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OVERLAPPING BLOCK-BALANCED CANONICAL FORMS AND PARAMETRIZATIONS: THE STABLE SISO CASE*

BERNARD HANZON† AND RAIMUND J. OBER‡

Abstract. The balanced canonical form and parametrization of Ober for the case of SISO stable systems are extended to block-balanced canonical forms and related input-normal forms and parametrizations. They form an overlapping atlas of parametrizations of the manifold of stable SISO systems of given order. This extends the usefulness of these parametrizations, e.g., in gradient algorithms for system identification. As an implication of our construction it follows that each of the subsets of the parametrization of [R. Ober, *Internat. J. Control*, 46 (1987), pp. 643–670] corresponding to a choice for the structural indices is in fact an imbedded submanifold of the manifold of stable SISO systems of fixed order.

Key words. linear dynamical systems, differentiable manifolds, stable systems, canonical forms, atlas, system identification

AMS subject classifications. 93XX, 53XX, 15XX

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1. Introduction.

In [18], [19] a canonical state-space form was presented for the set of asymptotically stable linear systems, with the property that it is balanced; i.e., for each system represented in canonical form, the corresponding observability and controllability Gramians are equal and diagonal (and positive definite). One motivation for studying balanced realizations and balanced canonical forms is their close relation to model reduction (see [19] and the references given there), which is in turn closely related to robust control theory (see, e.g., [20], [3]). Another motivation mentioned in [19] is the potential usefulness of balanced realizations for system identification, as indicated by [15]. In many cases, in system identification as well as in related areas, one can reduce the problem at hand to an optimization problem in which some criterion function is optimized over a set of systems. Very often one cannot solve the optimization problem analytically and has to use search algorithms (e.g., gradient algorithms), in which an initial point in the set of systems is adapted iteratively to give, ideally, a good approximation of the optimal system. In such search algorithms one often uses a parametrization of the set of relevant systems. The balanced parametrization of [19] has the advantage that by construction, problems of identifiability are to a large extent avoided in such a search algorithm. The parametrization has the property that it contains structural indices (i.e., discrete-valued parameters), and to each possible choice of values for these indices corresponds a particular subset of systems, for which a parametrization in terms of real-valued parameters is given. (In fact it will be shown in section 6 that these subsets are in fact submanifolds.) To each system corresponds a unique set of structural indices. Since the structural indices can take a large number of values, even for rather low order systems (the number of possibilities increases fast with increasing order of the system), this means that in a search algorithm one has either to identify the structural indices by other means

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†De Boelelaan 1105, 1081 HV Amsterdam, the Netherlands (bhanzon@econ.vu.nl).

‡Center for Engineering Mathematics, University of Texas at Dallas, Richardson, TX 75083-06688 (ober@utdallas.edu).

or to apply the search algorithm to a large number of parametrized submanifolds of systems. This is due to the fact that the parametrizations are disjoint.

Several authors (e.g. [4, 2, 10, 11, 20, 5, 6, 21, 22]) have investigated the possibility of using so-called overlapping parametrizations (in differential geometric terms: an atlas of coordinate charts). If one uses overlapping parametrizations, one does not have to search through each and every of the submanifolds but instead can search through the manifold as a whole, using the parametrizations to describe the manifold locally and changing from one parametrization to another when required. In case the search algorithm is of the gradient type, one can make sure that the decision rule for changing from one parametrization to another has little effect on the search algorithm by using a Riemannian gradient with respect to some suitable Riemannian metric on the manifold (cf. [7, 6, 8, 22, 9, 21]).

In view of this it would be very desirable if the balanced parametrization of [19] could be extended to give a set of overlapping parametrizations. In this paper such an extension, will be presented for the case of SISO stable systems. In the extension, balancedness of the realization no longer holds for all realizations. Instead block-balanced realizations and the corresponding input-normal realizations are used. A block-balanced canonical form is a canonical form for which the observability and controllability Gramians are equal and block-diagonal (and of course positive definite).

In section 2 some basic definitions are presented, including the concept of block-balanced realizations. In section 3 we present a Schwarz-like canonical form which will be a building block in the block-balanced canonical forms and the corresponding input-normal canonical forms that are treated in section 4. In section 5 it is shown how this leads to a set of overlapping block-balanced canonical forms and a corresponding atlas for the manifold of stable SISO input-output systems of a fixed order, and remarks are made as to how this atlas can be used if one wants to work with balanced and "almost balanced" realizations in search algorithms in system identification, for example. In section 6 the imbedded submanifolds structure of the original balanced parametrization is analyzed, using the atlas of the previous section.

2. Canonical forms, balanced realizations, and block-balanced realizations. In this section to a large extent the setup of [19] is followed. Let us consider continuous-time SISO systems of the form

$$\dot{x}_t = Ax_t + bu_t, \tag{1}$$

$$y_t = cx_t, \tag{2}$$

with $t \in \mathbf{R}, u_t \in \mathbf{R}, x_t \in \mathbf{R}^n, y_t \in \mathbf{R}, A \in \mathbf{R}^{n\times n}, b \in \mathbf{R}^{1\times n}, c \in \mathbf{R}^{n\times 1}$, and (A, b, c) a minimal triple.

For each $n \in \{1, 2, 3, \ldots\}$ let the set C_n be given by $C_n = \{(A, b, c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n} | (A, b, c)$ minimal and the spectrum of A is contained in the open left half plane$\}$.

As is well known, two minimal system representations (A_1, b_1, c_1) and (A_2, b_2, c_2) have the same transfer function, $g(s) = c_1(sI - A_1)^{-1}b_1 = c_2(sI - A_2)^{-1}b_2$, and therefore describe the same input-output behavior iff there exists an $n \times n$ matrix $T \in Gl_n(\mathbf{R})$ such that $A_1 = TA_2T^{-1}, b_1 = Tb_2, c_1 = c_2T^{-1}$. In that case we say that (A_1, b_1, c_1) and (A_2, b_2, c_2) are i/o-equivalent. This is clearly an equivalence relation; write $(A_1, b_1, c_1) \sim (A_2, b_2, c_2)$. A unique representation of a linear system can be obtained by deriving a canonical form.

