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LINEAR ALGEBRA AND ITS APPLICATIONS

Overlapping block-balanced canonical forms for various classes of linear systems

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Abstract

Through the use of balanced realizations it has been possible to derive parametrizations and canonical forms for various classes of minimal linear systems of given dimension. A possible problem of these parametrizations is that they are not overlapping. This could be a drawback for the application of balanced parametrizations in such areas as system identification, model reduction and optimization. It is the topic of this paper to derive overlapping parametrizations which are closely related to the existing balanced parametrizations. We first introduce input-normal canonical forms which are defined through a novel way of choosing nice selections of columns of the reachability matrix. These canonical forms provide overlapping parametrizations in the sense that they form a real analytic atlas of the manifold of systems which are considered. Then we introduce so-called block-balanced input normal forms which use the previously constructed input normal forms as building blocks. The classes of systems for which such parametrizations are given are the stable minimal systems, positive-real minimal systems, bounded-real minimal systems and the class of all minimal systems of given McMillan degree. The results include both the single-input single-output and the multivariable case. In the single-input single-output case, however, the issue of choosing nice selections of columns does not occur. Therefore in this case the derivation and presentation of the results is considerably simplified.

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1. Introduction

In [10,23–25] canonical forms and parametrizations were introduced for various classes of linear systems using the respective types of balanced realizations. While these balanced canonical forms and parametrizations have a number of interesting properties, a possible disadvantage for their use in areas such as system identification and model reduction is that they are true canonical forms with more than one and usually a quite large number of disjoint 'cells'. In geometric language this means that they do not give an atlas of the corresponding manifold of systems. This is a problem for the implementation of iterative algorithms for system identification or model reduction, like gradient-type search algorithms (see, e.g., [19,8,28]), continuation methods (see, e.g., [1]) etc. To cope with this problem an overlapping canonical form was introduced in [13,12] for the class of stable linear single-input single-output systems. In this paper overlapping canonical forms will be derived for multivariable stable systems, positive real systems, bounded real systems and the class of all multivariable systems of given McMillan degree. In the case of stable systems this gives a generalization of the single-input single-output results of [13]. The results include the case of single-input single-output (SISO) systems. In the SISO case the construction simplifies considerably because the issue of choosing a nice selection of columns, which is treated in Sections 2 and 3, is not of relevance.

The systems considered will be defined over the field $\mathbb{K}$, where $\mathbb{K}$ can be taken to be the real field $\mathbb{K} = \mathbb{R}$ and the complex field $\mathbb{K} = \mathbb{C}$. So both real-valued and complex-valued systems are treated.

The set of all continuous-time minimal state space systems of McMillan degree n with m-dimensional input and p-dimensional output space is denoted by $L_n^{p,m}$. Each such system can be represented by a quadruple $(A, B, C, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$. The subset of all stable systems is denoted by $S_n^{p,m}$. A matrix A is called *stable*, if all eigenvalues of A are in the *open* left half plane. A stable matrix is elsewhere also referred to as asymptotically stable. A system (A, B,C,D) is called *stable*, if A is stable. (Here we work with continuous time-systems; but see the remark at the end of this section concerning discrete-time systems.) The subset of $S_n^{p,m}$ of bounded real systems is denoted by $B_n^{p,m}$. A system (A,B,C,D) in $S_n^{p,m}$ is called bounded real if $I - G(\mathrm{i}w)^* G(\mathrm{i}w) > 0$, $w \in \mathbb{R} \cup \{\pm\infty\}$, where G is the transfer function, $G(s) = C(sI - A)^{-1}B + D$. If $p = m$, then P_n^m stands for the subset of $S_n^{m,m}$ of positive real systems. A system (A,B,C,D) in $S_n^{m,m}$ is called positive real if $G(\mathrm{i}w)^* + G(\mathrm{i}w) > 0$, $w \in \mathbb{R} \cup \{\pm\infty\}$, where G is the transfer function of the system.

Our aim is to study canonical forms and parametrizations for these classes of systems in terms of balanced realizations. We call two systems (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) in $L_n^{p,m}$ *(input–output) equivalent* and write $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2)$ if there exists a non-singular T such that $(A_1, B_1, C_1, D_1) = (TA_2T^{-1}, TB_2, C_2T^{-1}, D_2)$.

Definition 1.1. *A canonical form* for (input–output) equivalence on a subset $\mathscr{A} \subseteq L_n^{p,m}$ is a map

$$\Gamma : \mathscr{A} \to \mathscr{A},$$

such that
1. $\Gamma(a) \sim a$ for all $a \in \mathscr{A}$.
2. if $a, b \in \mathscr{A}$, and $a \sim b$ then $\Gamma(a) = \Gamma(b)$.

Often we will also refer to $\Gamma(a)$ as the canonical form of $a \in L_n^{p,m}$. A canonical form is called continuous if $\mathscr{A}$ is given a topology and Γ is continuous.

Remark. The topology that is typically being put on the various spaces of linear systems is given as follows. Let $L_n^{p,m}$ be embedded into $\mathbb{K}^{n\times n+n\times m+p\times n+p\times m}$ in the natural way. The space $L_n^{p,m}$ is then given the subspace topology in $\mathbb{K}^{n\times n+n\times m+p\times n+p\times m}$. The space $L_n^{p,m}/\sim$ of (input–output) equivalence classes in $L_n^{p,m}$ is given the quotient topology. In the same way the topologies for the other classes of systems are defined.

For many purposes it turns out that using one canonical form is rather restrictive. For example for the multivariable case it was shown by Hazewinkel [14] that there does not exist a continuous canonical form on $L_n^{p,m}$. If one looks for a continuous canonical form with additional properties, then it is not uncommon to find that even in the single-input single-output case such a canonical form does not exist. Therefore we will make use of the concept of an overlapping set of (continuous) canonical forms.

Definition 1.2. Let $\mathscr{A} \subseteq L_n^{p,m}$ and let $\mathscr{A}$ be given a topology. Let $\mathscr{A}_i$, $i \in I$, be subsets of the topological space $\mathscr{A}$. A set of (continuous) canonical forms

$$\{\Gamma_i : \mathscr{A}_i \to \mathscr{A}_i \mid i \in I\}$$

is said to be an *overlapping set of (continuous) canonical forms covering* $\mathscr{A}$ if

$$\bigcup_{i\in I} \operatorname{int}(\mathscr{A}_i) = \mathscr{A},$$

where $\operatorname{int}(\mathscr{A}_i)$ denotes the open interior of the set $\mathscr{A}_i$, $i \in I$.

We now recall the various types of balancing. The principle behind the definition of the various types of balancing is that associated with each class of systems there is a natural pair of Riccati or Lyapunov equations. A system is then called balanced if specified solutions of each of the two equations are identical and diagonal. Realizations that are closely related to balanced realizations are what we will call balanced input-normal realizations. Input-normal realizations are defined by demanding that the Riccati or Lyapunov equation in which the elements of BB^* occur in the constant term (i.e. in the (matrix) term of degree zero with respect to the entries of the unknown matrices) has the identity matrix as its stabilizing solution (see, e.g., [21,15,27,30,3,24]). This is a generalization of the concept of input-normality in the stable case, to the other classes of systems treated here. Balanced input-normal realizations are realizations which have as the solutions of the associated pair of algebraic Riccati or Lyapunov equations, a diagonal matrix and the identity matrix. Most of the properties of balanced realizations carry over to these balanced input-normal realizations, ncluding truncation properties. A balanced input-normal realization can be transformed into a balanced realization by a simple scaling of the entries of the state vector and the same set of parameters can be used to parametrize this form. Therefore the terminology *balanced input-normal* that is introduced in the following definition appears to be justified. After the definition some remarks are made concerning the solutions of the Riccati and Lyapunov equations that are fundamental to the definitions and the resulting possible constraints of the singular values that are defined.

Definition 1.3. 1. (*LQG-balancing*) Consider the system $(A, B, C, D) \in L_n^{p,m}$ and let Y and Z denote the stabilizing solutions to the control and filter algebraic Riccati equations,

$$0 = A_L^* Y + YA_L - YBR_L^{-1}B^*Y + C^*S_L^{-1}C, \qquad \text{(L1)}$$

$$0 = A_L Z + ZA_L^* - ZC^*S_L^{-1}CZ + BR_L^{-1}B^*, \qquad \text{(L2)}$$

where

$$A_L = A - BR_L^{-1}D^*C, \quad R_L = I + D^*D, \quad S_L = I + DD^*. \qquad \text{(L3)}$$

The condition that Y is a stabilizing solution of (L1) is equivalent to the requirement that $A_L - BR_L^{-1}B^*Y$ is stable; the condition that Z is a stabilizing solution of (L2) is equivalent to the requirement that $A_L - ZC^*S_L^{-1}C$ is stable.

(a) The system $(A, B, C, D) \in L_n^{p,m}$ is called *LQG-balanced* if Y and Z are equal and diagonal, such that

$$\Sigma_L := Y = Z = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n (>0)$. The matrix Σ_L is called the *LQG-gramian* of the system. The positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ are called the *LQG-singular values* of the system.

(b) The system is called *LQG-input normal* if $Z = I$. Similarly, the system is called *LQG-σ-input-normal* if $Z = \sigma I$, $\sigma > 0$.

(c) The system is called *LQG-balanced σ-input normal* if $Z = \sigma I$, $\sigma > 0$, and Y is diagonal with nonincreasing diagonal entries. In fact in that case

$$Y = \sigma^{-1} \operatorname{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) = \sigma^{-1}\Sigma_L^2,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the LQG singular values defined in (a).

2. (*Lyapunov-balancing*) Consider the system $(A, B, C, D) \in S_n^{p,m}$ and let Y and Z be the solutions to the Lyapunov equations,

$$0 = A^*Y + YA + C^*C, \qquad \text{(S1)}$$

$$0 = AZ + ZA^* + BB^*. \qquad \text{(S2)}$$

(a) The system $(A, B, C, D) \in S_n^{p,m}$ is called *Lyapunov-balanced* if Y and Z are equal and diagonal, such that

$$\Sigma_S := Y = Z = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n (>0)$. The matrix Σ_S is called the *Lyapunov-gramian* of the system. The positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ are called the *Lyapunov singular values* of the system.

(b) The system is called *Lyapunov input-normal* if $Z = I$. Similarly, the system is called *Lyapunov σ-input-normal* if $Z = \sigma I$, $\sigma > 0$.

(c) The system is called *Lyapunov balanced σ-input-normal* if $Z = \sigma I$, $\sigma > 0$, and Y is diagonal with nonincreasing diagonal entries. In fact in that case

$$Y = \sigma^{-1} \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) = \sigma^{-1}\Sigma_S^2,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the Lyapunov singular values of the system.

3. (*Bounded-real-balancing*) Consider the system $(A, B, C, D) \in B_n^{p,m}$ and let Y and Z denote the stabilizing solutions to the control and filter bounded-real Riccati equations,

$$0 = A_B^*Y + YA_B + YBR_B^{-1}B^*Y + C^*S_B^{-1}C, \qquad \text{(B1)}$$

$$0 = A_BZ + ZA_B^* + ZC^*S_B^{-1}CZ + BR_B^{-1}B^*, \qquad \text{(B2)}$$

where

$$A_B = A + BR_B^{-1}D^*C, \quad R_B = I - D^*D, \quad S_B = I - DD^*. \qquad \text{(B3)}$$

The condition that Y is a stabilizing solution of (B1) is equivalent to the requirement that $A_B + BR_B^{-1}B^*Y$ is stable; the condition that Z is a stabilizing solution of (B2) is equivalent to the requirement that $A_B + ZC^*S_B^{-1}C$ is stable.

(a) The system is called *bounded-real-balanced* if Y and Z are equal and diagonal such that

$$\Sigma_B := Y = Z = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

with $(1 >)\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n (> 0)$. The matrix Σ_B is called the *bounded-real-gramian* of the system. The numbers $\sigma_1, \sigma_2, \ldots, \sigma_n \in (0, 1)$ are called the *bounded-real singular values* of the system.

(b) The system is called *bounded-real input normal* if $Z = I$. Similarly, the system is called *bounded-real σ-input-normal* if $Z = \sigma I$, $\sigma > 0$.

(c) The system is called *bounded-real balanced σ-input-normal* if $Z=\sigma I$, $\sigma > 0$, and Y is diagonal with non-increasing diagonal entries. In fact in that case

$$Y = \sigma^{-1}\operatorname{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) = \sigma^{-1}\Sigma_S^2,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the bounded-real singular values defined in (a).

4. (*Positive-real-balancing*) Consider the system $(A, B, C, D) \in P_n^m$ and let Y and Z be the stabilizing solutions to the control and filter positive-real Riccati equations,

$$0 = A_P^* Y + Y A_P + Y B R_P^{-1} B^* Y + C^* R_P^{-1} C, \qquad \text{(P1)}$$

$$0 = A_P Z + Z A_P^* + Z C^* R_P^{-1} C Z + B R_P^{-1} B^*, \qquad \text{(P2)}$$

where

$$A_P = A - B R_P^{-1} C, \quad R_P = D + D^*. \qquad \text{(P3)}$$

The condition that Y is a stabilizing solution of (P1) is equivalent to the requirement that $A_P + B R_P^{-1} B^* Y$ is stable; the condition that Z is a stabilizing solution of (P2) is equivalent to the requirement that $A_P + Z C^* R_P^{-1} C$ is stable.

(a) The system is called *positive real balanced* if Y and Z are equal and balanced, such that

$$\Sigma_P := Y = Z = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

with $(1 >)\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n(>)0$. The matrix Σ_P is called the *positive-real-gramian* of the system. The numbers $\sigma_1, \sigma_2, \ldots, \sigma_n \in (0, 1) \subset \mathbb{R}$ are called the *positive-real-singular values* of the system.

(b) The system is called *positive-real input-normal* if $Z = I$. Similarly, the system is called *positive-real σ-input-normal* if $Z = \sigma I, \sigma > 0$.

(c) The system is called *positive-real balanced σ-input-normal* if $Z = \sigma I, \sigma > 0$, and Y is diagonal with non-increasing diagonal entries. In fact in that case

$$Y = \sigma^{-1}\operatorname{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) = \sigma^{-1}\Sigma_P^2,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the positive real singular values defined in (a).

There are a number of properties of the solutions of the Riccati and Lyapunov equations appearing in the Definition above, that are implicitly used in the Definition or will be used later. For proofs of the results one can refer, e.g., to [31,4,18]. The hermitian solutions of the Riccati equations involved are not necessarily strictly positive definite in general. In the LQG case the two Riccati equations in $L_n^{p,m}$ each have one strictly positive definite hermitian solution and this is precisely the unique stabilizing solution. It follows that the corresponding LQG singular values are strictly positive. In the Lyapunov case the two Lyapunov equations for a system in $S_n^{p,m}$ each have a unique solution and it is strictly positive definite hermitian. Hence the Lyapunov singular values, also known as the Hankel singular values, are strictly positive. In the bounded real case and the positive real case all the hermitian solutions (not only the stabilizing solutions) of the Riccati equations for a system in $B_n^{p,m}$ respectively P_n^m are strictly positive definite. Hermitian solutions Y and Z are the stabilizing solutions if and only if $Z^{-1} - Y$ is strictly positive definite. This is again equivalent to the property that the (strictly positive) singular values are all strictly less than one.

In [10,23–25] canonical forms and parametrizations were obtained for the various classes of linear systems using balanced realizations. The canonical forms are all 'non-overlapping' and even in the single-input single-output case they lead to 'cell decompositions' of the classes of systems with a large number of 'cells' [24]. For iterative search algorithms on the manifold of systems of given McMillan degree this could potentially lead to problems. If a parameter estimate approaches the 'boundary' of such a 'cell' it may not be obvious, how and in which other cell the path should be continued. Moreover, numerical problems could be expected since, e.g., for the class of stable second order systems the determinant of the L^2-induced Riemannian metric tensor of the balanced parametrized systems with distinct singular values can be calculated to be

$$b_1^2 b_2^2 \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right),$$

where λ_1, λ_2 are the eigenvalues of the corresponding Hankel operator and b_1, b_2 are the entries of the b-vector. A 'boundary' of a 'cell' is encountered, e.g., if λ_1 is nearly $-\lambda_2$, where it is seen that the parametrization is ill-conditioned. Using suitably chosen overlapping parametrizations it is possible to 'paste over' such boundaries.

