c^* \mathcal{O}_c = \mathcal{R}_c \mathcal{R}_c^*$, where $\mathcal{O}_c$ is the observability operator and $\mathcal{R}_c$ is the reachability operator. As in the discrete time case the duality properties of a completely symmetric observable/reachable system imply that such a system is par-balanced. We have the following result on the uniqueness of completely symmetric or equivalently par-balanced realizations.

Lemma 3.7 *let G_c be the transfer function of a completely symmetric system. Then*

1. *(A_c, B_c, C_c, D_c) is an observable/reachable completely symmetric realization of G_c if and only if (A_c, B_c, C_c, D_c) is a par-balanced realization of G_c.*

2. *if (A_c, B_c, C_c, D_c) is an observable/reachable completely symmetric (par-balanced) realization of G_c, then all other observable/reachable completely symmetric (par-balanced) realization are given by $(UA_cU^*, UB_c, C_cU^*, D_c)$, where U is a unitary operator.*

Proof: The proof follows from Lemma 2.5 and Lemma 3.1 since the bilinear transform preserves complete symmetry, par-balancedness and unitary equivalence of systems ([10]). □

In the following corollary the boundedness of the input and output operators is investigated.

Corollary 3.2 *Let (A_c, B_c, C_c, D_c) be a completely symmetric reachable/observable continuous time system with defining measure μ. Then*

1. *the input operator $B_c : U \to X$ is bounded if and only if $C_c : X \to Y$ is bounded.*

2. *the input operator B_c /the output operator C_c is bounded if and only if μ is a finite measure.*

3. *if (A_c, B_c, C_c, D_c) is the realization given in Theorem 3.4 and if B_c is bounded then B_c can be represented as*
$$B_c : U \to X; \quad u \mapsto \chi_{]-\infty,0]}u.$$

Proof: By the previous Lemma we can assume that the system (A_c, B_c, C_c, D_c) is the realization given in Theorem 3.4.
1.) The statement is a consequence of the duality between B_c and C_c.
2.) Assume that μ is finite then using the Cauchy-Schwarz inequality, for $x \in D(A_c)$,

$$|C_cx| = |\int_{]-\infty,0]} x(t)d\mu(t)| \le \left(\int_{]-\infty,0]} d\mu(t)\right)^{1/2} \|x\|_{L^2(]-\infty,0],\mu)}.$$

Hence C_c is bounded on $(D(A_c), \|\cdot\|_X)$. Since $D(A_c)$ is dense in $(X, \|\cdot\|_X)$, C_c can be extended to a bounded operator on X.

Assume that C_c extends to a bounded operator on $(X, \|\cdot\|_X)$ but μ is not finite. Then there exists $x \in X = L^2(]-\infty,0],\mu)$ such that $x \notin L^1(]-\infty,0],\mu)$. The integral representation of C_c implies that C_c is not bounded.
3.) If B_c acts as a bounded operator then for $u \in U$, $x \in D(A_c)$,

$$B_c(u)[x] = \int_{]-\infty,0]} \overline{x(t)}d\mu(r)u = < \chi_{]-\infty,0]}u, x >$$

which implies the claim. □

We can now establish the spectral minimality of observable/reachable completely symmetric systems.

Corollary 3.3 *Let μ be a positive regular Borel measure on* $]-\infty,0]$ *such that* $\int_{]-\infty,0]}\frac{1}{1-r}d\mu(r)$ *exists and* $\mu([r,0]) = O(r)$ *as* $r \to 0-$. *Let*

$$G_c(s) = \int_{]-\infty,0]} \frac{1}{s-r} d\mu(r)$$

for $s \in C \setminus supp(\mu)$. *If* (A_c, B_c, C_c, D_c) *is a par-balanced respectively observable/reachable completely symmetric realization of* G_c, *then*

$$\sigma(G_c) = \sigma(A_c) = supp(\mu),$$

where $\sigma(G_c)$ *denotes the set of singularities of* G_c. *Moreover, the spectrum of* A_c *has only simple multiplicity.*

Proof: The follows from the discrete time result, the spectral mapping theorem for selfadjoint operators, Lemma 3.4 part 4 and Lemma 3.4 part 7. □

The stability properties of completely symmetric systems are now considered again.

Corollary 3.4 *Let* (A_c, B_c, C_c, D_c) *be a completely symmetric observable/reachable continuous time system with defining measure* μ. *Then*

1. *the system is asymptotically stable.*
2. *the system is exponentially stable if and only if*

$$supp(\mu) \subseteq]-\infty, -\alpha]$$

   for some $\alpha > 0$.

3. *if the system is exponentially stable, the Hankel operator* $H_{G_c,RHP}$ *is compact, where* G_c *is the transfer function of the system.*

Proof: 1.) Follows from Proposition 3.2 .
2.) Follows from Proposition 3.2 and the spectral minimality of the realization (Corollary 3.3).
3.) Under this condition the transfer function is analytic and bounded in the right half plane and is continuous on the extended imaginary axis. The result therefore follows by Hartmann's theorem (see e.g. [11]). □

We now consider conditions for the boundedness of A_c.

Corollary 3.5 *Let* (A_c, B_c, C_c, D_c) *be a completely symmetric continuous time reachable/observable system with defining measure* μ. *Then*

1. A_c *is bounded if and only if*

$$supp(\mu) \subseteq [-\alpha, 0]$$

   for some $\alpha > 0$.

2. if A_c is bounded then B_c and C_c is bounded.

Proof: Since the system is observable and reachable we have by Corollary 3.3 that $\sigma(\mu) = \sigma(A_c)$. If A_c is bounded then $\sigma(A_c)$ is compact. 1.) now follows since a selfadjoint operator is bounded if and only if the spectrum is bounded.
2.) Follows from 1.) and Corollary 3.2) part2. □

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Center for Engineering Mathematics EC 35
University of Texas at Dallas
Richardson, TX 75083
USA
email: ober@utdallas.edu

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