DEFINITION 2.1. *A canonical form for an equivalence relation* $\sim$ *on a set* X *is a map*

$$\Gamma : X \to X$$

which satisfies, for all $x, y \in X$,

(i) $\Gamma(x) \sim x$;

(ii) $x \sim y \Longrightarrow \Gamma(x) = \Gamma(y)$.

Equivalently a canonical form can be given by the image set $\Gamma(X)$; *a subset* $B \subseteq X$ *describes a canonical form if for each* $x \in X$ *there is precisely one element* $b \in B$ *such that* $b \sim x$. *The mapping* $X \to B, x \mapsto b$ *then describes a canonical form.*

Let $(A, b, c) \in C_n$. The controllability Gramian W_c is the positive definite matrix that is given by the integral

$$W_c = \int_0^\infty \exp(At) bb^T \exp(A^T t) \mathrm{d}t.$$

As is well known, W_c can be obtained as the unique solution of the following Lyapunov equation:

$$AW_c + W_c A^T = -bb^T. \tag{3}$$

In a dual fashion, the observability Gramian W_o is the positive definite matrix that is given by the integral

$$W_o = \int_0^\infty \exp(A^T t) c^T c \exp(At) \mathrm{d}t.$$

This matrix is the unique solution of the following Lyapunov equation:

$$A^T W_o + W_o A = -c^T c. \tag{4}$$

DEFINITION 2.2. *Let* $(A, b, c) \in C_n$. *Then* (A, b, c) *is called* balanced *if the corresponding observability and controllability Gramians are equal and diagonal; i.e., there exist positive numbers* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *such that*

$$W_o = W_c = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) =: \Sigma. \tag{5}$$

The numbers $\sigma_1, \ldots, \sigma_n$ *are called the (Hankel) singular values of the system.*

The singular values are known to be uniquely determined by the input-output behavior of the system.

THEOREM 2.3 (see [17]). *Let* $(A, b, c) \in C_n$ *with*

$$\Sigma = \mathrm{diag}(\sigma_1 I_{n(1)}, \ldots, \sigma_k I_{n(k)}), \quad \sigma_1 > \sigma_2 > \ldots \sigma_k > 0, \quad \text{and} \quad \sum_{i=1}^{k} n(i) = n.$$

Then (A, b, c) *is unique within its i/o-equivalence class up to an orthogonal state-space transformation of the form*

$$Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_k)$$

with orthogonal $Q_i \in \mathbf{R}^{n(i) \times n(i)}, i = 1, \ldots, k$.

DEFINITION 2.4. *Let* $(A, b, c) \in C_n$. *Then* (A, b, c) *is called* input-normal *if* $W_c = I_n$ *and will be called* σ-input-normal *if* $W_c = \sigma I_n$.

Similarly (A, b, c) *is called* output-normal *if* $W_o = I_n$ *and* σ-output-normal *if* $W_o = \sigma I_n$.

It is not difficult to show that an input-normal realization is unique up to an arbitrary orthogonal state-space transformation.

The following definition is basic to our considerations in this paper.

DEFINITION 2.5. *Let* $(A, b, c) \in C_n$. *Then* (A, b, c) *will be called* block-balanced, *with indices* $n(i) \in \mathbf{N}, i = 1, \dots, k$, *adding up to* n, *if the observability Gramian and the controllability Gramian are equal and block-diagonal; i.e., there exist* $n(i) \times n(i)$ *positive definite matrices* $\Sigma_i, i = 1, \dots, k$, *such that*

$$W_o = W_c = \operatorname{diag}(\Sigma_1, \dots, \Sigma_k).$$

It will be convenient to call an arbitrary system representation $(A, b, c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$ *block-balanced if the pair of Lyapunov equations* $A\Sigma + \Sigma A^T = -bb^T$, $A^T\Sigma + \Sigma A = -c^Tc$ *has a positive definite solution of the form* $\Sigma = \operatorname{diag}(\Sigma_1, \dots, \Sigma_k)$ *(assuming neither asymptotic stability nor minimality).*

Remark. The matrices $\Sigma_i, i = 1, \dots, k$, are in general *not* uniquely determined by the input-output behavior of the system. However, the eigenvalues $\lambda_1(\Sigma_i) \geq \lambda_2(\Sigma_i) \geq \dots \geq \lambda_{n(i)}(\Sigma_i)$ of the matrices $\Sigma_i, i = 1, \dots, k$, together form the set of Hankel singular values of the system, which are uniquely determined by the input-output behavior of the system, as remarked before.

THEOREM 2.6. *Suppose that* $(A, b, c) \in C_n$ *is block-balanced with indices* $n(j) \in \mathbf{N}, j = 1, \dots, k, \sum_{j=1}^k n(j) = n$, *and the additional property* $\lambda_1(\Sigma_1) \geq \lambda_{n(1)}(\Sigma_1) > \lambda_1(\Sigma_2) \geq \lambda_{n(2)}(\Sigma_2) > \dots > \lambda_1(\Sigma_k) \geq \lambda_{n(k)}(\Sigma_k) > 0$.

This uniquely determines (A, b, c) *within its i/o-equivalence class up to an orthogonal state-space transformation of the form*

$$Q = \operatorname{diag}(Q_1, \dots, Q_k)$$

with orthogonal $Q_i \in \mathbf{R}^{n(i)\times n(i)}, i = 1, \dots, k$.

Proof. First note that if an *orthogonal* state-space transformation Q is applied to the system representation, then both Gramians transform in the same way, and therefore if they were equal before the orthogonal state-space transformation, then they will also be equal after the transformation.

Now consider two i/o-equivalent systems $(A_1, b_1, c_1), (A_2, b_2, c_2)$, which are both block-balanced with the same indices $n(j), j = 1, \dots, k$, and with Gramians $W_o^{(i)} = W_c^{(i)} = \operatorname{diag}(\Sigma_1^{(i)}, \dots, \Sigma_k^{(i)}), i = 1, 2$, with the property that $\lambda_1(\Sigma_1^{(i)}) \geq \lambda_{n(1)}(\Sigma_1^{(i)}) > \lambda_1(\Sigma_2^{(i)}) \geq \lambda_{n(2)}(\Sigma_2^{(i)}) > \dots > \lambda_1(\Sigma_k^{(i)}) \geq \lambda_{n(k)}(\Sigma_k^{(i)}) > 0, i = 1, 2$.