The purpose of this paper is therefore to develop overlapping parametrizations for the various classes of systems. This will be done using input-normal realizations. We will show that the canonical forms based on so-called block-balanced input-normal realizations are in a straightforward bijective correspondence with block-balanced realizations, which in turn are closely related to the corresponding balanced canonical forms. This is important from the following practical point of view. For example in system identification algorithms ideally a balanced canonical form is used because of its advantages with respect to the parametrization of systems. Should the parameter estimate come close to the boundary of a 'cell' in which the algorithm was started it would be advisable to change to a suitably chosen block-balanced input-normal canonical form until the 'boundary' is passed and the estimate is away from the 'boundary' of the new 'cell'.

The overlapping forms for the multivariable case are constructed using the classical way of employing nice selections of columns of the reachability matrix (see, e.g., [14,8]). But we introduce an important modification. We will be working with reordered columns of the reachability matrix. The reordering allows us to obtain a description of the canonical form in terms of the state space matrices which is of importance for the desired parametrization results. In fact one of the guiding ideas is to make the reordered reachability matrix upper triangular in a suitable fashion. The result is that a reordering of the columns of B and A will then also be upper triangular.

Precise definitions and a well-known Lemma concerning various concepts of upper triangularity that will be used, follow.

Definition 1.4. Let $M \in \mathbb{K}^{n\times l}$.

(a) M is called positive upper triangular if there exist n indices $i_1, i_2, \dots, i_n$, with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant l$ such that

$$M = \begin{pmatrix} 0 & \dots & m_{1i_1} & * & \dots & \dots & \dots & \dots & * & & & \\ 0 & \dots & 0 & 0 & \dots & m_{2i_2} & \dots & \dots & \dots & \dots & \dots & \\ \vdots & & \vdots & \vdots & & & \ddots & & & & \dots & \\ 0 & \dots & 0 & 0 & \dots & 0\dots & \dots & 0 & m_{ni_n} & * & & \end{pmatrix},$$

with $m_{ji_j} > 0$ for all $j \in \{1, 2, \dots, n\}$.

(b) M is called simple positive upper triangular if M is positive upper triangular and $i_n = n$, i.e. M can be partitioned as $M = [M_1, M_2]$, where M_1 is a square $n \times n$ positive upper triangular matrix and M_2 an arbitrary $n \times (l - n)$ matrix.

The following lemma is well-known (see, e.g., [23]).

Lemma 1.1. *Let $M \in \mathbb{K}^{n\times l}$, rank$(M) = n \leqslant l$. There exists a unitary matrix $Q_0 \in \mathbb{K}^{n\times n}$ such that the matrix Q_0M is positive upper triangular. The matrix Q_0 is unique and so Q_0M is uniquely determined.*

Let us now give a short description of the contents of the various sections. The notion of ordered dynamical indices is introduced in Section 2, where also the necessary material is reviewed concerning nice selections. In Section 3 the specific way in which the columns of the reachability matrix are ordered is introduced. In this section also one of the main results of the paper is established which shows that bringing the reordered columns of the pair $[B,A]$ into simple positive upper triangular form is equivalent to bringing the reordered columns of the reachability matrix in the nice selection into simple positive upper triangular form. In Sections 4 and 5, the canonical forms and parametrizations will be analyzed. More specifically, in Section 4 we discuss input-normal forms.

The results of this section will then be used in Section 5 to derive results concerning block-balanced input normal forms which are a generalization and refinement of input-normal forms.

Remark. In this paper we work with continuous time systems; for discrete time systems one can use the parametrizations and canonical forms presented here by employing the well-known bilinear transform which brings a continuous-time system into a discrete-time system (cf., e.g., [7,8,23]).

1.1. Notation

The letter $\mathbb{K}$ stands for either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. The symbols $L_n^{p,m}$, $S_n^{p,m}$, $B_n^{p,m}$ and P_n^m stand for various classes of systems as defined in the Introduction. The symbol $\sim$ stands for system equivalence (see Introduction). Definition 1.1 contains an explanation of the symbol Γ which stands for a canonical form and Definition 1.2 contains an explanation of 'int' which denotes the open interior of a set. The symbols A_L, A_B, A_P, R_L, R_B, R_P, S_L, S_B and S_P are explained in Definition 1.3. The symbol $\#A$ denotes the number of elements in the set A. The set of (n,m)-nice selections $\mathcal{N}(n;m)$ is defined in Definition 2.1. The set of (n,m)-dynamical indices $\mathcal{D}(n;m)$ is defined in Definition 2.2. The ordering permutation π of a sequence of dynamical indices is introduced in Section 2. The set $\mathcal{OD}(n;m)$ of ordered dynamical indices is defined in Definition 2.3. The set of (n,m)-step sizes $S(n;m)$ is defined in Definition 2.4. What is meant by a sequence of (n,m)-reversed step sizes is defined in Definition 2.5. The reachability matrix $R(A,B)$ of the n-dimensional system $(A,B,C,D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ is given by $R(A,B) = [B, AB, A^2B, \ldots, A^{n-1}B]$. The symbols $N(A,B;d)$ and $M(A,B;d)$ are explained in Section 3. The symbol $\tilde{N}(A,B;d)$ is explained in Section 5. The sets of systems $L_{n;d}^{p,m}, S_{n;d}^{p,m}, B_{n;d}^{p,m}, P_{n;d}^{m}$ are defined in Definition 4.1 and the sets of systems $\sigma\text{-}IL_{n;d}^{p,m}, \sigma\text{-}IS_{n;d}^{p,m}, \sigma\text{-}IB_{n;d}^{p,m}, \sigma\text{-}IP_{n;d}^{m}$ are defined in Definition 4.2. The sets of systems $\mathrm{Skew}_{n;d}^{p,m}, \tilde{\Theta}_{\sigma,d,L}^{p,m}, \tilde{\Theta}_{\sigma,d,S}^{p,m}, \tilde{\Theta}_{\sigma,d,B}^{p,m}, \tilde{\Theta}_{\sigma,d,P}^{m}, \Theta_{\sigma,d,L}^{p,m}, \Theta_{\sigma,d,S}^{p,m}, \Theta_{\sigma,d,B}^{p,m}, \Theta_{\sigma,d,P}^{m}$ are also defined in Section 4. The canonical forms $\Gamma_{\sigma;d}^{L}, \Gamma_{\sigma;d}^{S}, \Gamma_{\sigma;d}^{B}$, and $\Gamma_{\sigma;d}^{P}$ are defined in Lemma 4.3. The sets of feedthrough terms Δ, Δ_b and Δ_p are defined in Section 4. The parametrization maps $\phi_{\sigma,d,L}, \phi_{\sigma,d,S}, \phi_{\sigma,d,B}$ and $\phi_{\sigma,d,P}$ are also introduced in Section 4. The sets of systems $L_{n_1,n_2,\ldots,n_k}^{p,m}, S_{n_1,n_2,\ldots,n_k}^{p,m}, B_{n_1,n_2,\ldots,n_k}^{p,m}$ and $P_{n_1,n_2,\ldots,n_k}^{m}$ are defined in Section 5. The set of eigenvalues of a matrix A is denoted by spec(A). If for two sets of real numbers A and B we have that each element of A is larger than each element of B we write that $A \succ B$ (see Section 5). For an explanation of the notation $L_{n_1,n_2,\ldots,n_k;D_k}^{p,m}, S_{n_1,n_2,\ldots,n_k;D_k}^{p,m}, B_{n_1,n_2,\ldots,n_k;D_k}^{p,m}, P_{n_1,n_2,\ldots,n_k;D_k}^{m}, \sigma\text{-}IL_{n_1,n_2,\ldots,n_k;D_k}^{p,m}, \sigma\text{-}IS_{n_1,n_2,\ldots,n_k;D_k}^{p,m}, \sigma\text{-}IB_{n_1,n_2,\ldots,n_k;D_k}^{p,m}$, and $\sigma\text{-}IP_{n_1,n_2,\ldots,n_k;D_k}^{m}$ see Definition 5.3. The canonical forms $\Gamma_{\sigma,D_k}^{L}, \Gamma_{\sigma,D_k}^{S}, \Gamma_{\sigma,D_k}^{B}, \Gamma_{\sigma,D_k}^{P}$ are defined in Lemma 5.3. Refer to Section 5 for the definition of the sets of systems $\tilde{\Theta}_{\sigma,n_1,\ldots,n_k;D_k,L}^{p,m}, \tilde{\Theta}_{\sigma,n_1,\ldots,n_k;D_k,S}^{p,m}, \tilde{\Theta}_{\sigma,n_1,\ldots,n_k;D_k,B}^{p,m}, \tilde{\Theta}_{\sigma,n_1,\ldots,n_k;D_k,P}^{m}, \Theta_{\sigma,n_1,\ldots,n_k;D_k,L}^{p,m}, \Theta_{\sigma,n_1,\ldots,n_k;D_k,S}^{p,m}, \Theta_{\sigma,n_1,\ldots,n_k;D_k,B}^{p,m}$ and, $\Theta_{\sigma,n_1,\ldots,n_k;D_k,P}^{m}$. The parametrization maps $\phi_{\sigma,n_1,\ldots,n_k;D_k,L}$, $\phi_{\sigma,n_1,\ldots,n_k;D_k,S}$, $\phi_{\sigma,n_1,\ldots,n_k;}$ D_k, B, $\phi_{\sigma,n_1,\ldots,n_k;D_k,P}$ are also defined in Section 5. SISO system stands for single-input single-output system and MIMO (or multivariable) system stands for multi-input and/or multi-output system. The real part of a complex number λ is denoted by $\mathrm{Re}(\lambda)$.

2. Nice selection and dynamical indices

The purpose of this section is to recall the notions of nice selections and dynamical indices (see, e.g., [8]) and to introduce the new concept of an ordering permutation for a sequence of dynamical indices. This new concept will be central in the construction of the canonical forms for multi-input multi-output systems. This section is not of relevance, however, for the construction of the canonical forms for single-input single-output systems.

In order to fix the notation we now recall the well-known definitions of a nice selection and dynamical indices.

Definition 2.1. Let $n, m \geqslant 1$. Then $(v_1, v_2, \ldots, v_n)$ is a *nice selection* or (n, m)*-nice selection* if
1. $v_i \in \{1, 2, \ldots, nm\}, i = 1, \ldots, n$.
2. $v_1 \leqslant v_2 \leqslant \cdots v_n \leqslant nm$
3. for $i = 1, \ldots, n$, either

$$v_i \leqslant m$$

or

$$v_i - m = v_j$$

for some $1 \leqslant j \leqslant i$, i.e. $v_i - m$ is also in the nice selection.

The set of (n, m)-nice selections is denoted by $\mathcal{N}(n; m)$.

Let $(A, B, C, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ be a system and let $R := R(A, B) := [B, AB, \ldots, A^{n-1}B]$ be its reachability matrix. Let $v_1, \ldots, v_n$ be column numbers of R which form a (n, m)-nice selection. For $i = 1, \ldots, m$ let d_i be given by

$$d_i = \#\{j \in \{0, \ldots, n-1\} \mid i + jm \in \{v_1, v_2, \ldots, v_n\}\}.$$

The numbers $d_i, i = 1, 2, \ldots, m$, are called the *dynamical indices* corresponding to the nice selection $v_1, \ldots, v_n$. It follows immediately from the construction of the dynamical indices that $\sum_{i=1}^{m} d_i = n$. This motivates the following well-known definition.

Definition 2.2. Let $n, m \geqslant 1$. The sequence $(d_1, \ldots, d_m)$ is called a sequence of (n, m)-dynamical indices if

1. $d_i \in \{0, 1, \dots, n\}$, $i = 1, \dots, m$.
2. $\sum_{i=1}^{m} d_i = n$.

The set of all sequences of (n, m)-dynamical indices is denoted by $\mathscr{D}(n; m)$.

The above construction associates to each (n, m)-nice selection a sequence of (n, m)-dynamical indices. This induces a map T_d from $\mathscr{N}(n; m)$ to $\mathscr{D}(n; m)$, which is in fact a bijection.

Lemma 2.1. *The map*

$$T_d : \mathscr{N}(n; m) \to \mathscr{D}(n; m)$$

is a bijection.

For a given sequence of (n, m)-dynamical indices $(d_1, d_2, \dots, d_m)$ consider the unique permutation π on $(1, 2, \dots, m)$ with the following properties.
1. $d_{\pi(1)} \geqslant d_{\pi(2)} \geqslant \cdots \geqslant d_{\pi(m)}$;
2. for each i, $j \in \{1, 2, \dots, m\}$, $i < j$, with $d_{\pi(i)} = d_{\pi(j)}$ one has

$$\pi(i) < \pi(j).$$

The sequence $(d_{\pi(1)}, d_{\pi(2)}, \dots, d_{\pi(m)})$ is called the sequence of ordered (n, m)-dynamical indices corresponding to the sequence of (n, m) dynamical indices $(d_1, d_2, \dots, d_m)$. The permutation π is called the *ordering permutation.*

For the subsequent discussions it will also be useful to have a notation for the multiplicities of the dynamical indices. Let h be the number of distinct dynamical indices $d_1, \dots, d_m$ or $d_{\pi(1)}, \dots, d_{\pi(m)}$ and let $\delta_1, \delta_2, \dots, \delta_h$ denote the multiplicities such that

$$\begin{aligned} d_{\pi(1)} = \cdots = d_{\pi(\delta_1)} > d_{\pi(\delta_1+1)} = \cdots = d_{\pi(\delta_1+\delta_2)} > d_{\pi(\delta_1+\delta_2+1)} = \cdots \\ = d_{\pi(\delta_1+\delta_2+\delta_3)} > \cdots > d_{\pi(\delta_1+\delta_2+\cdots\delta_{n-1}+1)} = \cdots = d_{\pi(m)}. \end{aligned}$$

In some situations it is convenient to have a notation available for the values which the ordered dynamical indices take on, but disregarding multiplicities. We denote this sequence by

$$(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_h).$$

Example 2.1. Let $n = 18$, $m = 7$ and $(d_1, d_2, \dots, d_7) = (3, 1, 3, 2, 4, 4, 1)$ then

$$(d_{\pi(1)}, d_{\pi(2)}, \dots, d_{\pi(7)}) = (4, 4, 3, 3, 2, 1, 1)$$

and

$$(\pi(1), \pi(2), \dots, \pi(7)) = (5, 6, 1, 3, 4, 2, 7).$$

Note that for the example the permutation

$$(\tilde{\pi}(1), \tilde{\pi}(2), \ldots, \tilde{\pi}(7)) = (6, 5, 1, 3, 4, 2, 7)$$

would also produce

$$(d_{\tilde{\pi}(1)}, d_{\tilde{\pi}(2)}, \ldots, d_{\tilde{\pi}(7)}) = (4, 4, 3, 3, 2, 1, 1)$$

but would not satisfy the second property, since $d_{\tilde{\pi}(1)} = d_{\tilde{\pi}(2)} = 4$, but $\tilde{\pi}(1) \not< \tilde{\pi}(2)$.

The multiplicities of the dynamical indices are

$$(\delta_1, \delta_2, \delta_3, \delta_4) = (2, 2, 1, 2)$$

with $h = 4$. The sequence of ordered dynamical indices disregarding multiplicities is

$$(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4) = (4, 3, 2, 1).$$

Definition 2.3. Let $n, m \geqslant 1$. The set of all sequences $(d_1, \ldots, d_m)$ of ordered (n, m)-dynamical indices, i.e. all sequences $(d_1, \ldots, d_m)$ of (n, m)-dynamical indices for which $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$, is denoted by $\mathcal{OD}(n; m)$.

The following set of numbers $s_1, \ldots, s_l$, $l \leqslant n$, is naturally associated with an (n, m)-nice selection of the columns of the reachability matrix R of the system $(A, B, C, D) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n} \times \mathbb{K}^{p \times m}$. Let s_i be the number of columns in the nice selection that were picked from $A^{i-1}B$, $i = 1, \ldots, n$. Let l be such that $s_l > 0$ and $s_{l+1} = 0$. By construction of the nice selection it is clear that

$$m \geqslant s_1 \geqslant s_2 \geqslant \cdots \geqslant s_l > s_{l+1} = 0.$$

Since the number of columns of the (n, m)-nice selection of R is n we have

$$\sum_{i=1}^{l} s_i = n.$$

The indices $s_1, \ldots, s_l$ are called the (n, m)-step sizes corresponding to the (n, m)-nice selection. In a similar way the step sizes can be deduced from a sequence of (n, m)-dynamical indices $(d_1, \ldots, d_m)$. Then s_i is the number of dynamical indices such that $d_j \geqslant i$, i.e.

$$s_i = \#\{j \mid d_j \geqslant i\}.$$

Definition 2.4. The sequence $(s_1, s_2, \ldots, s_l)$ is called a sequence of (n, m)-*step sizes* if
1. $s_i \in \{1, \ldots, m\}$, $i = 1, \ldots, l$.
2. $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_l > 0$.
3. $\sum_{i=1}^{l} s_i = n$.

The set of all sequences of (n, m)-step sizes is denoted by $\mathcal{S}(n; m)$.

It is clear that two different nice selections or two different sequences of dynamical indices can give rise to the same sequence of step sizes. But there is a bijection between ordered dynamical indices and step sizes.

Lemma 2.2. *Let $n, m \geqslant 1$. Define the map*

$$T_\rho : \mathscr{O}\mathscr{D}(n;m) \to \mathscr{S}(n;m); \quad (d_1, \dots, d_m) \mapsto (s_1, s_2, \dots, s_l),$$

where $s_i := \#\{j \mid d_j \geqslant i\}$ and l is such that $s_l > 0, s_{l+1} = 0$. Then T_ρ is a bijection.

An easy way to see how the step sizes and the ordered dynamical indices are related and that they are bijectively related is by way of a so-called *Young diagram*. One possible way to describe it is as a 0–1-matrix: Form the $m \times n$ matrix $Y = (y_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ such that

$$y_{ij} = \begin{cases} 1 & \text{for } j \leqslant d_{\pi(i)}, \\ 0 & \text{for } j > d_{\pi(i)}. \end{cases}$$

Then the sum of the entries in the ith row is clearly $d_{\pi(i)}$ for each $i \in \{1, \dots, m\}$. But also the sum of the entries of the jth column is equal to $s_j, j = 1, \dots, n$. It is not hard to see that $l = \max_{1 \leqslant i \leqslant m} d_i$ and $s_j = 0$ for $j > l$. Furthermore the sequence $(s_l, s_{l-1} - s_l, \dots, s_1 - s_2, s_0 - s_1)$ from which the zero elements are removed (set $s_0 := m$), forms the sequence of multiplicities $(\delta_1, \delta_2, \dots, \delta_h)$ of the ordered dynamical indices. Clearly

$$\delta_1 + \delta_2 + \cdots + \delta_h = s_l + (s_{l-1} - s_l) + \cdots + (s_1 - s_2) + (s_0 - s_1) = s_0 = m.$$

That the step sizes $s_1, \dots, s_l$ and the ordered dynamical indices are bijectively related can be read off easily from the Young diagram with matrix Y.

Example 2.2. Let again $n = 18$, $m = 7$ and

$$(d_{\pi(1)}, d_{\pi(2)}, \dots, d_{\pi(7)}) = (4, 4, 3, 3, 2, 1, 1).$$

The Young diagram can be represented by the 0–1-matrix Y,

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The row-sums are 4,4,3,3,2,1,1 respectively, i.e. equal to $d_{\pi(1)}, d_{\pi(2)}, \dots, d_{\pi(7)}$, while the column-sums are (7,5,4,2,0,0,0) and so $l = 4$ and

$s_1 = 7, s_2 = 5, s_3 = 4, s_4 = 2$. The multiplicities of the ordered dynamical indices are obtained from $(s_4, s_3 - s_4, s_2 - s_3, s_1 - s_2, s_0 - s_1) = (2, 2, 1, 2, 0)$ by removing the zero elements, therefore $h = 4$ and $\delta_1 = 2$, $\delta_2 = 2$, $\delta_3 = 1$, $\delta_4 = 2$. This can of course be seen directly from the dynamical indices (4,4,3,3,2,1,1) as well.

In later sections it will be convenient to have a notation available for sequences obtained by reversing the order of a sequence of step sizes.

Definition 2.5. The sequence $(r_1, r_2, \dots, r_l)$ is called a sequence of (n, m)-reversed step sizes if
1. $r_i \in \{1, 2, \dots, m\}$, $i = 1, \dots, l$.
2. $0 < r_1 \leqslant r_2 \leqslant \dots \leqslant r_l$.
3. $\sum_{i=1}^{n} r_i = n$.

If h' is the number of distinct elements in a sequence of (n, m) reversed step sizes $(r_1, \dots, r_l)$, the multiplicities are denoted by

$$\rho_1, \dots, \rho_{h'}.$$

The sequence of reversed step sizes, ignoring multiplicities is denoted by

$$\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_{h'}.$$

Clearly, there is a bijective relationship between sequences of (n, m)-step sizes and sequences of reversed (n, m)-step sizes; the bijection is given simply be a reversion of the sequence. Combining this with the bijection between sequences of ordered dynamical indices and sequences of step sizes (cf. Lemma 2.2) it follows of course that with each ordered sequence of dynamical indices there corresponds a unique reversed sequence of step sizes.

The following lemma summarizes some useful facts which will be important in the subsequent sections.

Lemma 2.3. *Let $m \geqslant 1$, $n \geqslant 1$ and let $(d_1, \dots, d_m)$ be a sequence of (n, m)-dynamical indices with sequence of (n, m)-step sizes $(s_1, s_2, \dots, s_l)$. Let π be the ordering permutation. Then*
1. $d_{\pi(s_i)} \geqslant i$ *for* $i = 1, \dots, l$.
2. *If for some* $1 \leqslant i < l - 1, s_{i+1} < s_i$, *then* $d_{\pi(s_i)} = i$.
3. $d_{\pi(i)} = 0$ *for* $i = s_1 + 1, \dots, m$.
4. $d_{\pi(i)} = l$ *for* $i = 1, 2, \dots, s_l$.
5. $l = \max_{1 \leqslant i \leqslant m} d_i$.
6. *The sequence*

$$(s_l, s_{l-1} - s_l, \dots, s_1 - s_2, s_0 - s_1)$$

from which the zero elements are removed, forms the sequence of multiplicities $(\delta_1, \delta_2, \dots, \delta_h)$ *of the ordered dynamical indices. Here we have set* $s_0 := m$.

7. *Let* h' *be the number of distinct elements in the sequence of step sizes* $(s_1, s_2, \dots, s_l)$ *and let* h *be the number of different elements in the sequence of dynamical indices* $(d_1, \dots, d_m)$. *Then*

$$h' = \begin{cases} h & \text{if } d_{\pi(m)} \neq 0, \\ h-1 & \text{if } d_{\pi(m)} = 0. \end{cases}$$

8. *Let* $\tilde{r}_1, \dots, \tilde{r}_{h'}$ *be the reversed step sizes, ignoring multiplicities, that correspond to the given sequence of step sizes. Then for* $i = 1, \dots, h'$,

$$\delta_1 + \cdots + \delta_i = \tilde{r}_i.$$

Proof. (1): Let $1 \leqslant i \leqslant l$. Recall that $s_i = \#\{j \mid d_j \geqslant i\}$. Hence for $k = 1, \dots, s_i$ it follows that $d_{\pi(k)} \geqslant i$.

(2): Let $1 \leqslant i \leqslant l-1$ be such that $s_{i-1} < s_i$. Assume that $d_{\pi(s_i)} > i$. Since $d_{\pi(k)} \geqslant d_{\pi(s_i)} \geqslant i+1$ for all $1 \leqslant k \leqslant s_i$ it follows that $s_{i+1} = s_i$ which is a contradiction. Hence $d_{\pi(s_i)} = i$.

(3)–(5): Follows from the definition of the step sizes.

(6): Since by definition $s_i = \#\{j \mid d_j \geqslant i\}$, $i = 1, \dots, l$, it follows that

$$s_i - s_{i+1} = \#\{j \mid d_j = i\},$$

$i = 1, \dots, l-1$. Note that

$$\#\{j \mid d_j = 0\} = m - \#\{j \mid d_j \geqslant 1\} = s_0 - s_1$$

and

$$\#\{j \mid d_j = l\} = \#\{j \mid d_j \geqslant l\} = s_l.$$

(7): The number of distinct elements h' in the sequence $(s_1, s_2, \dots, s_l)$ is given by the number of non-zero elements in the sequence

$$(s_l, s_{l-1} - s_l, \dots, s_1 - s_2).$$

Hence

$$\begin{aligned} h' &= \begin{cases} h & \text{if } s_0 = s_1, \\ h-1 & \text{if } s_0 \neq s_1, \end{cases} \\ &= \begin{cases} h & \text{if } d_{\pi(m)} \neq 0, \\ h-1 & \text{if } d_{\pi(m)} = 0. \end{cases} \end{aligned}$$

(8): This is a consequence of 6. □

3. Nice selections of columns of the reachability matrix

If a column of a reachability matrix is in fact the ith column, $i \in \{1, \dots, nm\}$, then we will say that this column has *column label i*. If a system $(A, B, C, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ is reachable, then its reachability matrix $R(A, B)$ has full rank and as is well-known, the labels of the first n independent columns of $R(A, B)$ form a nice selection, called the *Kronecker selection.* This shows that there exists at least one nice selection $(v_1, \dots, v_n)$ such that the corresponding submatrix of $R(A, B)$ consisting of the columns with the labels $v_1, \dots, v_n$ has full rank. The Kronecker selection will not play a special role in this paper. Instead, arbitrary nice selections will be used under the condition that the corresponding submatrix of the reachability matrix has full rank. The existence of the Kronecker selection establishes that there is at least one such selection for each reachable system.

In the study of canonical forms the nice selections have often played an important role. Given a nice selection of columns of $R(A, B)$ the submatrix of $R(A, B)$ is considered which is obtained by deleting the columns of $R(A, B)$ whose labels are not in the nice selection. One of the key innovations of the present paper is not to work with this submatrix, but with a closely related matrix which is obtained by reordering the columns of this submatrix in a specific way.

We now give the details of the construction of this matrix. Let $(A, B, C, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ and let $d = (d_1, d_2, \dots, d_m)$ be a sequence of (n, m)-dynamical indices. Let $(s_1, \dots, s_l)$ be the corresponding sequence of step sizes. Let $(r_1, r_2, \dots, r_l)$ be the associated sequence of reversed step sizes with multiplicities $\rho_1, \dots, \rho_{h'}$ and let $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_{h'}$ be the same sequence but ignoring multiplicities. Let π be the ordering permutation of the sequence of dynamical indices $(d_1, \dots, d_m)$. Let $(\delta_1, \dots, \delta_h)$ be the corresponding multiplicities and let $(\tilde{d}_1, \dots, \tilde{d}_h)$ be the corresponding sequence of ordered dynamical indices ignoring the multiplicities. For $j = 1, \dots, l$, let

$$N_{l+1-j} := [A^{d_{\pi(1)}-j} Be_{\pi(1)}, A^{d_{\pi(2)}-j} Be_{\pi(2)}, \dots, A^{d_{\pi(s_j)}-j} Be_{\pi(s_j)}]$$

and let

$$N(A, B; d) := [N_1, N_2, \dots, N_l].$$

An interesting reformulation of $N(A, B; d)$ is given in the next lemma. To formulate the lemma we need the following notation which will also be used later. Let

$$\tilde{B} := [Be_{\pi(1)}, Be_{\pi(2)}, \dots, Be_{\pi(m)}]$$

and partition $\tilde{B}$ as

$$\tilde{B} = [\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_h],$$

where $\tilde{B}_j \in \mathbb{K}^{n\times\delta_j}, j = 1, 2, \ldots, h$.

Lemma 3.1. *With the above notation*

1. $N_1 = [Be_{\pi(1)}, Be_{\pi(2)}, \ldots, Be_{\pi(s_l)}] = \tilde{B}_1$.
2. *For* $j = 1, \ldots, l-1$,
   (a) *if* $s_j = s_{j+1}$, *then*
   $$N_{l+1-j} = AN_{l-1-(j+1)};$$
   (b) *if* $s_j > s_{j+1}$, *then*
   $$N_{l+1-j} = [AN_{l+1-(j-1)}, \tilde{B}_q],$$

where q is such that $\tilde{B}_q = [Be_{\pi(s_{j-1}+1)}, \ldots, Be_{\pi(s_j)}]$.

Proof. (1): Note that since $d_{\pi(i)} = l$ for $i = 1, 2, \ldots, s_l$

$$N_1 = [Be_{\pi(1)}, Be_{\pi(2)}, \ldots, Be_{\pi(s_l)}].$$

(2): Let $1 \leqslant j \leqslant l-1$ and assume that $s_j = s_{j+1}$. Then by Lemma 2.3 $d_{\pi(s_j)} \geqslant j+1$ and hence

$$\begin{aligned} N_{l+1-j} &= [A^{d_{\pi(1)}-j}Be_{\pi(1)}, A^{d_{\pi(2)}-j}Be_{\pi(2)}, \ldots, A^{d_{\pi(s_j)}-j}Be_{\pi(s_j)}] \\ &= A[A^{d_{\pi(1)}-(j+1)}Be_{\pi(1)}, A^{d_{\pi(2)}-(j+1)}Be_{\pi(2)}, \ldots, A^{d_{\pi(s_j)}-(j+1)}Be_{\pi(s_{j+1})}] \\ &= AN_{l+1-(j+1)}. \end{aligned}$$

If $1 \leqslant j \leqslant l-1$ and $s_j > s_{j+1}$ then by Lemma 2.3 $d_{\pi(s_j)} = j$ and hence

$$\begin{aligned} N_{l+1-j} &= [A^{d_{\pi(1)}-j}Be_{\pi(1)}, \ldots, A^{d_{\pi(s_{j+1})}-j}Be_{\pi(s_j+1)}, A^{d_{\pi(s_{j+1}+1)}-j}Be_{\pi(s_{j+1}+1)}, \ldots, \\ &\quad A^{d_{\pi(s_j)}-j}Be_{\pi(s_j)}] \\ &= [A^{d_{\pi(1)}-j}Be_{\pi(1)}, \ldots, A^{d_{\pi(s_{j+1})}-j}Be_{\pi(s_{j+1})}, Be_{\pi(s_{j+1}+1)}, \ldots, Be_{\pi(s_j)}] \\ &= [A[A^{d_{\pi(1)}-(j+1)}Be_{\pi(1)}, \ldots, A^{\pi(s_{j+1})-(j+1)}Be_{\pi(s_{j+1})}], \tilde{B}_q] \\ &= [AN_{l+1-(j+1)}, \tilde{B}_q], \end{aligned}$$

where q is such that $\tilde{B}_q = [Be_{\pi(s_{j+1}+1)}, \ldots, Be_{\pi(s_j)}]$. □

Example 3.1. Continuing the earlier example leaving the entries of (A, B) unspecified but writing $B = [b_1, \ldots, b_m]$, i.e. $b_i = Be_i$ is the ith column of B, one obtains

$$R(A, B) = [B, AB, A^2B, \ldots, A^{17}B],$$

$$N(A, B; d) = [N_1, N_2, N_3, N_4],$$

with

$$N_1 = [b_5, b_6]$$

an 18×2 matrix,

$$N_2 = [Ab_5, Ab_6, b_1, b_3]$$

an 18×4 matrix,

$$N_3 = [A^2b_5, A^2b_6, Ab_1, Ab_3, b_4]$$

an 18×5 matrix,

$$N_4 = [A^3b_5, A^3b_6, A^2b_1, A^2b_3, Ab_4, b_2, b_7]$$

an 18×7 matrix, where $(2, 4, 5, 7) = (r_1, r_2, r_3, r_4)$ is the reversed sequence of step sizes.

A simple method to decide whether a column $A^{j-1}Be_{\pi(i)}$ is in the nice selection and to find its *column label* in $N(A, B; d)$, proceeds as follows. Consider the Young diagram, represented here by the matrix Y. Consider the last row containing unit entries and change the last unit entry to n. Then proceed to the row above and change its last unit entry into $n - 1$. Proceed like this until the first row is reached, changing its last unit element into $n - s_1 + 1$. In the matrix that is obtained in this way go to the last row that contains units, which is the s_2th row and change its last unit element to $n - s_1$, etc. Proceed like this until the (1,1) element of the matrix is reached, which in fact remains 1.