Because $\Sigma_j^{(i)}$ is symmetric positive definite for any $i = 1, 2, j = 1, \dots, k$, there exists an orthogonal matrix $Q_j^{(i)}$ such that $Q_j^{(i)}\Sigma_j^{(i)}(Q_j^{(i)})^T = \operatorname{diag}(\lambda_1(\Sigma_j^{(i)}), \lambda_2(\Sigma_j^{(i)}), \dots, \lambda_{n(j)}(\Sigma_j^{(i)}))$. Therefore, the state-space transformation $Q^{(i)} := \operatorname{diag}(Q_1^{(i)}, \dots, Q_k^{(i)})$ applied to the system representation (A_i, b_i, c_i) brings it into *balanced form with non-increasing singular values*, $i = 1, 2$. We can therefore apply Theorem 2.3 to the transformed system representations, and it follows that there exists an orthogonal state-space transformation of the form $Q = \operatorname{diag}(Q_1, \dots, Q_k)$ with $Q_i \in \mathbb{R}^{n(i)\times n(i)}, i = 1, 2, \dots, k$, that transforms (A_1, b_1, c_1) into (A_2, b_2, c_2) (and vice versa). □

The following theorem will be fundamental for our results.

THEOREM 2.7 (Pernebo and Silverman [24], Kabamba [12]). *Let* $(A,b,c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$ *be conformally partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & c_2 \end{pmatrix},$$

with $A_{ii} \in \mathbf{R}^{n(i)\times n(i)}, i = 1,2$, *and let* (A,b,c) *be block-balanced with indices* $n(1), n(2)$ *such that* $\Sigma_1, \Sigma_2 > 0$ *have no eigenvalues in common.*

Then $(A,b,c) \in C_n \Leftrightarrow (A_{ii}, b_i, c_i) \in C_{n(i)}, i = 1,2.$

3. The case k=1: A Schwarz-like canonical form for stable SISO systems in continuous time.

THEOREM 3.1. *Consider the set* B_n *of all* $(A,b,c) \in C_n$ *of the following form:*

$$A = \begin{pmatrix} a_{11} & -\alpha_1 & & 0 \\ \alpha_1 & 0 & \ddots & \\ & \ddots & \ddots & -\alpha_{n-1} \\ 0 & & \alpha_{n-1} & 0 \end{pmatrix}, \qquad a_{11} = -\frac{b_1^2}{2} < 0,$$

$$\alpha_i > 0, \qquad i = 1,\ldots,n-1,$$

$$b = \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b_1 > 0,$$

$$c = \begin{pmatrix} c_1 & \gamma_1 & \ldots & \gamma_{n-1} \end{pmatrix}, \qquad c_1 \in \mathbf{R},\ \gamma_j \in \mathbf{R},\ j = 1,\ldots,n-1.$$

Each triple $(A,b,c) \in B_n$ *is input-normal.*

Let S_n *be the set of values of the vector of parameters* $(b_1, \alpha_1, \ldots, \alpha_{n-1}, c_1, \gamma_1, \ldots, \gamma_{n-1})$ *such that the corresponding triple* $(A,b,c) \in B_n$, *i.e., such that* $b_1 > 0, \alpha_i > 0, i = 1,\ldots,n$, *and* $c_1, \gamma_1, \ldots, \gamma_{n-1}$ *such that the pair* (c,A) *is observable.*

The set B_n *describes a real analytic (hence continuous) canonical form, and the parametrization mapping* $S_n \longrightarrow B_n$, *which maps each parameter vector to the corresponding triple* (A,b,c), *is a real analytic diffeomorphism (hence a homeomorphism).*

If $(\gamma_1, \ldots, \gamma_{n-1}) \neq 0 \in \mathbf{R}^{n-1}, n \geq 2$, *then the system has several different singular values.*

Proof. The requirement that a realization is input-normal reduces the freedom of choosing a basis of the state space to the freedom of choosing an *orthonormal* basis, i.e., to the freedom of choosing an element from the orthogonal group.

Now consider the controllability matrix of a triple $(A,b,c) \in B_n$. It is easily seen to be positive upper triangular. According to [19] there is a unique element in the orthogonal group that transforms a controllability matrix to a positive upper triangular matrix. Therefore the form presented here is canonical indeed.

Next let us show the smoothness properties. The mapping $S_n \longrightarrow B_n$, which maps a parameter vector from S_n to its corresponding triple (A,b,c), is polynomial, hence real analytic.

Now consider the mapping $C_n \longrightarrow S_n$, which maps *any* triple $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n$ to the corresponding parameter vector describing the *canonical form* of the system. Clearly the coefficients of the characteristic polynomial of $\tilde{A}$ depend polynomially on $\tilde{A}$, and therefore the parameters $a_{11}, \alpha_1, \ldots, \alpha_{n-1}$ depend real analytically on $\tilde{A}$, as they are rational functions of these characteristic polynomial coefficients (cf. [18]).

It remains to show that the parameter vector $c = (c_1, \gamma_1, \dots, \gamma_{n-1})$ depends real analytically on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$. Let (A, b, c) denote the canonical form of the system and $g(z) := \frac{p(z)}{q(z)} := c(zI - A)^{-1}b = \tilde{c}(zI - \tilde{A})^{-1}\tilde{b}$ denote the (rational) transfer function of the system, with monic polynomial denominator $q(z) := \det(zI - A) = \det(zI - \tilde{A})$ and polynomial numerator $p(z)$. It is easy to see that the coefficients of $p(z)$ depend real analytically on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$. Let $M(z)$ denote the polynomial matrix of cofactors of $(zI - A)$. Then one has

$$p(z) = cM(z)^T b. \tag{6}$$

Consider $m_{1i}(z)$, which is $(-1)^{1+i}$ times the determinant of the matrix that is obtained from $zI - A$ by leaving out the first row and ith column, $i \in \{1, \dots, n\}$:

$$
\begin{aligned}
m_{1i}(z) &= (-1)^{1+i} \\
&\times \left|
\begin{array}{cccc|cccccc}
\alpha_1 & * & \dots & * & * & \dots & \dots & \dots & \dots & * \\
0 & \ddots & & \vdots & \vdots & & & & & \vdots \\
\vdots & & \ddots & * & \vdots & & & & & \vdots \\
0 & \dots & 0 & \alpha_{i-1} & * & \dots & \dots & \dots & \dots & * \\
\hline
0 & \dots & \dots & 0 & z & -\alpha_{i+1} & 0 & \dots & \dots & 0 \\
\vdots & & & \vdots & \alpha_{i+1} & z & -\alpha_{i+2} & 0 & \dots & 0 \\
\vdots & & & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & \vdots & \vdots & & \ddots & \ddots & \ddots & -\alpha_{n-1} \\
0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & \alpha_{n-1} & z
\end{array}
\right| \\
&= (-1)^{1+i} \times \left(\prod_{j=1}^{i-1} \alpha_j \right) z^{n-i} + \text{terms of lower degree in } z,
\end{aligned}
$$

where $i \in \{1, \dots, n\}$; if $i = 1$, the product $\prod_{j=1}^{i-1} \alpha_j$ is taken to be equal to one by convention. Because $\prod_{j=1}^{i-1} \alpha_j$ is unequal to zero (and in fact positive) for each $i \in \{1, \dots, n\}$ the polynomials $m_{11}(z), \dots, m_{1n}(z)$ form a basis of the linear vector space of polynomials of degree $< n$ over $\mathbf{R}$. Therefore (6), which can be rewritten as

$$c_1 m_{11}(z) + \gamma_1 m_{12}(z) + \cdots + \gamma_{n-1} m_{1n}(z) = \frac{p(z)}{b_1}, \tag{7}$$

has a unique solution $c = (c_1, \gamma_1, \gamma_2, \dots, \gamma_{n-1})$, which depends real analytically on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$ and the parameters $b_1, \alpha_1, \dots, \alpha_{n-1}$. Since these parameters themselves depend real analytically on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$, the real analyticity of all parameters on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$ follows. This completes the proof of the smoothness properties.

The remaining statements follow from the fact that for $\gamma = 0$, the form is a canonical form for systems with only *one* positive Hankel singular value (i.e., all nonzero Hankel singular values coincide); cf. [19], [18]. □

Remarks. (i) The fact that *if* the asymptotically stable matrix A can be brought into the presented form by a basis change of the state space, then the resulting matrix is unique, also follows from the fact mentioned in the proof that for $\gamma = 0, c_1 \neq 0$ the form is a canonical form for systems with only one positive Hankel singular value; cf. [19], [18]. Note that here we use a different sign convention for the off-diagonal elements of the matrix A than in those papers. This corresponds to consideration of the dual state-space representation.

(ii) If $c_1 \neq 0$, we define $\sigma := |\frac{c_1}{b_1}| > 0$, which we will call a pseudosingular value. If the vector $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ is close enough to zero, the pseudosingular value will be close to the true singular values of the system, because of continuity of the singular values as a function of γ and the fact that if $\gamma = 0$, the system has only one singular value and its value is σ. If $c_1 \neq 0$, the system can be brought simply into σ-input-normal form by multiplying c by $\sigma^{-\frac{1}{2}}$ and b by $\sigma^{\frac{1}{2}}$. The resulting σ-input-normal form is a *canonical* form locally around $\gamma = 0$, but not globally because the systems which have $c_1 = 0$ in the previous canonical form cannot be represented in this way. (It would lead to $\sigma = 0$, and therefore one cannot transform back to the input-normal case, etc.) Locally around $\gamma = 0$ it takes the following form:

$$A = \begin{pmatrix} a_{11} & -\alpha_1 & & 0 \\ \alpha_1 & 0 & \ddots & \\ & \ddots & \ddots & -\alpha_{n-1} \\ 0 & & \alpha_{n-1} & 0 \end{pmatrix},$$

$$a_{11} = -\frac{b_1^2}{2\sigma} < 0,$$

$$\alpha_i > 0, \qquad i = 1, \ldots, n-1,$$

$$b = \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b_1 > 0$$

$$c = \begin{pmatrix} sb_1 & \gamma_1 & \ldots & \gamma_{n-1} \end{pmatrix}, \qquad s \in \{-1, 1\},\ \gamma_j \in \mathbf{R},\ j = 1, \ldots, n-1.$$

(iii) Because the canonical form is input-normal, if one starts with an arbitrary input-normal realization $(\tilde{A}, \tilde{b}, \tilde{c})$ of the system, it takes an orthogonal state-space transformation Q in order to obtain the canonical form of the system involved. The same holds for the (local) σ-input-normal canonical form.

(iv) Clearly the canonical forms presented are controllable (because they are input-normal; resp., σ-input-normal), but observability will fail for certain choices of c; the observability Gramian will be singular for such a choice of c. If $\gamma = 0, c_1 \neq 0$, the system is observable, because the observability Gramian will be $\sigma^2 I$ (resp., σI). (In that case the system representation is σ^2-output-normal; resp., σ-output-normal.) Therefore, also in some open neighborhood around such a system, observability will still hold. (This follows from the continuity of the determinant of the observability Gramian as a function of the parameters.)

(v) This canonical form is closely related to the so-called Schwarz canonical form; cf. [13], [14], [25].

(vi) A canonical form can be interpreted as a choice of basis of the state space for each system. In this case the basis can be obtained as follows. Define an inner product on the state space by the inverse of the reachability Gramian. Take the first n columns of the reachability matrix, and apply the Gram–Schmidt orthogonalization procedure to it, with respect to the inner product. With respect to the resulting set of n vectors as the basis of the state space the system has the canonical form. This observation can in fact be used to obtain an alternative proof of the smoothness properties stated in the theorem.