Let the resulting matrix be denoted by $\tilde{Y} = (\tilde{y}_{ij})$. Then $A^{j-1}Be_{\pi(i)}$ is in the nice selection if $y_{ij} \neq 0$ and in fact $y_{ij} \neq 0$ is the column number of $A^{j-1}Be_{\pi(i)}$ in $N(A, B; d)$.

Example 3.2. Continuing the same example the following matrix is obtained,

$$\tilde{Y} = \begin{bmatrix} 1 & 3 & 7 & 12 \\ 2 & 4 & 8 & 13 \\ 5 & 9 & 14 & \\ 6 & 10 & 15 & \\ 11 & 16 & & \\ 17 & & & \\ 18 & & & \end{bmatrix},$$

where the empty entries are to be read as zeros. Recalling that

$$(\pi(1), \pi(2), \ldots, \pi(7)) = (5, 6, 1, 3, 4, 2, 7)$$

it turns out for example that $A^2Be_6 = A^2Be_{\pi(2)}$ is the 8th column of $N(A,B;d)$ because $\tilde{y}_{23} = 8$ while $A^4Be_1 = A^4Be_{\pi(3)}$ is not in the nice selection because $\tilde{y}_{53} = 0$.

Another matrix which is important for our development is obtained by a suitable reordering of the columns of the matrix $[B,A]$, where $(A,B,C,D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$. The precise reordering depends on a chosen sequence of (n,m)-dynamical indices $(d_1,d_2,\dots,d_m)$. Partition A into h' matrices

$$A = [A_1, A_2, \dots, A_{h'}],$$

where $A_i \in \mathbb{K}^{n\times p_i\tilde{r}_i}, i = 1,\dots,h'$. Then set

$$M(A,B;d) := [\tilde{B}_1, A_1, \tilde{B}_2, A_2, \dots, A_{h'}, \tilde{B}_h]$$

if $h' = h - 1$ and set

$$M(A,B;d) := [\tilde{B}_1, A_1, \tilde{B}_2, A_2, \dots, A_{h'-1}, \tilde{B}_h, A_{h'}]$$

if $h' = h$.

Let (A,B,C,D) be a n-dimensional system and let $d = (d_1,d_2,\dots,d_m)$ be a sequence of (n,m)-dynamical indices. In the following we are going to establish a close relationship between $N(A,B;d)$ and $M(A,B;d)$.

Theorem 3.1. *Let $(A,B,C,D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$. Then the following two statements are equivalent.*
1. *$N(A,B;d)$ is positive upper triangular.*
2. *$M(A,B;d)$ is simple positive upper triangular.*

Proof. In the course of this proof we will use the following notation. We denote by $U(v, v-1+w)$ the set of all matrices with w columns and n rows, such that if u_{ij} is the i,j-entry of $U(v,v-1+w)$ then

$$u_{ij} = \begin{cases} 0 & \text{if } i > j+v-1, \\ > 0 & \text{if } i = j+v-1, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Note that if a matrix is simple positive upper triangular, then the submatrix consisting of the w columns with labels $v, v+1, \dots, v-1+w$ is an element of $U(v, v-1+w)$.

We first show that (1) implies (2). Assume that $N(A,B;d)$ is positive upper triangular. We will recursively construct a matrix $\overline{M}_l$ which is simple positive upper triangular. The proof of the implication will then be finished by noting that $M(A,B;d)[I_n,0]^{\mathrm{T}} = \overline{M}$. Since $N_1 = \tilde{B}_1$ and N_1 is by assumption in $U(1,r_1)$ we have that

$$\overline{M}_1 := \tilde{B}_1 \in U(1, r_1).$$

If $l = 1$ then we are done. Suppose $l > 1$.

Let $1 \leqslant i \leqslant l-1$ and assume ('induction hypothesis') that $\overline{M}_i \in U(1, r_1 + \cdots + r_i)$ and $\overline{M}_i$ contains the matrices $\overline{A}_1, \ldots, \overline{A}_{i-1}$ that will be constructed recursively.

Case 1: $r_{i+1} = r_i$. By Lemma 3.1 we have that

$$N_{i+1} = AN_i,$$

where by assumption $N_i \in U(r_1 + \cdots + r_{i-1} + 1, r_1 + \cdots + r_i)$ (set $r_0 := 0$) and $N_{i+1} \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i+1})$. Write

$$A =: [A_{\mathscr{L}} \ \overline{A}_i \ A_{\mathscr{R}}]$$

with $A_{\mathscr{L}} = [\overline{A}_1, \ldots, \overline{A}_{i-1}]$ a $n \times (r_1 + \cdots + r_{i-1})$ matrix (if $i = 1$ then $A_{\mathscr{L}}$ is empty) and $\overline{A}_i$ is a $n \times r_i$ matrix. Because of the special structure of N_i we have that

$$N_{i+1} = AN_i = [A_{\mathscr{L}} \ \overline{A}_i \ A_{\mathscr{R}}]N_i = [A_{\mathscr{L}} \ \overline{A}_i \ 0]N_i.$$

For $i = 1$ the matrix $A_{\mathscr{L}}$ is empty and hence

$$\overline{A}_i \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i+1}).$$

For $i > 1$, it follows from the induction hypothesis that the matrix $A_{\mathscr{L}}$ is such that all rows are zero with the possible exception of some rows amongst the first $r_1 + \cdots + r_i$ rows. The above identity therefore also implies that for $i > 1$,

$$\overline{A}_i \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i+1}).$$

Hence

$$\overline{M}_{i+1} := [\overline{M}_i \ \overline{A}_i] \in U(1, r_1 + \cdots + r_{i+1}).$$

Case 2: $r_{i+1} > r_i$. By Lemma 3.1 we have that

$$N_{i+1} = \left[AN_i \ \tilde{B}_q\right],$$

for some q. By assumption $N_i \in U(r_1 + \cdots + r_{i-1} + 1, r_1 + \cdots + r_i)$ (set $r_0 := 0$) and $N_{i+1} \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i+1})$. Write

$$N_{i+1} =: [N_{i+1}^L \ N_{i+1}^R]$$

with N_{i+1}^L a $n \times r_i$ matrix and let $A =: [A_{\mathscr{L}} \ \overline{A}_i \ A_{\mathscr{R}}]$ as above. Then

$$N_{i+1}^L = AN_i$$

with $N_{i+1}^L \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i-1} + 2r_i)$. By the same argument as above we obtain that $\overline{A}_i \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i-1} + 2r_i)$. Since

$$N_{i+1} = [AN_i \ \tilde{B}_q] = [N_{i+1}^L \ \tilde{B}_q] \in U(r_1 + \cdots + r_i + 1, r_1 + \cdots + r_{i+1})$$

this implies that $\tilde{B}_q \in U(r_1 + \cdots + r_{i-1} + 2r_i + 1, r_1 + \cdots + r_{i+1})$. Hence

$$\overline{M}_{i+1} := [\overline{M}_i \ \overline{A}_i \ \tilde{B}_q] \in U(1, r_1 + \cdots + r_{i+1}).$$

We have therefore recursively constructed a matrix $\overline{M}_l \in U(1, r_1 + \cdots + r_l)$, i.e. $\overline{M}_l$ is simple positive upper triangular. It remains to note that $M(A, B; d) = [\overline{M}_l | *]$, because for each $i = 1, 2, \ldots, h'$, one has $A_i = [\overline{A}_{\rho_1 + \cdots + \rho_{i-1}+1}, \ldots, \overline{A}_{\rho_1 + \cdots + \rho_i}]$; these matrices all have $\tilde{r}_i$ columns, and so the total number of columns of A_i is indeed $\rho_i \tilde{r}_i$.

We now prove the converse direction, i.e. that (2) implies (1). We will show recursively that $N(A, B; d)$ is positive upper triangular. Note that by assumption $\tilde{B}_1 \in U(1, r_1)$. Hence $N_1 = \tilde{B}_1 \in U(1, r_1)$.

Case 1. Assume now that we have shown that for $1 \leqslant i \leqslant l - 1$

$$[N_1, N_2, \ldots, N_i] \in U(1, \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1} \tilde{r}_{j-1} + \gamma \tilde{r}_j)$$

for some choice of j with $1 \leqslant j \leqslant h'$ and γ with $1 \leqslant \gamma < \rho_j$; it follows that $r_{i+1} = r_i$. Hence by Lemma 3.1

$$N_{i+1} = AN_i.$$

Partition

$$A = [A_{\mathscr{L}} \ A_{\mathscr{M}} \ A_{\mathscr{R}}]$$

with $A_{\mathscr{L}}$ a $n \times (\rho_1 \tilde{r}_1 + \cdots + \rho_{j-1} \tilde{r}_{j-1} + (\gamma - 1)\tilde{r}_j)$ matrix and $A_{\mathscr{M}}$ a $n \times \tilde{r}_j$ matrix. By assumption $A_{\mathscr{M}} \in U(\tilde{r}_j + \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + (\gamma - 1)\tilde{r}_j + 1, \ \tilde{r}_j + \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + \gamma \tilde{r}_j)$ and $A_{\mathscr{L}}$ is such that all rows are zero with the possible exception of the first $\tilde{r}_j + \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + (\gamma - 1)\tilde{r}_j$ rows. Note that because of the special structure of N_i

$$\begin{aligned} N_{i+1} = AN_i = [A_{\mathscr{L}} \ A_{\mathscr{M}} \ A_{\mathscr{R}}]N_i = [A_{\mathscr{L}} \ A_{\mathscr{M}} \ 0]N_i \in U(\tilde{r}_j + \rho_1 \tilde{r}_1 \\ + \cdots + \rho_{j-1}\tilde{r}_{j-1} + (\gamma - 1)\tilde{r}_j + 1, \tilde{r}_j + \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + \gamma \tilde{r}_j). \\ = U(\rho_1 \tilde{r}_1 + \cdots + \gamma \tilde{r}_j + 1, \rho_1 \tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + (\gamma + 1)\tilde{r}_j). \end{aligned}$$

Hence

$$[N_1 \ \cdots \ N_{i+1}] \in U(1, \rho_1 \tilde{r}_1 + \cdots + (\gamma + 1)\tilde{r}_j)$$

which was to be shown.

Case 2. Assume now that we have shown that for $1 \leqslant i \leqslant l - 1$

$$[N_1 \ \cdots \ N_i] \in U(1, \rho_1 \tilde{r}_l + \cdots + \rho_j \tilde{r}_j),$$

for some $1 \leqslant j \leqslant h' - 1$. This implies that $r_{i+1} > r_i$. Hence by Lemma 3.1

$$N_{i+1} = \left[AN_i \ \tilde{B}_q\right],$$

where q is such that $\tilde{B}_q = \left[Be_{\pi(r_{i+1})}, \ldots, Be_{\pi(r_{i+1})}\right]$. In fact it follows that $r_i = \tilde{r}_j \ r_{i+1} = \tilde{r}_{j+1}$ and $\tilde{B}_q$ is the $n \times (\tilde{r}_{j+1} - \tilde{r}_j) = n \times \delta_{j+1}$ matrix $\tilde{B}_{j+1}$. The fact that $\tilde{r}_{j+1} - \tilde{r}_j = \delta_{j+1}$ for $1 \leqslant j \leqslant h' - 1$ follows from Lemma 2.3. Partition

$$A = [A_{\mathscr{L}} \ A_{\mathscr{M}} \ A_{\mathscr{R}}],$$

where $A_{\mathscr{L}}$ is a $n \times \rho_1\tilde{r}_1 + \cdots + (\rho_j - 1)\tilde{r}_j$ matrix and $A_{\mathscr{M}}$ is a $n \times \tilde{r}_j$ matrix. By assumption

$$A_{\mathscr{M}} \in U(\tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + \rho_{j-1}\tilde{r}_{j-1} + (\rho_j - 1)\tilde{r}_j + 1, \tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j)$$

and $A_{\mathscr{L}}$ is such that all rows are zero with the possible exception of some of the first $\tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + (\rho_j - 1)\tilde{r}_j$ rows. Because of the special structure of A we have that

$$\begin{aligned} AN_i &= [A_{\mathscr{L}} \ A_{\mathscr{M}} \ A_{\mathscr{R}}]N_i = [A_{\mathscr{L}} \ A_{\mathscr{M}} \ 0]N_i \\ &\in U(\tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + (\rho_j - 1)\tilde{r}_j + 1, \tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j) \\ &= U(\rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + 1, \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + \tilde{r}_j). \end{aligned}$$

By assumption

$$\tilde{B}_q \in U(\tilde{r}_j + \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + 1, \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + \tilde{r}_{j+1}).$$

Therefore

$$N_{i+1} = \left[AN_i, \tilde{B}_q\right] \in U(\rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + 1, \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + \tilde{r}_{j+1})$$

and hence

$$[N_1 \ N_2 \ \cdots \ N_{i+1}] \in U(1, \rho_1\tilde{r}_1 + \cdots + \rho_j\tilde{r}_j + \tilde{r}_{j+1}),$$

which completes this last part of the proof. □

4. Input-normal canonical forms

In this section we are going to introduce overlapping σ-input-normal canonical forms for the state space systems in the various classes, where the σ-input-normality is defined with respect to the relevant pair of Riccati or Lyapunov equations in each of the classes.

These canonical forms are the building blocks for the construction of block-balanced forms in Section 5. However the canonical forms presented here are of interest in their own right as canonical forms for the various classes. They can be considered as a far-reaching generalization of the Schwarz-like canonical form for stable single-input single-output systems (cf. [22,13]). For the class of stable systems (both single-input single-output and multivariable) the canonical form has the property that the C and D matrices can be chosen almost independently of the A and B with the only exception that the resulting system has to be observable. Therefore in optimization problems of a quadratic nature over the class of stable systems, C and D can be solved explicitly in terms of the solutions for A and B. For the class of SISO systems this has been exploited for example in [11]. These canonical forms are closely related to canonical forms for stable all-pass systems and we will show how overlapping *balanced* canonical forms for multivariable stable all-pass systems are obtained from the constructions presented here.

The existence and uniqueness properties of σ-input-normal realizations are established in the usual way [21].

Lemma 4.1. *Let $\sigma > 0$ and let $(A,B,C,D) \in L_n^{p,m}(S_n^{p,m}, B_n^{p,m}, P_n^m)$. Then there exists an equivalent LQG-σ-input-normal (Lyapunov-σ-input-normal, bounded-real-σ-input-normal, positive-real-σ-input-normal) system (A_1, B_1, C_1, D). All equivalent LQG-σ-input-normal (Lyapunov-σ-input-normal, bounded-real-σ-input-normal, positive-real-σ-input-normal) systems are given by $(QA_1Q^*, QB_1, C_1Q^*, D)$, Q unitary.*

In order to define a canonical form for the various classes of systems in terms of the respective input-normal canonical form we need to impose a further condition on input-normal systems.

Definition 4.1. Let $d = (d_1, \dots, d_m)$ be a sequence of (n,m)-dynamical indices. The set of all $(A,B,C,D) \in L_n^{p,m}(S_n^{p,m}, B_n^{p,m}, P_n^m)$ such that $N(A,B;d)$ is non-singular is denoted by $L_{n;d}^{p,m}(S_{n;d}^{p,m}, B_{n;d}^{p,m}, P_{n;d}^m)$.

In the following lemma some basic properties are collected of the sets $L_{n;d}^{p,m}, S_{n;d}^{p,m}, B_{n;d}^{p,m}$ and $P_{n;d}^m$.