4. An input-normal and a block-balanced canonical form. Let $n(1), \ldots, n(k) \in \{1, 2, \ldots, n\}, \sum_{j=1}^{k} n(j) = n$, denote a partition of n as before. Let $C_{n(1),n(2),\ldots,n(k)}$ denote the subset of all systems in C_n, with the property that their n Hankel singular values (multiplicities *included*) $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(n) > 0$ can be partitioned into k disjoint sets of singular values (again with multiplicities included) in the following way:

$$\begin{aligned}
&\sigma(1) \geq \cdots \geq \sigma(n(1)) > \sigma(n(1)+1) \\
&\quad \geq \cdots \geq \sigma(n(1)+n(2)) > \sigma(n(1)+n(2)+1) \\
&\quad \geq \cdots \geq \sigma\left(\sum_{j=1}^{l} n(j)\right) > \sigma\left(\left(\sum_{j=1}^{l} n(j)\right)+1\right) \\
&\quad \geq \cdots > 0. \qquad (8)
\end{aligned}$$

So we require that $\sigma(\sum_{j=1}^{l} n(j)) > \sigma((\sum_{j=1}^{l} n(j)) + 1)$ for $l = 1, 2, \ldots, k-1$ and $\sigma(n) > 0$, of course. Note that the notation is consistent with the fact that C_n denotes the set of stable systems which have as their only "restriction" that there are n positive singular values (multiplicities included), i.e., that the order of the system is n.

The other extreme is $C_{1,1,\ldots,1}$, which denotes the set of nth-order stable systems with n *distinct* singular values. For this set of systems a balanced canonical form was derived in [12].

Remark. The set $C_{n(1),\ldots,n(k)}$ should not be confused with the subset of C_n consisting of the systems which have k distinct singular values $\sigma_1 > \cdots > \sigma_k > 0$ with multiplicities $n(1), \ldots, n(k)$. Of course these systems are included in $C_{n(1),\ldots,n(k)}$, but they generally form only a (thin) subset.

Next we will present a canonical form on $C_{n(1),\ldots,n(k)}$.

THEOREM 4.1. *Consider the set $B_{n(1),\ldots,n(k)}$ of triples (A, b, c) of the following form:*

$$A = (A(i,j))_{1 \leq i,j \leq k},$$

$$A(i,j) \in \mathbf{R}^{n(i) \times n(j)}, \qquad i, j \in \{1, \ldots, k\},$$

$$b = \begin{pmatrix} b(1) \\ b(2) \\ \vdots \\ b(k) \end{pmatrix}, \qquad b(i) \in \mathbf{R}^{n(i)},\ i = 1, \ldots, k,$$

$$c = (c(1), \ldots, c(k)), \qquad c(j)^T \in \mathbf{R}^{n(j)},\ j = 1, \ldots, k,$$

$$A(i,i) = \begin{pmatrix} a(i,i)_{11} & -\alpha(i)_1 & 0 & \ldots & 0 \\ \alpha(i)_1 & 0 & -\alpha(i)_2 & \ddots & \vdots \\ 0 & \alpha(i)_2 & & \ddots & 0 \\ \vdots & \ddots & \ddots & & -\alpha(i)_{n(i)-1} \\ 0 & \ldots & 0 & \alpha(i)_{n(i)-1} & 0 \end{pmatrix},$$

$$a(i,i)_{11} = -\frac{b_i^2}{2},$$
$$\alpha(i)_j > 0, \qquad j = 1, \ldots, n(i) - 1,$$
$$b(i) = \begin{pmatrix} b_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b_i > 0,$$
$$c(i) = \left(c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}\right), \qquad i = 1, \ldots, k,$$

where the parameters are to be taken such that the corresponding observability Gramians $\Sigma_i^2, i = 1, \ldots, k$, *which satisfy the observability Lyapunov equations*

$$\Sigma_i^2 A(i,i) + A(i,i)^T \Sigma_i^2 = -c(i)^T c(i) \tag{9}$$

are fulfilling the following inequalities:

$$\lambda_1(\Sigma_1^2) \geq \lambda_{n(1)}(\Sigma_1^2) > \lambda_1(\Sigma_2^2) \geq \lambda_{n(2)}(\Sigma_2^2) > \cdots > \lambda_1(\Sigma_k^2) \geq \lambda_{n(k)}(\Sigma_k^2) > 0. \tag{10}$$

For each pair $(i,j), i \neq j$, *the matrices* $A(i,j), A(j,i)$ *are determined (uniquely!) from the following pair of linear matrix equations:*

$$\begin{aligned} A(i,j) + A(j,i)^T &= -b(i)b(j)^T, \\ \Sigma_i^2 A(i,j) + A(j,i)^T \Sigma_j^2 &= -c(i)^T c(j). \end{aligned} \tag{11}$$

The set $B_{n(1),\ldots,n(k)}$ *describes a real analytic (hence continuous) canonical form on* $C_{n(1),\ldots,n(k)}$. *The* $2n$ *"free" parameters of the canonical form are*

$$b_i, \alpha(i)_1, \ldots, \alpha(i)_{n(i)-1}, c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}, \qquad i = 1, \ldots, k.$$

Let $S_{n(1),\ldots,n(k)} \subset \mathbf{R}^{2n}$ *be the set of all values of the parameter vector for which the corresponding triple* $(A,b,c) \in B_{n(1),\ldots,n(k)}$, *i.e., for all* $i \in \{1, \ldots, k\}$ $b_i > 0$, $\alpha(i)_j > 0$, $j = 1, \ldots, n(i) - 1$, *and* $c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}$ *such that the matrices* $\Sigma_i, i = 1, \ldots, k$, *found in* (9) *satisfy the inequalities* (10). *The mapping* $S_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$ *which maps a parameter vector to the corresponding triple* (A,b,c) *is a real analytic diffeomorphism.*

The form is input-normal, i.e.,

$$A + A^T = -bb^T, \tag{12}$$

and has block-diagonal observability Gramian $\Sigma^2 := \operatorname{diag}(\Sigma_1^2, \ldots, \Sigma_k^2) > 0$.

Let $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(n) > 0$ *denote the* n *positive Hankel singular values of the system (with their multiplicities). If for some* $i \in \{1, \ldots, k\}$ *the vector* $\gamma(i) = 0$, *then* Σ_i^2 *is a scalar matrix*

$$\Sigma_i^2 = \sigma^2\left(1 + \sum_{j=1}^{i-1} n(j)\right) \times I_{n(i)}, \tag{13}$$

and

$$\sigma\left(\sum_{j=1}^{i-1} n(j)\right)$$
$$> \sigma\left(1+\sum_{j=1}^{i-1} n(j)\right) = \sigma\left(2+\sum_{j=1}^{i-1} n(j)\right) = \cdots = \sigma\left(\sum_{j=1}^{i} n(j)\right)$$
$$> \sigma\left(1+\sum_{j=1}^{i} n(j)\right).$$

The observability Gramian is diagonal if and only if for all $i \in \{1,\ldots,k\}, \gamma(i)=0$.

Remark. A block-balanced realization can be obtained from the presented canonical form by applying a state-space transformation

$$T := \Sigma^{\frac{1}{2}} = \operatorname{diag}\left(\Sigma_1^{\frac{1}{2}},\ldots,\Sigma_k^{\frac{1}{2}}\right) > 0. \tag{14}$$

The corresponding controllability and observability Gramians will both be equal to

$$\Sigma = \operatorname{diag}(\Sigma_1,\ldots,\Sigma_k) > 0.$$

Proof. (i) To start we will show that the form presented is canonical on $C_{n(1),\ldots,n(k)}$. Consider a system which can be represented by a triple in $C_{n(1),\ldots,n(k)}$. A balanced realization of the system is also in block-balanced form with partitioning indices $n(1),\ldots,n(k)$. So one can find a block-balanced realization (A,b,c) of the system with these partitioning indices. It follows from Theorem 2.6 that the requirement that (A,b,c) is block-balanced with these partitioning indices uniquely determines (A,b,c) up to an orthogonal state-space transformation of the form $Q = \operatorname{diag}(Q_1,Q_2,\ldots,Q_k)$, with orthogonal matrices $Q_i \in \mathbf{R}^{n(i)\times n(i)}$. If (A,b,c) is in block-balanced form, it can be brought into input-normal form with block-diagonal observability Gramian by the state-space transformation T^{-1}, where T is as defined in (14). It follows easily that if (A,b,c) is in input-normal form with block-diagonal controllability Gramian $\Sigma^2 = \operatorname{diag}(\Sigma_1^2,\ldots,\Sigma_k^2)$, with $\lambda_1(\Sigma_1^2) \geq \lambda_{n(1)}(\Sigma_1^2) > \lambda_1(\Sigma_2^2) \geq \lambda_{n(2)}(\Sigma_2^2) > \cdots > \lambda_1(\Sigma_k^2) \geq \lambda_{n(k)}(\Sigma_k^2) > 0$, $\Sigma_i^2 \in \mathbf{R}^{n(i)\times n(i)}$, then (A,b,c) is uniquely determined up to an orthogonal state-space transformation of the form $Q = \operatorname{diag}(Q_1,Q_2,\ldots,Q_k)$. If such a transformation is applied, then $(A(i,i),b(i),c(i))$ is transformed to $\left(Q_iA(i,i)Q_i^T, Q_ib(i), c(i)Q_i^T\right)$. Note that $(A(i,i),b(i),c(i)) \in C_{n(i)}$ because of Theorem 2.7, and therefore it follows from Theorem 3.1 that there is a unique choice for Q_i which brings $\left(Q_iA(i,i)Q_i^T, Q_ib(i), c(i)Q_i^T\right)$ into the required canonical form.

We need only to check that by using the solutions $A(i,j), A(j,i)$ of (11) the Gramians indeed have the required block structure, which is straightforward and left to the reader.

(ii) Second, we will show the smoothness properties. Clearly the mapping $S_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$, which maps any parameter vector in $S_{n(1),\ldots,n(k)}$ to the corresponding triple $(A,b,c) \in B_{n(1),\ldots,n(k)}$, is real analytic.

Now consider the mapping $C_{n(1),\ldots,n(k)} \longrightarrow S_{n(1),\ldots,n(k)}$, which maps a triple $(\tilde{A},\tilde{b},\tilde{c})$ to the parameter vector of the corresponding canonical form.

The map which assigns to $(\tilde{A},\tilde{b},\tilde{c})$ the coefficients of the characteristic polynomial of the product of the Gramians is real analytic. The zeroes of this polynomial are the squared singular values. Now consider the polynomial

$$a(z) = \prod_{j=n(1)+1}^{n} (z - \sigma(j)^2).$$

Because on $C_{n(1),\ldots,n(k)}$ the inequality $\sigma(n(1)) > \sigma(n(1)+1)$ holds, the coefficients of $a(z)$ depend real analytically on those of the characteristic polynomial of the product of the Gramians (see, e.g., [16]).

Let $\Sigma^2 = W_c^{\frac{1}{2}} W_o W_c^{\frac{1}{2}}$, where W_c and W_o are the controllability and observability Gramians, respectively, of $(\tilde{A}, \tilde{b}, \tilde{c})$; W_c and W_o depend real analytically on $(\tilde{A}, \tilde{b}, \tilde{c})$. The matrix $a(\Sigma^2)$ has as its range space an $n(1)$-dimensional linear subspace of $\mathbf{R}^n$ which clearly depends real analytically on $(\tilde{A}, \tilde{b}, \tilde{c})$. The corresponding orthogonal projection matrix Π_1, which maps an arbitrary vector $x \in \mathbf{R}^n$ to its orthogonal projection in the linear subspace spanned by the columns of $a(\Sigma^2)$ (i.e., the linear subspace which is obtained by taking the direct sum of the eigenspaces of the largest $n(1)$ eigenvalues $\sigma(1)^2, \ldots, \sigma(n(1))^2$ of Σ^2), depends real analytically on $a(\Sigma^2)$.