Lemma 4.2. *Let $d = (d_1, \dots, d_m)$ be a sequence of (n,m)-dynamical indices. The sets $L_{n;d}^{p,m}(S_{n;d}^{p,m}, B_{n;d}^{p,m}, P_{n;d}^m)$ are open in $L_n^{p,m}(S_n^{p,m}, B_n^{p,m}, P_n^m)$. The union $\bigcup_{d\in\mathscr{D}(n;m)} L_{n;d}^{p,m}$ is covering all minimal systems of order n, i.e. $\bigcup_{d\in\mathscr{D}(n;m)} L_{n;d}^{p,m} = L_n^{p,m}$ and, similarly, $\bigcup_{d\in\mathscr{D}(n;m)} S_{n;d}^{p,m} = S_n^{p,m}$, $\bigcup_{d\in\mathscr{D}(n;m)} B_{n;d}^{p,m} = B_n^{p,m}$, $\bigcup_{d\in\mathscr{D}(n;m)} P_{n;d}^{m} = P_n^m$.*

Proof. Fix a sequence d of (n,m)-dynamical indices and consider det $N(A,B;d)$. It is a polynomial, hence continuous, function of the entries of (A,B) and the non-singularity of $N(A,B;d)$ can be expressed as det $N(A,B;d) \neq 0$. Therefore $L_{n;d}^{p,m}$ is an open subset of $L_n^{p,m}$. Similarly, $S_{n;d}^{p,m}, B_{n;d}^{p,m}$ and $P_{n;d}^m$ are open subsets of $S_n^{p,m}, B_n^{p,m}$ and P_n^m respectively.

Consider a system $(A,B,C,D) \in L_n^{p,m}(S_n^{p,m}, B_n^{p,m}, P_n^m)$. Its Kronecker reachability indices form a nice selection. Let d be the corresponding sequence of dynamical indices. Then the matrix $N(A,B;d)$ will be non-singular and therefore $(A,B,C,D) \in L_{n;d}^{p,m}(S_{n;d}^{p,m}, B_{n;d}^{p,m}, P_{n;d}^m)$. This shows the covering property. □

For each class of systems $L_{n;d}^{p,m}(S_{n;d}^{p,m}, B_{n;d}^{p,m}$, and $P_{n;d}^m)$, we now define an input-normal canonical form.

Definition 4.2. Let $\sigma > 0$, let $d = (d_1, d_2, \ldots, d_m)$ be a sequence of (n,m)-dynamical indices and let $(A,B,C,D) \in L^{p,m}_{n;d}(S^{p,m}_{n;d}, B^{p,m}_{n;d}, P^{m}_{n;d})$. Then the system is said to be in σ-input-normal LQG (Lyapunov, bounded-real, positive-real) canonical form corresponding to the dynamical indices $(d_1, \ldots, d_m)$ if

1. it is LQG σ-input-normal (Lyapunov σ-input-normal, bounded-real σ-input-normal, positive-real σ-input-normal), and
2. the square matrix $N(A,B;d)$ is positive upper triangular.

The set of all such systems is denoted by $\sigma\text{-}IL^{p,m}_{n;d}(\sigma\text{-}IS^{p,m}_{n;d}, \sigma\text{-}IB^{p,m}_{n;d}, \sigma\text{-}IP^{m}_{n;d})$.

That the term input-normal canonical form is justified is established in the following lemma.

Lemma 4.3. *Let $\sigma > 0$, let $d = (d_1, d_2, \ldots, d_m)$ be a sequence of (n,m)-dynamical indices and let $(A,B,C,D) \in L^{p,m}_{n;d}(S^{p,m}_{n;d}, B^{p,m}_{n;d}, P^{m}_{n;d})$. There exists a unique $(A_0, B_0, C_0, D_0) \in \sigma\text{-}IL^{p,m}_{n;d}(\sigma\text{-}IS^{p,m}_{n;d}, \sigma\text{-}IB^{p,m}_{n;d}, \sigma\text{-}IP^{m}_{n;d})$ that is equivalent to (A,B,C,D). The induced map $\Gamma^L_{\sigma;d}(\Gamma^S_{\sigma;d}, \Gamma^B_{\sigma;d}, \Gamma^P_{\sigma;d})$ that maps the system (A,B,C,D) to the system (A_0,B_0,C_0,D_0) is a canonical form.*

In the real case the canonical form $\Gamma^L_{\sigma;d}(\Gamma^S_{\sigma;d}, \Gamma^B_{\sigma;d}, \Gamma^P_{\sigma;d})$ is real analytic and hence continuous. In the complex case the canonical form is real analytic and hence continuous in the real parameters that are obtained by taking real and imaginary parts of the quantities involved.

Proof. Let $(A,B,C,D) \in L^{p,m}_{n;d}$. By Lemma 4.1 there exists an equivalent LQG σ-input-normal system (A_1,B_1,C_1,D_1). All other equivalent LQG σ-input-normal systems are given by $(QA_1Q^*, QB_1, C_1Q^*, D_1)$, where Q is unitary. The reachability matrices of these systems are given by $Q[B_1, A_1B_1, \ldots, A_1^{n-1}B_1]$. Since the matrix $N(A_1,B_1;d)$ is a permutation of a square submatrix of $R(A,B)$ which has full rank, it follows from Lemma 1.1 that there exists a unique unitary Q_0 such that $Q_0N(A_1,B_1;d)$ is square positive upper triangular. Let $(A_0,B_0,C_0,D_0) := (Q_0A_1Q_0^*, Q_0B_1, C_1Q_0^*, D_1)$. By the uniqueness of Q_0 it follows that (A_0,B_0,C_0,D_0) is the unique system that is equivalent to (A,B,C,D) and is in $\sigma\text{-}IL^{p,m}_{n;d}$. This implies the result for the case of systems in $L^{p,m}_{n;d}$. The result for the other classes of systems follows analogously.

We now need to consider the continuity and smoothness statements. To do this consider the above two-step procedure to obtain the canonical form, starting with a quadruple $(A,B,C,D) \in L^{p,m}_{n;d}$.

Let Z be the stabilizing solution of (L2). Then Z depends real-analytically on the entries of (A,B,C,D) (cf. [2,18]) and $(((1/\sigma)Z)^{-1/2}A((1/\sigma)Z)^{1/2}, ((1/\sigma)Z)^{-1/2}B, C((1/\sigma)Z)^{1/2}, D)$ is LQG σ-input normal. Here $Z^{1/2}$ is the positive hermitian square root of the matrix Z. The entries of $Z^{1/2}$ are real analytic functions of the entries of Z. This can be seen as follows.

Consider an arbitrary positive definite matrix Z_0. Let λ_0 be a real number larger than half the largest eigenvalue of Z_0. Consider the Taylor series expansion $z^{1/2} = \lambda^{1/2} + \sum_{k=1}^{\infty} \alpha_k (z-\lambda)^k$, $z \in (0, 2\lambda)$. It is well-known that this Taylor series has radius of convergence λ and therefore converges on the given interval. From [5] (vol. I, Section V.4, Theorem 2), it now follows that the power series expansion

$$Z^{1/2} = \lambda^{1/2} I + \sum_{k=1}^{\infty} \alpha_k (Z - \lambda I)^k$$

holds for any hermitian matrix Z which has as its largest eigenvalue a number smaller than 2λ, and therefore, if we take $\lambda = \lambda_0$, for all hermitian matrices in an open neighborhood of Z_0. So the real analyticity of $Z^{1/2}$ follows. Since Z and therefore $Z^{1/2}$ is invertible in an open neighborhood of Z_0, we also have the real analyticity of $Z^{-1/2}$.

The second step consists in the calculation of Q_0 from the matrix $N = N(((1/\sigma)Z)^{-1/2} A((1/\sigma)Z)^{1/2}, ((1/\sigma)Z)^{-1/2} B; d)$ such that $Q_0 N$ is square positive upper triangular. In fact, as can be shown easily by induction on the number of columns, Q_0^* is obtained from N by Gram–Schmidt orthonormalization of the columns $1, 2, \ldots, n$ of N. The Gram–Schmidt orthonormalization procedure only involves multiplication, addition, substraction, division, taking square roots of strictly positive quantities and matrix inversion, all of which are real analytic operations. The mapping $Q_0^* \mapsto Q_0$ is also real analytic in terms of the real and imaginary parts of the entries of Q_0^*. Therefore the canonical form is real analytic. □

We now formulate some properties of systems in σ-input-normal canonical form which will be used in the sequel to derive a parametrization.

Lemma 4.4. *Let $\sigma > 0$, let $d = (d_1, \ldots, d_m)$ be a sequence of (n, m)-dynamical indices. Let $(A, B, C, D) \in \sigma\text{-}IL_{n;d}^{p,m}$ ($\sigma\text{-}IS_{n;d}^{p,m}$, $\sigma\text{-}IB_{n;d}^{p,m}$, $\sigma\text{-}IP_{n;d}^{m}$). Then the statements* (1) *and* (2) *below hold simultaneously.*

1. *$M(A, B; d)$ is a simple positive upper triangular matrix, i.e. B and A have the following structure*:

B-matrix: *B is such that*

$$\tilde{B} := [Be_{\pi(1)}, \ldots, Be_{\pi(m)}] = [\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_h],$$

with $\tilde{B}_j$ an $n \times \delta_j$ matrix of the form:

$$\tilde{B}_1 = \begin{bmatrix} \tilde{b}_{11} & * & \cdots & * & * \\ 0 & \tilde{b}_{22} & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & \tilde{b}_{\tilde{r}_1,\tilde{r}_1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

with $\tilde{b}_{ii} > 0,\ i = 1,\dots,\tilde{r}_1,$

$$\tilde{B}_j = \begin{bmatrix} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ & * & \cdots & \cdots & * \\ \tilde{b}_{\tilde{r}_{j-1}+1+\rho_1\tilde{r}_1+\cdots+\rho_{j-1}\tilde{r}_{j-1},\tilde{r}_{j-1}+1} & * & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & \tilde{b}_{\tilde{r}_j+\rho_1\tilde{r}_1+\cdots+\rho_{j-1}\tilde{r}_{j-1},\tilde{r}_j} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

with $\tilde{b}_{\tilde{r}_{j-1}+i+\rho_1\tilde{r}_1+\cdots+\rho_{j-1}\tilde{r}_{j-1},\tilde{r}_{j-1}+i} > 0$ *for* $i = 1,\dots,\delta_j$, *for* $j = 1,\dots,h$. *If* $m > s_1 = \tilde{r}_{h'}$ *then* $\tilde{B}_h$ *is an* $n \times \delta_h$ *matrix without special structure.*

A-matrix:

$$A = [A_1,\dots,A_{h'}],$$

with A_j *an* $n \times (\rho_j\tilde{r}_j)$ *matrix* $j = 1,\dots,h'$. *For* $j = 1,\dots,h'-1$, A_j *is of the form*

$$A_j = \begin{bmatrix} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ & * & \cdots & \cdots & * \\ a_{\tilde r_j+\rho_1\tilde r_1+\cdots+\rho_{j-1}\tilde r_{j-1}+1,\rho_1\tilde r_1+\cdots+\rho_{j-1}\tilde r_{j-1}+1} & * & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & a_{\tilde r_j+\rho_1\tilde r_1+\cdots+\rho_j\tilde r_j,\rho_1\tilde r_1+\cdots+\rho_j\tilde r_j} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $a_{\tilde r_j+\rho_1\tilde r_1+\cdots+\rho_{j-1}\tilde r_{j-1}+i,\rho_1\tilde r_1+\cdots+\rho_{j-1}\tilde r_{j-1}+i} > 0$ for $i = 1, \ldots, \rho_j\tilde r_j$.

If $m > s_1 = \tilde r_{h'}$ then $A_{h'}$ can be partitioned as

$$A_{h'} = \left[A^l_{h'}, A^r_{h'}\right],$$

where $A^r_{h'}$ is an $n \times s_1 = n \times \tilde r_{h'}$ matrix without any special structure and $A^l_{h'}$ is an $n \times (\rho_{h'} - 1)\tilde r_{h'}$ matrix which is empty if $\rho_{h'} = 1$ and otherwise of the form

$$A^l_{h'} = \begin{bmatrix} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ & * & \cdots & \cdots & * \\ a_{\tilde r_{h'}+\rho_1\tilde r_1+\cdots+\rho_{h'-1}\tilde r_{h'-1}+1,\rho_1\tilde r_1+\cdots+\rho_{h'-1}\tilde r_{h'-1}+1} & * & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & a_{n,n-\tilde r_{h'}} \end{bmatrix}$$

with $a_{\tilde r_{h'}+\rho_1\tilde r_1+\cdots+\rho_{h'-1}\tilde r_{h'-1}+1,\rho_1\tilde r_1+\cdots+\rho_{h'-1}\tilde r_{h'-1}+1} > 0, \ldots, \tilde a_{n,n-\tilde r_{h'-1}} > 0$.

If $m = s_1 = \tilde r_{h'}$, then $A_{h'}$ is an $n \times \rho_{h'}\tilde r_{h'}$ matrix that can be partitioned as $A_{h'} = (A^l_{h'}, A^r_{h'})$ with $A^r_{h'}$ an unstructured $n \times m$ matrix and $A^l_{h'}$ an $n \times (\rho_{h'} - 1)m$ matrix which is empty if $\rho_{h'} = 1$ and otherwise is of the form

$$A_{h'}^{l} = \begin{bmatrix} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ & * & \cdots & \cdots & * \\ a_{m+\rho_1\tilde{r}_1+\cdots+\rho_{h'-1}\tilde{r}_{h'-1}+1,\rho_1\tilde{r}_1+\cdots+\rho_{h'-1}\tilde{r}_{h'-1}+1} & * & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & a_{n,n-m} \end{bmatrix}$$

with $a_{m+\rho_1\tilde{r}_1+\cdots+\rho_{h'-1}\tilde{r}_{h'-1}+i,\rho_1\tilde{r}_1+\cdots+\rho_{h'-1}\tilde{r}_{h'-1}+i} > 0$, *for* $i = 1, \ldots, (\rho_{h'} - 1)m$.

The entries in the matrices that are denoted by $*$ *are unique but not further specified.*

2. *A satisfies the following equations.*

(a) *In the* σ-$IL_{n;d}^{p,m}$*-case*:

$$A + A^* = BR_L^{-1}D^*C + C^*DR_L^{-1}B^* + \sigma C^*S_L^{-1}C - \frac{1}{\sigma}BR_L^{-1}B^*,$$

where R_L *and* S_L *are as in* (L3).

(b) *In the* σ-$IS_{n;d}^{p,m}$*-case*:

$$A + A^* = -\frac{1}{\sigma}BB^*.$$

(c) *In the* σ-$IB_{n;d}^{p,m}$*-case*:

$$A + A^* = -BR_B^{-1}D^*C - C^*DR_B^{-1}B^* - \sigma C^*S_B^{-1}C - \frac{1}{\sigma}BR_B^{-1}B^*,$$

where R_B *and* S_B *are as in* (B3).

(d) *In the* σ-$IP_{n;d}^{m}$*-case*:

$$A + A^* = BR_P^{-1}C + C^*R_P^{-1}B^* - \sigma C^*R_P^{-1}C - \frac{1}{\sigma}BR_P^{-1}B^*,$$

where R_P *as in* (P3).

Proof. This follows directly from Definition 4.2 and Theorem 3.1. □

We now come to consider parametrization issues. The main problem to be solved is to combine the simple positive upper triangular structure of $M(A, B; d)$ with the requirement that A solves the equation that determines input-normality for the particular class of systems which is being considered.

In fact the input-normality equations presented in Lemma 4.4 for the various classes of systems show a remarkable similarity. We will exploit this similarity in the construction of parametrizations for these classes. This can in fact be done by introducing the unique decomposition of A into a skew-hermitian matrix $\tilde{A}$ and an upper triangular matrix V with real entries on the main diagonal. The reason why this works is that the requirement that $M(A,B;d)$ is simple positive upper triangular puts no restrictions on the upper triangular part of A, only on the strictly lower triangular part of A.

Let $d = (d_1,\ldots,d_m)$ be a sequence of (n,m)-dynamical indices. Let $\mathrm{Skew}^{p,m}_{n;d}$ be the set of all $(\tilde{A},B,C,D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ such that $N(\tilde{A},B;d)$ is positive upper triangular and $\tilde{A}$ is skew-hermitian. Note that all systems in $\mathrm{Skew}^{p,m}_{n;d}$ are reachable but not necessarily observable. All systems in $\mathrm{Skew}^{p,m}_{n;d}$ are parametrized in an obvious way which is analogous to the matrix structures obtained in Lemma 4.4. Therefore in the real case $\mathrm{Skew}^{p,m}_{n;d}$ can be identified with $\mathbb{R}^n_+ \times \mathbb{R}^{n(m-1)+p(n+m)}$ and in the complex case with $\mathbb{R}^n_+ \times (\mathrm{i}\mathbb{R})^n \times \mathbb{C}^{n(m-1)+p(n+m)}$, where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$. For reasons of notational convenience we are going to consider the various classes of systems to be parametrized by elements in $\mathrm{Skew}^{p,m}_{n;d}$. But as just mentioned such a parametrization can be translated in a straightforward way to a parametrization by vectors in a Euclidean space.

We also introduce some notation to denote the allowable classes of feedthrough terms. Let Δ be the set of all $p \times m$ matrices with entries in $\mathbb{K}$; let Δ_b be the subset of all $D \in \Delta$, such that $I - D^*D > 0$ and let Δ_p be the set of all $m \times m$ matrices D, such that $D + D^* > 0$.

In order to obtain parametrizations for the different classes of systems we need to consider these classes separately. Given a system $(\tilde{A},B,C,D) \in \mathrm{Skew}^{p,m}_{n;d}$ we need to construct an upper triangular matrix V whose diagonal entries are real to give the finally parametrized matrix $A := \tilde{A} + V$. Let $\sigma > 0$.

1. In the $L^{p,m}_n$-case for $(\tilde{A},B,C,D) \in \mathrm{Skew}^{p,m}_{n;d}$, V is the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = BR_L^{-1}D^*C + C^*DR_L^{-1}B^* + \sigma C^*S_L^{-1}C - \frac{1}{\sigma}BR_L^{-1}B^*, \tag{1}$$

where R_L, S_L are as in (L3). Let $\tilde{\Theta}^{p,m}_{\sigma,d,L}$ be the subset of systems $(\tilde{A},B,C,D) \in \mathrm{Skew}^{p,m}_{n;d}$, such that $(A,B,C,D) := (\tilde{A}+V,B,C,D)$ is observable. Denote by $\Theta^{p,m}_{\sigma,d,L}$ the set of all such systems (A,B,C,D) and by $\phi_{\sigma,d,L}$ the parametrization map:

$$\begin{aligned} \phi_{\sigma,d,L} : \tilde{\Theta}^{p,m}_{\sigma,d,L} &\rightarrow \Theta^{p,m}_{\sigma,d,L} \subseteq L^{p,m}_n; \\ (\tilde{A},B,C,D) &\mapsto (A,B,C,D) := (\tilde{A}+V,B,C,D). \end{aligned} \tag{2}$$

2. In the $S^{p,m}_n$-case for $(\tilde{A},B,C,D) \in \mathrm{Skew}^{p,m}_{n;d}$, V is the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = -\frac{1}{\sigma} BB^*. \tag{3}$$

Let $\tilde{\Theta}^{p,m}_{\sigma,d,S}$ be the subset of systems $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{p,m}_{n;d}$ such that $(A, B, C, D) = (\tilde{A} + V, B, C, D)$ is observable. Denote by $\Theta^{p,m}_{\sigma,d,S}$ the set of all such systems (A, B, C, D) and by $\phi_{\sigma,d,S}$ the parametrization map:

$$\phi_{\sigma,d,S} : \tilde{\Theta}^{p,m}_{\sigma,d,S} \to \Theta^{p,m}_{\sigma,d,S} \subseteq S^{p,m}_n;$$
$$(\tilde{A}, B, C, D) \mapsto (A, B, C, D) := (\tilde{A} + V, B, C, D).$$

3. In the $B^{p,m}_n$-case for $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{p,m}_{n;d}$, with $D \in \Delta_b$, V is the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = -BR_B^{-1}D^*C - C^*DR_B^{-1}B^* - \sigma C^*S_B^{-1}C - \frac{1}{\sigma}BR_B^{-1}B^*, \tag{4}$$

where R_B, S_B are as in (B3). Let $\tilde{\Theta}^{p,m}_{\sigma,d,B}$ be the subset of systems $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{p,m}_{n;d}$ such that $D \in \Delta_b$, $(A, B, C, D) = (\tilde{A} + V, B, C, D)$ is observable and such that σI is the stabilizing solution of the bounded-real Riccati equation, i.e. such that $A + (BD + \sigma C^*)S_B^{-1}C$ is stable. Denote by $\Theta^{p,m}_{\sigma,d,B}$ the set of all such systems (A, B, C, D) and by $\phi_{\sigma,d,B}$ the parametrization map:

$$\phi_{\sigma,d,B} : \tilde{\Theta}^{p,m}_{\sigma,d,B} \to \Theta^{p,m}_{\sigma,d,B} \subseteq B^{p,m}_n;$$
$$(\tilde{A}, B, C, D) \mapsto (A, B, C, D) := (\tilde{A} + V, B, C, D).$$

4. In the P^m_n-case for $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{p,m}_{n;d}$, with $p = m$ and with $D \in \Delta_p$, V is the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = BR_P^{-1}C + C^*R_P^{-1}B^* - \sigma C^*R_P^{-1}C - \frac{1}{\sigma}BR_P^{-1}B^*, \tag{5}$$

where R_P as in (P3). Let $\tilde{\Theta}^{m}_{\sigma,d,P}$ be the subset of systems $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{m,m}_{n;d}$ such that $D \in \Delta_p$, $(A, B, C, D) = (\tilde{A} + V, B, C, D)$ is observable and such that σI is the stabilizing solution of the positive-real Riccati equation, i.e. such that $A - (B - \sigma C^*)R_P^{-1}C$ is stable. Denote by $\Theta^{m}_{\sigma,d,P}$ the set of all such systems (A,B,C,D) and by $\phi_{\sigma,d,P}$ the parametrization map:

$$\phi_{\sigma,d,P} : \tilde{\Theta}^{m}_{\sigma,d,P} \to \Theta^{m}_{\sigma,d,P} \subseteq P^{m}_n;$$
$$(\tilde{A}, B, C, D) \mapsto (A, B, C, D) := (\tilde{A} + V, B, C, D).$$

Note that in each of the above presented cases $\tilde{A}$ does not appear in the right-hand side of the equations that specify V and that V depends only on the parameters in B, C, D.

The fact that the inclusions $\Theta^{p,m}_{\sigma,d,L} \subseteq L^{p,m}_n$, $\Theta^{p,m}_{\sigma,d,S} \subseteq S^{p,m}_n$, $\Theta^{p,m}_{\sigma,d,B} \subseteq B^{p,m}_n$, and $\Theta^{m}_{\sigma,d,P} \subseteq P^m_n$ hold, so that minimality holds is actually shown in Lemma 4.5, together with a number of other properties. In fact we map into the σ-input normal forms of each of these classes.

Lemma 4.5. 1. *Let $\sigma > 0$. Let $d = (d_1, \ldots, d_m)$ be a sequence of (n, m)-dynamical indices and let (A, B, C, D) be in $\Theta^{p,m}_{\sigma,d,L}$. Then*
(a) *$N(A, B; d)$ is positive upper triangular and the system is reachable,*
(b) *$Z = \sigma I$ is a solution of the algebraic Riccati equation*

$$A_L Z + Z A_L^* - Z C^* S_L^{-1} C Z + B R_L^{-1} B^* = 0,$$

where A_L, R_L and S_L as in (L3),
(c) *$A - (BD^* + ZC^*)S_L^{-1}C$ is stable,*
(d) *$(A, B, C, D) \in \sigma\text{-}IL^{p,m}_{n:d}$.*

2. *Let $\sigma > 0$ and let (A, B, C, D) be in $\Theta^{p,m}_{\sigma,d,S}$. Then*
(a) *$N(A, B; d)$ is positive upper triangular and the system is reachable,*
(b) *$Z = \sigma I$ is a solution to the Lyapunov equation*

$$AZ + ZA^* = -BB^*.$$

(c) *A is stable,*
(d) *$(A, B, C, D) \in \sigma\text{-}IS^{p,m}_{n:d}$.*

3. *Let $\sigma > 0$ and let (A, B, C, D) be in $\Theta^{p,m}_{\sigma,d,B}$. Then*
(a) *$N(A, B; d)$ is positive upper triangular and the system is reachable,*
(b) *$Z = \sigma I$ is a solution of the bounded real algebraic Riccati equation*

$$A_B Z + Z A_B^* + Z C^* S_B^{-1} C Z + B R_B^{-1} B^* = 0,$$

where A_B, R_B and S_B are as in (B3),
(c) *A, A_B and $A_B + \sigma C^* S_B^{-1} C$ are stable.*
(d) *$(A, B, C, D) \in \sigma\text{-}IB^{p,m}_{n:d}$.*

4. *Let $\sigma > 0$ and let (A, B, C, D) be in $\Theta^{m}_{\sigma,d,P}$. Then*
(a) *$N(A, B; d)$ is positive upper triangular and the system is reachable,*
(b) *$Z = \sigma I$ is a solution of the positive real algebraic Riccati equation*

$$A_P Z + Z A_P^* + Z C^* R_P^{-1} C Z + B R_P^{-1} B^* = 0,$$

where A_P, R_P are as in (P3),
(c) *A, A_p and $A_P + \sigma C^* R_P^{-1} C$ are stable.*
(d) *$(A, B, C, D) \in \sigma\text{-}IP^{m}_{n:d}$.*

Proof. 1(a), 2(a), 3(a), 4(a): Let $(A, B, C, D) \in \Theta^{p,m}_{\sigma,d,L}$ ($\Theta^{p,m}_{\sigma,d,S}$; $\Theta^{p,m}_{\sigma,d,B}$; $\Theta^{m}_{\sigma,d,P}$) and let $\tilde{A}$ be skew-hermitian and V upper triangular with real diagonal entries such that $A = \tilde{A} + V$. By assumption *$M(\tilde{A}, B; d)$ is simple positive upper triangular. Since V* is upper triangular with real diagonal entries we have that $M(A, B; d)$ is also simple positive upper triangular and hence by Theorem 3.1 $N(A, B; d)$ is positive upper triangular and therefore non-singular. This shows that the system (A, B, C, D) is reachable.

1(b), (2)(b), (3)(b), (4)(b) is true by construction.

1(c): As just noted in 1(b), $Z = \sigma I$ solves the equation

$$A_L Z + Z A_L^* - Z C^* S_L^{-1} C Z + B R_L^{-1} B^* = 0.$$

This equation can be rewritten as

$$0 = (A_L - ZC^*S_L^{-1}C)Z + Z(A_L - ZC^*S_L^{-1}C)^* \\ + (ZC^*S_L^{-1/2},\ BR_L^{-1/2})\,(ZC^*S_L^{-1/2},\ BR_L^{-1/2})^*.$$

By a standard Lemma on the Lyapunov equation (see, e.g., [29]), since Z is positive definite, $(A_L - ZC^*S_L^{-1}C)$ is stable if and only if the system

$$\big((A_L - ZC^*S_L^{-1}C)\,(ZC^*S_L^{-1},\ BR_L^{-1})\big)$$

is reachable. Assume that this system is not reachable, then there exists a vector $x \neq 0$ such that for some scalar λ

$$x^*(A_L - ZC^*S_L^{-1}C) = \lambda x^*, \qquad x^*(ZC^*S_L^{-1},\ BR_L^{-1}) = 0.$$

Hence $x^*ZC^*S_L^{-1} = 0$, $x^*BR_L^{-1} = 0$ and hence $x^*B = 0$. Therefore

$$\lambda x^* = x^*(A_L - ZC^*S_L^{-1}C) = x^*A_L = x^*A,$$

which implies that (A,B) is not reachable, which is a contradiction. Hence 1(c).

2(c): This statement follows immediately from the well-known result on the Lyapunov equation that A is stable if and only if (A,B) is reachable.

3(c): Note that the bounded real equation can be rewritten as

$$0 = AZ + ZA^* + BB^* + (ZC^* + BD^*)S_B^{-1}(ZC^* + BD^*)^* \\ = AZ + ZA^* + \left(B,\ (ZC^* + BD^*)S_B^{-1/2}\right)\left(B,\ (ZC^* + BD^*)S_B^{-1/2}\right)^*.$$

Since Z is positive definite, A is stable if $(A,(B,\ (ZC^* + BD^*)S_B^{-1/2}))$ is reachable, which is the case since (A,B) is reachable. That A_B is stable follows from the analogous argument applied to the equation

$$A_B Z + Z A_B^* + ZC^*S_B^{-1}CZ + BR_B^{-1}B^* = 0.$$

The stability of $A_B + \sigma C^* S_B^{-1} C$ follows directly from the definition of $\Theta^{p,m}_{\sigma,d,B}$.

4(c): These statements follow analogously to the statements in 3(c).

1(d), 2(d), 3(d), 4(d): In each of the four cases, by construction the system is observable and hence minimal, using the results in parts (a). Therefore the system is indeed of order n. Furthermore $N(A,B;d)$ is non-singular. In case 2(d) it follows directly from 2(c) that the system is stable. In case 3(d) it follows from the existence of a positive definite stabilizing solution of the bounded-real Riccati equation, that the system is bounded-real (see, e.g., [31]). Analogously in case 4(d) it follows that the system is positive real.

From parts (b) it follows in cases 1(d), 2(d), 3(d), 4(d) respectively that the system is LQG σ-input-normal, Lyapunov σ-input-normal, bounded-real σ-input-normal, positive-real σ-input-normal, respectively. Finally from (a) we have that $N(A,B;d)$ is in fact positive upper triangular, so in case 1(d), 2(d), 3(d), 4(d), the system is in σ-input-normal LQG (Lyapunov, bounded-real, positive real) canonical form, i.e.

$$(A,B,C,D) \in \sigma\text{-}IL^{p,m}_{n;d}(\sigma\text{-}IS^{p,m}_{n;d}, \sigma\text{-}IB^{p,m}_{n;d}, \sigma\text{-}IP^{p,m}_{n;d}). \qquad \square$$

We can now conclude that we have constructed a well-defined parametrization of the σ-input-normal canonical forms:

Corollary 4.1. *The following equalities of sets hold:*

$$\Theta^{p,m}_{\sigma,d,L} = \sigma\text{-}IL^{p,m}_{n;d}, \quad \Theta^{p,m}_{\sigma,d,S} = \sigma\text{-}IS^{p,m}_{n;d}, \quad \Theta^{p,m}_{\sigma,d,B} = \sigma\text{-}IB^{p,m}_{n;d}, \quad \Theta^{p,m}_{\sigma,d,P} = \sigma\text{-}IP^{p,m}_{n;d}.$$

Proof. From Lemma 4.4 it follows in a straightforward manner that if $(A,B,C,D) \in \sigma\text{-}IL^{p,m}_{n;d}(\sigma\text{-}IS^{p,m}_{n;d}, \sigma\text{-}IB^{p,m}_{n;d}, \sigma\text{-}IP^{m}_{n;d})$ then one can construct an upper triangular matrix V with real diagonal entries as described in (Eq. (1), Eq. (3), Eq. (4), Eq. (5)) such that $A - V$ is skew-hermitian. From the structure of $M(A,B;d)$ the structure of $\tilde{A} = A - V$ follows, and clearly $(\tilde{A},B,C,D) \in \tilde{\Theta}^{p,m}_{\sigma,d,L}(\tilde{\Theta}^{p,m}_{\sigma,d,S}, \tilde{\Theta}^{p,m}_{\sigma,d,B}, \tilde{\Theta}^{p,m}_{\sigma,d,P})$ and $(A,B,C,D) \in \Theta^{p,m}_{\sigma,d,L}(\Theta^{p,m}_{\sigma,d,S}, \Theta^{p,m}_{\sigma,d,B}, \Theta^{p,m}_{\sigma,d,P})$. Using Lemma 4.5 the equalities follow. $\square$

Lemma 4.6 shows that the parameter sets $\tilde{\Theta}^{p,m}_{\sigma,d,L}(\tilde{\Theta}^{p,m}_{\sigma,d,S}, \tilde{\Theta}^{p,m}_{\sigma,d,B}, \tilde{\Theta}^{p,m}_{\sigma,d,P})$ are relatively open subsets in the set $\text{Skew}^{p,m}_{n;d}$. Therefore the corresponding parameter vectors lie in an open subset of Euclidean space. This will be of importance to show that one can obtain an atlas for the various manifolds of systems using these parameter sets.