Now consider $(\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1, \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}, \tilde{c} W_c^{\frac{1}{2}} \Pi_1)$ with corresponding controllability Gramian Π_1 and observability Gramian $\Pi_1 \Sigma^2 \Pi_1 = \Pi_1 \Sigma^2 = \Sigma^2 \Pi_1$. (Because of the way Π_1 is constructed, it commutes with Σ^2.) We can now apply the canonical form of Theorem 3.1 to find a basis for the range space of Π_1 (which corresponds to the state space there) depending real analytically on $(\tilde{A}, \tilde{b}, \tilde{c})$. The first basis vector is

$$\frac{\Pi_1 W_c^{-\frac{1}{2}} \tilde{b}}{\|\Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\|};$$

the second one (Gram–Schmidt orthonormalization) is obtained by normalization of the vector

$$\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1 W_c^{-\frac{1}{2}} \tilde{b} - \frac{\left(\tilde{b}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{\frac{1}{2}} \tilde{A}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\right)}{\left(\tilde{b}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\right)} \times \Pi_1 W_c^{-\frac{1}{2}} \tilde{b};$$

and so on. Clearly this choice of basis of the range space of Π_1 is real analytic. With respect to the resulting basis of the $n(1)$-dimensional state space the triple

$$\left(\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1, \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}, \tilde{c} W_c^{\frac{1}{2}} \Pi_1\right)$$

takes the form $(\tilde{A}(1,1), \tilde{b}(1), \tilde{c}(1))$, as described in Theorem 3.1:

$$\tilde{A}(1,1) = \begin{pmatrix} a(1,1)_{11} & -\alpha(1)_1 & 0 & \cdots & 0 \\ \alpha(1)_1 & 0 & -\alpha(1)_2 & \ddots & \vdots \\ 0 & \alpha(1)_2 & & \ddots & 0 \\ \vdots & \ddots & \ddots & & -\alpha(1)_{n(1)-1} \\ 0 & \cdots & 0 & \alpha(1)_{n(1)-1} & 0 \end{pmatrix},$$

$$a(1,1)_{11} = -\frac{b_1^2}{2},$$
$$\alpha(1)_j > 0, \qquad j = 1, \ldots, n(1)-1,$$
$$\tilde{b}(1) = \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b_1 > 0,$$
$$\tilde{c}(1) = \left(c_1, \gamma(1)_1, \ldots, \gamma(1)_{n(1)-1}\right),$$

and therefore this triple and the parameters describing it depend real analytically on $(\tilde{A}, \tilde{b}, \tilde{c})$. Similarly for any $i \in \{1, \ldots, k\}$ the matrix triple and the parameters describing it depend real analytically on $(\tilde{A}, \tilde{b}, \tilde{c})$. This proves the real analyticity of the mapping which maps $(\tilde{A}, \tilde{b}, \tilde{c})$ to the parameters of the canonical form.

(iii) The remaining statements follow from the results in [19]. □

5. An atlas of overlapping block-balanced canonical forms.

THEOREM 5.1. *Let the state-space dimension n be fixed. The canonical forms $C_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$, $n(j) \in \{1, \ldots, n\}, j = 1, \ldots, k$, $\sum_{j=1}^k n(j) = n$, $k \in \{1, \ldots, n\}$, form an overlapping set of real analytic (hence continuous) canonical forms covering C_n. Each of the sets $C_{n(1),\ldots,n(k)}$, $\sum_{j=1}^k n(j) = n$, is an open subset of C_n, and together they cover C_n.*

Proof. Let $P(n;k) := \{(n(1), \ldots, n(k)) | n(j) \in \{1, \ldots, n\}; j = 1, \ldots, k; \sum_{j=1}^k n(j) = n\}$, the set of partitions of n into k parts. It is trivial to show that

$$\bigcup_{k=1}^{n} \bigcup_{(n(1),\ldots,n(k)) \in P(n;k)} C_{n(1),\ldots,n(k)} = C_n, \tag{15}$$

because $C_{n(1),\ldots,n(k)} \subset C_n$ for each partition $(n(1), \ldots, n(k))$ of n and for $k = 1$ one has $n(1) = n$ and $C_{n(1)} = C_n$. Clearly for each partition $(n(1), \ldots, n(k))$ of n the set $C_{n(1),\ldots,n(k)}$ is an open subset of C_n. The remaining properties follow from Theorem 4.1. □

COROLLARY 5.2. *The set of mappings*

$$\phi : C_{n(1),\ldots,n(k)} / \sim \longrightarrow S_{n(1),\ldots,n(k)} \subset \mathbf{R}^{2n},$$
$$(n(1), \ldots, n(k)) \in P(n;k), \qquad k = 1, \ldots, n,$$

which map each equivalence class of triples to the corresponding parameter vector in the canonical form, forms an atlas for the real analytic manifold of stable SISO input-output systems of order n.

Proof. Any input-output system has a minimal state-space realization which is unique up to choice of basis of the state space. Therefore, the equivalence classes of (minimal!) triples in C_n can be identified with stable SISO input-output systems, and the result follows from the theorem. □

Remark. A motivation for using this atlas rather than, for example, just the Schwarz-like canonical form B_n is the following. Suppose one wants to use *balanced realizations.* Then one can use the balanced parametrization of [19]. However, this parametrization is discontinuous at all points of $C_n \setminus C_{1,\ldots,1}$, i.e., in all triples $(\tilde{A}, \tilde{b}, \tilde{c})$ which have two or more coinciding singular values. Also, the complement $C_{1,\ldots,1}$, of the set of discontinuity points consists of 2^n topological components, one component for each sign pattern of the vector c (which cannot have zero components in this case; cf. (9), (10) with $n(i) = 1, i = 1, 2, \ldots, k$); this should be compared to C_n, which has only $n + 1$ topological components (the Brockett components). It appears that this is a serious disadvantage if one wants to use balanced realizations and balanced parametrizations in, for example, search algorithms for system identification, because one has to find out first which is the right "cell" of the parametrization. Another difficulty is that the balanced parametrization will tend to become numerically ill behaved if two or more of the Hankel singular values of the system are close to each other. For example, for the class of second-order systems, the determinant of the L_2-induced Riemannian metric tensor of the balanced parametrization can be calculated (e.g., using a computer algebra package) to be

$$b_1^2 b_2^2 \left(\frac{s_1\sigma_1 - s_2\sigma_2}{s_1\sigma_1 + s_2\sigma_2} \right)^2$$

in the notation of [19]. Here the s_1 and s_2 are the sign parameters, which are either $+1$ or -1. It follows that if two Hankel singular values come close, for given values of b_1 and b_2, then the parametrization becomes *ill conditioned* in the sense that a small parameter change may lead to a large change in the system (in the L_2-sense) and/or a large parameter change may lead to only a small change in the system (again in the L_2-sense).