Lemma 4.6. *Let $\sigma > 0$ and let $d = (d_1, \ldots, d_m)$ be a sequence of (n,m)-dynamical indices. The parameter set $\tilde{\Theta}^{p,m}_{\sigma,d,L}(\tilde{\Theta}^{p,m}_{\sigma,d,S}, \tilde{\Theta}^{p,m}_{\sigma,d,B}, \tilde{\Theta}^{m}_{\sigma,d,P})$ is a relatively open subset of the set $\text{Skew}^{p,m}_{n;d}$ (with $p = m$ in the positive-real case). For the bounded-real and positive-real case we assume that the set $\text{Skew}^{p,m}_{n;d}$ refers to those systems $(A,B.C,D)$ in $\text{Skew}^{p,m}_{n;d}$ with $D \in \Delta_b$, $D \in \Delta_p$ respectively.*

Proof. The parametrization maps $\phi_{\sigma,d,L}$, $\phi_{\sigma,d,S}$, $\phi_{\sigma,d,B}$ and $\phi_{\sigma,d,P}$ were defined on the subsets $\tilde{\Theta}^{p,m}_{\sigma,d,L}$, $\tilde{\Theta}^{p,m}_{\sigma,d,S}$, $\tilde{\Theta}^{p,m}_{\sigma,d,B}$ and $\tilde{\Theta}^{m}_{\sigma,d,P}$ of $\text{Skew}^{p,m}_{n;d}$. Using the same constructions these maps can be considered to be defined on the whole set $\text{Skew}^{p,m}_{n;d}$

$$\text{Skew}^{p,m}_{n;d} \to \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m},$$

$$(\tilde{A},B,C,D) \mapsto (\tilde{A} + V, B, C, D).$$

Clearly these extensions are continuous maps.

Let $\mathcal{O}$ denote the open subset of observable quadruples in $\mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$. Because $\tilde{\Theta}^{p,m}_{\sigma,d,L}$ is the pre-image of $\mathcal{O}$ under the extension of the mapping $\phi_{\sigma,d,L}$, it is relatively open in $\mathrm{Skew}^{p,m}_{n;d}$. Similarly, because $\tilde{\Theta}^{p,m}_{\sigma,d,S}$ is the pre-image of $\mathcal{O}$ under the extension of the mapping $\phi_{\sigma,d,S}$, it is relatively open in $\mathrm{Skew}^{p,m}_{n;d}$. For the proof for the bounded real class one can use the fact that the mapping

$$s : (A, B, C, D) \mapsto \max\{\mathrm{Re}(\lambda) \mid \lambda \in \mathrm{spec}(A + (BD + \sigma C^*)S_B^{-1}C)\},$$

is continuous and the pre-image $\mathcal{S}_B := s^{-1}(]-\infty, 0[)$ is open. Because $\tilde{\Theta}^{p,m}_{\sigma,d,B}$ is the pre-image of the intersection $\mathcal{S}_B \cap \mathcal{O}$ under the extension of the mapping $\phi_{\sigma,d,B}$, this set is relatively open in $\mathrm{Skew}^{p,m}_{n;d}$. That $\tilde{\Theta}^{p,m}_{\sigma,d,P}$ is relatively open in $\mathrm{Skew}^{p,m}_{n;d}$ is shown similarly. □

Consider the set $L^{p,m}_{n;d}(S^{p,m}_{n;d}, B^{p,m}_{n;d}, P^{m}_{n;d})$. In the following Theorem a number of maps are shown to be real analytic. If the underlying field is $\mathbb{K} = \mathbb{C}$ then we identify $\mathbb{C}$ with $\mathbb{R}^2$ and interpret real analyticity in this way.

Theorem 4.1. *Let* $\sigma > 0$.

1. *The collection*

$$\left\{\Gamma^{L}_{\sigma;d} : L^{p,m}_{n;d} \to \sigma\text{-}IL^{p,m}_{n;d} \mid d \in \mathcal{D}(n;m)\right\}$$

forms an overlapping set of (real analytic) canonical forms covering $L^{p,m}_{n}$. *Similarly*

$$\left\{\Gamma^{S}_{\sigma;d} : S^{p,m}_{n;d} \to \sigma\text{-}IS^{p,m}_{n;d} \mid d \in \mathcal{D}(n;m)\right\},$$

$$\left\{\Gamma^{B}_{\sigma;d} : B^{p,m}_{n;d} \to \sigma\text{-}IB^{p,m}_{n;d} \mid d \in \mathcal{D}(n;m)\right\},$$

$$\left\{\Gamma^{P}_{\sigma;d} : P^{m}_{n;d} \to \sigma\text{-}IP^{m}_{n;d} \mid d \in \mathcal{D}(n;m)\right\},$$

respectively, form an overlapping set of (real analytic) canonical forms covering $S^{p,m}_{n}$, $B^{p,m}_{n}$, P^{m}_{n} *respectively.*

2. *The mappings*

$$\phi_{\sigma,d,L} : \tilde{\Theta}^{p,m}_{\sigma,d,L} \to \sigma\text{-}IL^{p,m}_{n;d}, \qquad \phi_{\sigma,d,S} : \tilde{\Theta}^{p,m}_{\sigma,d,S} \to \sigma\text{-}IS^{p,m}_{n;d},$$

$$\phi_{\sigma,d,B} : \tilde{\Theta}^{p,m}_{\sigma,d,B} \to \sigma\text{-}IB^{p,m}_{n;d}, \qquad \phi_{\sigma,d,P} : \tilde{\Theta}^{m}_{\sigma,d,P} \to \sigma\text{-}IP^{m}_{n;d}$$

are real analytic diffeomorphisms for all $d \in \mathcal{D}(n;m)$.

3. *Let*

$$\hat{\phi}_{\sigma,d,L} : \tilde{\Theta}^{p,m}_{\sigma,d,L} \to L^{p,m}_{n;d}/\sim, \qquad \theta \mapsto \phi_{\sigma,d,L}(\theta)/\sim$$

denote the mapping from the parameter space into the set of equivalence classes of state space systems. In an obvious notation, let $\hat{\phi}_{\sigma,d,S}$, $\hat{\phi}_{\sigma,d,B}$, $\hat{\phi}_{\sigma,d,P}$ *denote the* *analogous mappings for the spaces* $S^{p,m}_{n;d}/\sim, B^{p,m}_{n;d}/\sim, P^{m}_{n;d}/\sim$, *respectively. The collection*

$$\left\{\hat{\phi}^{-1}_{\sigma,d,L} : L^{p,m}_{n;d}/\sim \;\rightarrow\; \tilde{\Theta}^{p,m}_{\sigma,d,L} \mid d \in \mathscr{D}(n;m)\right\}$$

forms a real analytic atlas of the set of input-output systems $L^{p,m}_{n}/\sim$.

The collection

$$\left\{\hat{\phi}^{-1}_{\sigma,d,S} : S^{p,m}_{n;d}/\sim \;\rightarrow\; \tilde{\Theta}^{p,m}_{\sigma,d,S} \mid d \in \mathscr{D}(n;m)\right\}$$

forms a real analytic atlas of $S^{p,m}_{n}/\sim$;

$$\left\{\hat{\phi}^{-1}_{\sigma,d,B} : B^{p,m}_{n;d}/\sim \;\rightarrow\; \tilde{\Theta}^{p,m}_{\sigma,d,B} \mid d \in \mathscr{D}(n;m)\right\}$$

forms a real analytic atlas of $B^{p,m}_{n}/\sim$ *and*

$$\left\{\hat{\phi}^{-1}_{\sigma,d,P} : P^{m}_{n;d}/\sim \;\rightarrow\; \tilde{\Theta}^{m}_{\sigma,d,P} \mid d \in \mathscr{D}(n;m)\right\}$$

forms a real analytic atlas of $P^{m}_{n}/\sim$;

Proof. (1): This follows directly from Lemmas 4.2 and 4.3. Because the sets $L^{p,m}_{n;d}$ respectively $S^{p,m}_{n;d}$, $B^{p,m}_{n;d}$, $P^{m}_{n;d}$ are open we have an *overlapping* set of (real analytic) canonical forms.

(2): From Corollary 4.1 we know that the range space of the mapping $\phi_{\sigma,d,L}$, ($\phi_{\sigma,d,S}$, $\phi_{\sigma,d,B}$, $\phi_{\sigma,d,P}$) is equal to σ-$IL^{p,m}_{n;d}$, (σ-$IS^{p,m}_{n;d}$, σ-$IB^{p,m}_{n;d}$, σ-$IP^{m}_{n;d}$). The mappings $\phi_{\sigma,d,L}$, $\phi_{\sigma,d,S}$, $\phi_{\sigma,d,B}$, $\phi_{\sigma,d,P}$ as well as the corresponding inverse mappings are in fact rational mappings in several real variables which are by construction without singularities and are therefore real analytic and by construction they are bijective. Therefore the mappings are real analytic diffeomorphisms.

(3): From (1) and (2) it follows that the mappings $\hat{\phi}_{\sigma,d,L}$ ($\hat{\phi}_{\sigma,d,S}$, $\hat{\phi}_{\sigma,d,B}$, $\hat{\phi}_{\sigma,d,P}$) are bijections. To verify that we have a real analytic atlas it remains to check that coordinate changes are real analytic.

Let $d, \bar{d} \in \mathscr{D}(n;m), d \neq \bar{d}$, denote two distinct sequences of (n,m)-dynamical indices and let V denote the intersection $V = (L^{p,m}_{n;d}/\sim) \cap (L^{p,m}_{n;\bar{d}}/\sim)$ and $W := \hat{\phi}^{-1}_{\sigma,d,L}(V)$. Then $\phi_{\sigma,d,L}(W) = \sigma\text{-}IL^{p,m}_{n;d} \cap L^{p,m}_{n;\bar{d}}$, which is a relatively open set in σ-$IL^{p,m}_{n;d}$. Therefore, using (2) it follows that W is relatively open in $\tilde{\Theta}^{p,m}_{\sigma,d,L}$. Consider the following composition of mappings $\hat{\phi}^{-1}_{\sigma,\bar{d},L} \circ \hat{\phi}_{\sigma,d,L}|W = \phi^{-1}_{\sigma,\bar{d},L} \circ \phi_{\sigma,d,L}|W$. It follows from (2) that this coordinate change is a real analytic map. As d and $\bar{d}$ were arbitrary elements from $\mathscr{D}(n;m)$, it follows that the collection of mappings $\{\hat{\phi}^{-1}_{\sigma,d,L} : L^{p,m}_{n;d}/\sim \;\rightarrow\; \tilde{\Theta}^{p,m}_{\sigma,d,L} \mid d \in \mathscr{D}(n;m)\}$ forms a real analytic atlas for the set of input–output systems $L^{p,m}_{n}/\sim$. The proof for the other classes of systems is analogous. □

The class of systems with only one singular value, σ, say, forms precisely the class of systems for which the σ-input-normal canonical forms are balanced.

They will be characterized in Lemma 4.7. Note that the bounded-real singular values and the positive real singular values are known to be smaller than one (cf., e.g., [24]).

Lemma 4.7. *Let $\sigma > 0$. Let $d = (d_1, d_2, \dots, d_m)$ be a sequence of (n, m)*-dynamical indices and let $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{p,m}_{n;d}$.

1. *The matrix C is such that*

$$C^* S_L^{-1} C = B R_1^{-1} B^*,$$

where R_L, S_L as in (L3), *if and only if the system*

$$(A, B, C, D) := (\tilde{A} + V, B, C, D)$$

with V the upper triangular matrix with real diagonal entries such that

$$V + V^* = B R_L^{-1} D^* C + C^* D R_L^{-1} B^* + \sigma C^* S_L^{-1} C - \frac{1}{\sigma} B R_L^{-1} B^*,$$

where R_L, S_L are as in(L3) *is in $\Theta^{p,m}_{\sigma,d,L}$ and is LQG-balanced with LQG-gramian $\Sigma_L = \sigma I$.*

2. *The matrix C is such that*

$$C^* C = B B^*,$$

if and only if the system

$$(A, B, C, D) := (\tilde{A} + V, B, C, D)$$

with V the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = -\frac{1}{\sigma} B B^*$$

is in $\Theta^{p,m}_{\sigma,d,S}$ and is Lyapunov-balanced with Lyapunov-gramian $\Sigma_S = \sigma I$.

3. *Let $0 < \sigma < 1$ and $D \in \Delta_b$. Then the matrix C is such that*

$$C^* S_B^{-1} C = B R_B^{-1} B^*,$$

where R_B, S_B as in (B3), *if and only if the system*

$$(A, B, C, D) := (\tilde{A} + V, B, C, D)$$

with V the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = -B R_B^{-1} D^* C - C^* D R_B^{-1} B^* - \sigma C^* S_B^{-1} C - \frac{1}{\sigma} B R_B^{-1} B^*,$$

where R_B, S_B are as in (B3) *is in $\Theta^{p,m}_{\sigma,d,B}$ and is bounded-real-balanced with bounded real-gramian $\Sigma_B = \sigma I$.*

4. *Let $0 < \sigma < 1$* and *$D \in \Delta_p$. Then C is such that*

$$C^* R_P^{-1} C = B R_P^{-1} B^*,$$

where R_P is as in (P3), *if and only if the system*

$$(A, B, C, D) := (\tilde{A} + V, B, C, D)$$

with V the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = BR_P^{-1}C + C^*R_P^{-1}B^* - \sigma C^*R_P^{-1}C - \frac{1}{\sigma}BR_P^{-1}B^*,$$

where R_P as in (P3), is in $\Theta^m_{\sigma,d,P}$ and is positive-real-balanced with positive-real-gramian $\Sigma_P = \sigma I$.

Proof. The 'if'-part follows in all four cases straightforwardly from the balancedness with scalar gramians: substituting $Y = Z = \sigma I$ in (S1), (S2) one immediately obtains the equality $C^*C = BB^*$. A similar argument holds for the other cases; in the bounded real case and the positive real case one has to use $0 < \sigma < 1$. The 'only if' part can be shown as follows.

(1): By construction the system (A, B, C, D) is reachable and σI is a solution to the Riccati equation (L2). Since by assumption $C^*S_L^{-1}C = BR_L^{-1}B^*, \sigma I$ also solves the dual equation (L1). If we show that the system is also observable, it follows that the system is a minimal system in LQG-balanced form and therefore it follows easily (just as in [24]) that Y, Z are the stabilizing solutions of the Riccati equations (L1), (L2) and that the system is in $\Theta^{p,m}_{\sigma,d,L}$. In fact to show observability we will use stability of the matrix $A_L - \sigma C^*S_L^{-1}C$ (which implies directly that $Z = \sigma I$ is the stabilizing solution of (L2)). The proof of the stability of this matrix is completely analogous to the proof of Lemma 4.5, part 1 and is left to the reader. With $Z = \sigma I$ the algebraic Riccati equation can also be rewritten as

$$0 = (A_L - ZC^*S_L^{-1}C)Z + Z(A_L - C^*S_L^{-1}C)^* + ZC^*S_L^{-1}CZ + BR_L^{-1}B^*,$$

so

$$0 = (A_L - ZC^*S_L^{-1}C)Z + Z(A_L - C^*S_L^{-1}C)^* + ZC^*S_L^{-1}CZ + C^*S_L^{-1}C.$$

Assume that (A, B, C, D) is not observable. Then there exists $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ such that

$$Cx = 0, \quad Ax = \lambda x.$$

Therefore also $\lambda x = Ax = (A - BR_L^{-1}D^*C)x = A_L x$. Multiplying the above Riccati equation on the left by x^* and on the right by x we obtain that

$$\begin{aligned} 0 &= x^*\big((A_L - ZC^*S_L^{-1}C)Z + Z(A_L - C^*S_L^{-1}C)^* + ZC^*S_L^{-1}CZ + C^*S_L^{-1}C\big)x. \\ &= \sigma(\lambda x^*x + \bar{\lambda}x^*x) \\ &= 2\sigma \mathrm{Re}(\lambda)x^*x. \end{aligned}$$

Since $x \neq 0$, $\sigma \neq 0$ we have that $\mathrm{Re}(\lambda) = 0$. Note that

$$(A_L - \sigma C^* S_L^{-1} C)x = \lambda x$$

which is a contradiction to $A_L - \sigma C^* S_L^{-1} C$ being stable. Hence the system (A, B, C, D) is observable.