In order to overcome these difficulties one could use the overlapping block-balanced canonical forms as follows. If $(\tilde{A}, \tilde{b}, \tilde{c})$ has k distinct Hankel singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$ with respective multiplicities $n(1), \ldots, n(k)$, then one can use the block-balanced continuous canonical form on $C_{n(1),\ldots,n(k)}$ *locally around* $(\tilde{A}, \tilde{b}, \tilde{c})$. If one is moving away from $(\tilde{A}, \tilde{b}, \tilde{c})$ in a search algorithm, for example, one has to decide whether the canonical form corresponding to a different partition should be used: if the largest $n(1)$ singular values differ sufficiently from each other, one could use, e.g., $C_{1,\ldots,1,n(2),\ldots,n(k)}$ (where there are $n(1)$ ones in the subindex before $n(2)$), etc. In this way one would use balanced realizations and "almost-balanced" realizations while moving around in the set of nth-order systems, without encountering discontinuity points.

6. On the imbedded submanifolds structure of the balanced canonical form. Consider the balanced canonical form for C_n of [19]. For each $k \in \{1, \ldots, n\}$ and each partition $(n_1, n_2, \ldots, n_k) \in P(n; k)$ let $K_{n_1,\ldots,n_k}$ denote the subset of C_n of systems with k distinct singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_k$, which have multiplicities $n_1, n_2, \ldots, n_k$, respectively. Clearly $K_{n_1,\ldots,n_k} \subset C_{n_1,\ldots,n_k}$ and equality holds only if $k = n$, $n_i = 1$, $i = 1, \ldots, n$. The mapping $K_{n_1,\ldots,n_k} \longrightarrow B_{n_1,\ldots,n_k} \cap K_{n_1,\ldots,n_k}$ is a canonical form on $K_{n_1,\ldots,n_k}$, the restriction of the canonical form $C_{n_1,\ldots,n_k} \longrightarrow B_{n_1,\ldots,n_k}$ to $K_{n_1,\ldots,n_k}$. This canonical form on $K_{n_1,\ldots,n_k}$ is input-normal with diagonal observability Gramian W_o. If one applies the state-space transformation (14) (which is diagonal here), then one obtains the balanced canonical form of [19] restricted to $K_{n_1,\ldots,n_k}$. Clearly on $K_{n_1,\ldots,n_k}$ the balanced canonical form is smooth (real analytic), while it is of course not even continuous on C_n. Both the balanced canonical form and the corresponding input-normal form parametrize $K_{n_1,\ldots,n_k}/\sim$ by the parameters $b_i > 0, \alpha(i)_j > 0, j = 1, \ldots, n_i - 1, c_i \neq 0, i = 1, \ldots, k$. Because $(c_1, \ldots, c_k)$ has 2^k possible sign patterns, it follows that $K_{n_1,\ldots,n_k}/\sim$ has 2^k topological components, each real analytically diffeomorphic to $\mathbf{R}^{n+k}$. It follows clearly that $K_{n_1,\ldots,n_k}/\sim$ is a real analytic manifold. The question arises whether it is a *regular* submanifold of $C_n/\sim$ in the sense of [1] and therefore an imbedded submanifold (cf. [1], esp. Lemma 5.2).

The answer is affirmative and is a direct consequence of the construction developed in the previous sections.

THEOREM 6.1. *For each $k \in \{1, \ldots, n\}$ and each partition $(n_1, \ldots, n_k) \in P(n; k)$ the subset $K_{n_1,\ldots,n_k}/\sim$ is a regular submanifold of $C_n/\sim$ and therefore an imbedded submanifold with the inclusion as the imbedding map.*

Proof. It follows from [1, Chapter III, section 5] that it suffices to show the socalled *$n+k$-submanifold property* for $K_{n_1,\ldots,n_k}/\sim$. This property is said to hold if for each point $p \in K_{n_1,\ldots,n_k}/\sim$ there exists a coordinate neighborhood U, φ on $C_n/\sim$ with local coordinates $\xi_1, \ldots, \xi_{2n}$ such that (i) $\varphi(p) = (0, \ldots, 0)$, (ii) $\varphi(U) = \{(\xi_1, \ldots, \xi_{2n}) | -\epsilon < \xi_i < \epsilon, i = 1, \ldots, 2n\}$, and (iii) $\varphi(U \cap K_{n_1,\ldots,n_k}/\sim) = \{\xi \in \varphi(U) | \xi_{n+k+1} = \cdots = \xi_{2n} = 0\}$. The $n+k$-submanifold property can be shown to hold as follows. Suppose that the parameter values of point $p \in K_{n_1,\ldots,n_k}/\sim$ are $b_i^0, \alpha(i)_j^0 > 0, j = 1, \ldots, n_i - 1, c_i^0 \neq 0$; of course at p, $\gamma(i)_1 = \cdots = \gamma(i)_{n_i-1} = 0$. Now choose the local coordinates $\xi_1, \ldots, \xi_{2n}$, as follows: $(\xi_1, \ldots, \xi_{n+k}) = (b_1 - b_1^0, \alpha(1)_1 - \alpha(1)_1^0, \ldots, \alpha(1)_{n_1-1} - \alpha(1)_{n_1-1}^0, c_1 - c_1^0; b_2 - b_2^0, \alpha(2)_1 - \alpha(2)_1^0, \ldots, \alpha(2)_{n_2-1} - \alpha(2)_{n_2-1}^0, c_2 - c_2^0; \ldots; b_k - b_k^0, \alpha(k)_1 - \alpha(k)_1^0, \ldots, \alpha(k)_{n_k-1} - \alpha(k)_{n_k-1}^0, c_k - c_k^0)$, $(\xi_{n+k+1}, \ldots, \xi_{2n}) = (\gamma(1)_1, \ldots, \gamma(1)_{n_1-1}, \ldots, \gamma(k)_1, \ldots, \gamma(k)_{n_k-1})$. Clearly (i) holds. It follows from Theorem 4.1 that there exists a neighborhood U of p such that (ii) holds, and from Theorem 4.1, (iii) follows. □

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Note added in proof. In a forthcoming article by the present authors in *Linear Algebra and its Applications*, the results presented here are extended to various classes of SISO and multivariable systems.

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