(2)–(4): This follows by arguments analogous to those in (1). □

Remark [*Stable all-pass systems*]. A square stable system is called *stable all-pass* if the corresponding transfer matrix has the property $G(\mathrm{i}w)^* G(\mathrm{i}w) = I$ for all $w \in \mathbb{R}$. A well-known theorem, which can be found, e.g., in [6,7], states that if $p = m$, a Lyapunov balanced triple (A, B, C) has identical singular values $\sigma = 1$ if and only if there exists a matrix D such that (A, B, C, D) represents a stable all-pass system. In such a case D is unitary, i.e. $D^* = D^{-1}$, and $C = -DB^*$. Let $d = (d_1, \ldots, d_m)$ be a sequence of (n, m)-dynamical indices. Let $\tilde{\Theta}^m_{n,d,A}$ be the set of all $(\tilde{A}, B, C, D) \in \mathrm{Skew}^{m,m}_{n;d}$ such that
- $D \in \mathbb{K}^{m\times m}$ is unitary.
- $C = -DB^*$

Let $\Theta^m_{n,d,A}$ be the set of all

$$(A, B, C, D) = (\tilde{A} + V, B, C, D),$$

where $(\tilde{A}, B, C, D) \in \tilde{\Theta}^m_{n,d,A}$ and V is the unique upper triangular matrix with real diagonal entries such that

$$V + V^* = -BB^*.$$

Note that by Lemma 4.7 each system $(A, B, C, D) \in \Theta^m_{n,d,A}$ is stable and minimal, i.e. is in $S^{m,m}_n$ and is all-pass. We have therefore constructed a Lyapunov balanced canonical form and parametrization of all-pass systems with dynamical indices d. Because of the simple geometric structure of $\tilde{\Theta}^m_{n,d,A}$, this set is in fact of the form $\mathbb{R}^n_+ \times (\mathrm{i}\mathbb{R})^n \times \mathbb{C}^{n(m-1)} \times U(m)$ in the complex case, where $U(m)$ denotes the group of $m \times m$ unitary matrices. In the real case $\tilde{\Theta}^m_{n,d,A}$ is of the form $\mathbb{R}^n_+ \times \mathbb{R}^{n(m-1)} \times O(m)$, where $O(m)$ denotes the group of orthogonal $m \times m$ matrices. Note that despite the simple structure, no loss of minimality nor loss of stability occurs within $\tilde{\Theta}^m_{n,d,A}$ and all systems parametrized in this way are Lyapunov balanced stable all-pass systems. By considering all possible dynamical indices an overlapping canonical form and an atlas can be obtained for the manifold of $m \times m$ all-pass systems of McMillan degree n, where the charts can be obtained in a straightforward way from the sets $\tilde{\Theta}^m_{n,d,A}$, $d \in \mathscr{D}(n; m)$, using any atlas for the unitary or orthogonal group, respectively.

5. Block-balanced input-normal canonical forms

In the previous section σ-input-normal forms were constructed for the various classes of systems. From Lemma 4.7 it follows that in case the system has all its singular values coinciding the σ-input normal canonical forms are balanced, up to a scalar scaling transformation of the state vector. In fact in that case $Z=\sigma I$ and $Y=(\sigma_1^2/\sigma)I$, while $\sigma_1=\sigma_2=\cdots=\sigma_n$. Furthermore the σ-input-normal forms are continuous on an open set around such a point. This is not true in general for balanced canonical forms. In optimization tasks such as occuring in system identification and model reduction procedures, we therefore propose to use the σ-input normal forms in case all singular values are close together. If the singular values are further apart one may again use the balanced canonical form. However if only a number of singular values are clustered, then the idea is to use the corresponding σ-input normal form for the subsystem in which these singular values are clustered. In this section we are going to develop this idea of overlapping block-balanced canonical forms. An envisaged application for these overlapping block-balanced canonical forms that we will construct in this section is that in optimization tasks one can use the balanced canonical form until a number of singular values start to cluster, then one can switch to the appropriate block-balanced canonical form. The construction of the block-balanced forms is such that they are in fact balanced (up to a trivial scalar transformation) if the clustering singular values actually coincide.

The block-balanced input-normal canonical forms that we are going to study in this section can in general not be defined for the full classes of systems that were studied above. We therefore need to define the subclasses of systems for which these block forms can be introduced. Let $n_1,n_2,\ldots,n_k$ be positive integers such that $\sum_{i=1}^{k} n_i=n$.

In order to define the subclasses we need to put conditions on the singular values of the systems in such a subclass. It is well-known that the LQG-singular values (resp. Lyapunov singular values, bounded-real singular values, positive-real singular values) are system invariants. Therefore we can characterize systems using their singular values whether they are balanced or not. The subclasses will be defined by a partitioning property of the singular values, namely that the singular values can be partitioned into k subsets which can be ordered with respect to the value of their elements, because comparing any two sets, the smallest element in one of the sets is larger then the largest element in the other set. The precise definition is as follows. The set $L^{p,m}_{n_1,n_2,\ldots,n_k}$ of systems is defined as the set of all systems in $L^{p,m}_n$, for which the n positive LQG-singular values satisfy the strict inequalities

$$\sigma_{\sum_{i=1}^{j} n_i} > \sigma_{(\sum_{i=1}^{j} n_i)+1}, \quad j=1,\ldots,k-1, \tag{6}$$

as well as the usual ordering $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n$. The sets $S^{p,m}_{n_1,n_2,\ldots,n_k}$, $B^{p,m}_{n_1,n_2,\ldots,n_k}$, $P^{m}_{n_1,n_2,\ldots,n_k}$ are defined similarly, however with the $\sigma_1,\ldots,\sigma_n$ being respectively Lyapunov singular values, bounded real singular values, positive real singular values. A system (A,B,C,D) *is called partitioned according to* $n_1,n_2,\ldots,n_k$ if

$A = (A(i,j))_{1 \leqslant i,j \leqslant k,}$ with $A(i,j) \in \mathbb{K}^{n_i \times n_j}$, $1 \leqslant i,j \leqslant k$; $B = (B(i))_{1 \leqslant i \leqslant k}$, with $B(i) \in \mathbb{K}^{n_i \times m}$, $1 \leqslant i \leqslant k$; $C = (C(j))_{1 \leqslant j \leqslant k,} C(j) \in \mathbb{K}^{p \times n_j}$, $1 \leqslant j \leqslant k$.

We will now introduce the definition of block-balanced forms and block-balanced input-normal forms. In order to do that in a concise notation, we will extend the notion of inequality of numbers to spectra as follows: If V, W are hermitian matrices, not necessarily of the same size, and if spec(V), spec(W) denote the (real) spectra of these matrices, then we write spec(V) $\succ$ spec(W) to denote that each element of spec(V) is larger than each element of spec(W). This can be extended of course to any pair subsets of $\mathbb{R}$. It can also be extended to real numbers by identifying a real number with the atomic set containing that number.

Definition 5.1. Let $k \in \{1, \ldots, n\}$ and let $n_1, \ldots, n_k$ be such that $\sum_{i=1}^{k} n_i = n$. A system $(A,B,C,D) \in L^{p,m}_{n_1,n_2,\ldots,n_k}$ $(S^{p,n}_{n_1,n_2,\ldots,n_k}, B^{p,m}_{n_1,n_2,\ldots,n_k}, P^{m}_{n_1,n_2,\ldots,n_k})$ is said to be in *block balanced form with respect to* $n_1, n_2, \ldots, n_k$, if the stabilizing solutions Y and Z to the Riccati equations (L1) *and* (L2) (*resp.* (S1) *and* (S2), (B1) and (B2), (P1) and (P2)) are equal and block diagonal $Y = Z = \mathrm{diag}(Y_1, \ldots, Y_i, \ldots, Y_k)$, with $Y_i \in \mathbb{K}^{n_i \times n_i}, i = 1, \ldots, k$, and if for the spectra $\mathrm{spec}(Y_1), \mathrm{spec}(Y_2), \ldots, \mathrm{spec}(Y_k)$ the following inequalities hold:

$$\mathrm{spec}(Y_1) \succ \mathrm{spec}(Y_2) \succ \cdots \succ \mathrm{spec}(Y_k).$$

Remark. Whether solutions Y and Z of the Riccati equations (L1), (L2) (resp. (S1) and (S2); (B1) and (B2); (P1) and (P2)) are the stabilizing solutions can be determined from the singular values. In the present case, where $Y = Z = \mathrm{diag}(Y_1, \ldots, Y_k)$, one can conclude that Y and Z are the stabilizing solutions if

$$\mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0$$

in the LQG case,

$$1 \succ \mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0$$

in the bounded-real and positive-real case, while in the Lyapunov case there will be a unique solution of the Lyapunov equations and the solution will be positive definite:

$$\mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0.$$

Just as in the case of diagonal gramians, here we also define an input-normal counterpart to balancing and we propose to call this block-balanced input-normal:

Definition 5.2. Let $k \in \{1, \ldots, n\}$ and let $n_1, \ldots, n_k$ be such that $\sum_{i=1}^{k} n_i = n$. A system $(A, B, C, D) \in L^{p,m}_{n_1,n_2,\ldots,n_k}(S^{p,m}_{n_1,n_2,\ldots,n_k}, B^{p,m}_{n_1,n_2,\ldots,n_k}, P^{m}_{n_1,n_2,\ldots,n_k})$ is said to be in *block-balanced σ-input-normal form with respect to* $n_1, n_2, \dots, n_k, \sigma > 0$, *if it is* LQG σ-input-normal (Lyapunov σ-input-normal, bounded-real σ-input-normal, positive-real σ-input-normal) and if the stabilizing solution Y to the Riccati equation (L1) ((S1), (B1), (P1)) is block diagonal $Y = \mathrm{diag}(Y_1, \dots, Y_i, \dots, Y_k)$, with $Y_i \in \mathbb{K}^{n_i \times n_i}, i = 1, \dots, k$ such that

$$\mathrm{spec}(Y_1) \succ \mathrm{spec}(Y_2) \succ \cdots \succ \mathrm{spec}(Y_k).$$

Remark. Similarly to the previous remark, here again one can determine whether a given pair of solutions Y, Z of the Riccati equations involved are in fact the stabilizing solutions, by checking the spectra. The matrices $Y = \mathrm{diag}(Y_1, \dots, Y_k)$ and $Z = \sigma I$, with $\sigma > 0$, are the stabilizing solutions if

$$\mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0$$

in the LQG case, and

$$\sigma^{-1} \succ \mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0$$

in the bounded-real and positive-real case, while in the Lyapunov case there will be a unique solution of the Lyapunov equations and the solution will be positive definite:

$$\mathrm{spec}(Y_1) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0.$$

The following important result is a direct consequence of well-known results in the literature on balanced realizations (see [29,16,24,25]). It is a model reduction result in one way and an augmentation result, useful for parametrization issues, the other way.

Proposition 5.1. *Let* $k \in \{1, \dots, n\}$ *and let* $n_1, \dots, n_k$ *be such that* $\sum_{i=1}^{k} n_i = n$.

(a) *Let* $(A, B, C, D) \in L^{p,m}_{n_1,n_2,\dots,n_k}(S^{p,m}_{n_1,n_2,\dots,n_k}, B^{p,m}_{n_1,n_2,\dots,n_k}, P^{m}_{n_1,n_2,\dots,n_k})$. *Assume that* (A, B, C, D) *is in block-balanced σ-input-normal form with respect to* $n_1, n_2, \dots, n_k$. *Then, for* $1 \leqslant i \leqslant k$, *the diagonal subsystem* $(A(i,i), B(i), C(i), D)$ *is in* $L^{p,m}_{n_i}(S^{p,m}_{n_i}, B^{p,m}_{n_i}, P^{m}_{n_i})$.

(b) *Consider* $(A, B, C, D) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n} \times \mathbb{K}^{p \times m}$. *Let* (A, B, C, D) *be partitioned with respect to* $n_1, \dots, n_k$. *Suppose* (A, B, C, D) *is satisfying equations* (L2)((S2), (B2), P2)), *with* $Z = \sigma I, \sigma > 0$ *and is satisfying* (L1)((S1), (B1), (P1)) *with* $Y = \mathrm{diag}(Y_1, Y_2, \dots, Y_k)$, *such that*

$$\mathrm{spec}(Y_1) \succ \mathrm{spec}(Y_2) \succ \cdots \succ \mathrm{spec}(Y_k).$$

Suppose further that for $i = 1, \dots, k$, *the subsystem* $(A(i,i), B(i), C(i), D)$ *is an element of the set* $L^{p,m}_{n_i}(S^{p,m}_{n_i}, B^{p,m}_{n_i}, P^{m}_{n_i})$. *Then* (A, B, C, D) *is minimal and* $(A, B, C, D) \in L^{p,n}_{n_1,n_2,\dots,n_k}(S^{p,m}_{n_1,n_2,\dots,n_k}, B^{p,m}_{n_1,n_2,\dots,n_k}, P^{m}_{n_1,n_2,\dots,n_k})$ *and in block-balanced σ-input-normal form.*

Proof. (a) Consider $(A,B,C,D) \in L^{p,m}_{n_1,n_2,\dots,n_k}$ in LQG-block-balanced σ-input-normal form with respect to $n_1,n_2,\dots,n_k$. There exists a unique matrix Q such that (QAQ^*,QB,CQ^*,D) is in balanced σ-input-normal canonical form. In fact $Q = \operatorname{diag}(Q_1,\dots,Q_k)$, unitary, where Q_i is a unitary $n_i \times n_i$ matrix for $i = 1,\dots,k$. From [24], Corollary 9.2 (see also [15]) it follows that the truncated state space system $(Q_iA(i,i)Q_i^*, Q_iB(i), C(i)Q_i^*, D)$ is in LQG-balanced σ-input-normal canonical form for $i = 1,2,\dots,k$ and a member of the set $L^{p,m}_{n_i}$. The state space system $(A(i,i),B(i),C(i),D)$ is clearly input-output equivalent to $(Q_iA(i,i)Q_i^*, Q_iB(i), C(i)Q_i^*, D)$ and therefore also in $L^{p,m}_{n_i}$. This proves (a) for the LQG-case. The other cases follow similarly, also using Corollary 9.2 of [24] (see also [16,29] for the case concerning Lyapunov balanced systems).

(b) Consider the LQG-case. For each $i = 1,2,\dots,k$ there is a unique matrix Q_i, in fact unitary, such that $(Q_iA(i,i)Q_i^*, Q_iB(i), C(i)Q_i^*, D)$ is in balanced σ-input-normal canonical form, which implies that $Q_iY_iQ_i^*$ is diagonal. Let $Q = \operatorname{diag}(Q_1,\dots,Q_k)$ and consider (QAQ^*,QB,CQ^*,D). This will be balanced σ-input-normal with QYQ^* diagonal. From Theorem 4.1 of [24] it follows by inspection that (QAQ^*,QB,CQ^*,D) is in balanced σ-input-normal canonical form: this is quite straightforward to see for the blocks of (QAQ^*,QB,CQ^*, D) that coincide with $Q_iA(i,i)Q_i^*, Q_iB(i)$, and $C(i)Q_i^*$ for $i = 1,\dots,k$ and for D. For the 'off-diagonal blocks' of QAQ^*, the equality follows from the uniqueness of these off-diagonal blocks given the $Q_iA(i,i)Q_i^*, Q_iB(i)$, and $C(i)Q_i^*$ for $i = 1,\dots,k$, due to the balancedness and the pairwise disjointness of the spectra of $Y_1,Y_2,\dots,Y_k$. We can conclude that (QAQ^*,QB,CQ^*,D) is in $L^{p,m}_{n_1,\dots,n_k}$. Because (A,B,C,D) is input–output equivalent to (QAQ^*,QB,CQ^*,D), it must also be a member of the set $L^{p,m}_{n_1,\dots,n_k}$. Clearly (A,B,C,D) is in block-balanced σ-input-normal form. This proves (b) for the LQG-case. For the other cases the proof of (b) runs completely analogously, using Theorems 2.1, 5.1 and 6.1 of [24] for, respectively, the Lyapunov, bounded-real and positive-real case. □

The existence and uniqueness question of block balanced σ-input-normal realizations is addressed in the following lemma.

Lemma 5.1. *Let $k \in \{1,\dots,n\}$ and let $n_1,\dots,n_k$ be such that $\sum_{i=1}^k n_i = n$. Let $(A,B,C,D) \in L^{p,m}_{n_1,n_2,\dots,n_k}(S^{p,m}_{n_1,n_2,\dots,n_k}, B^{p,m}_{n_1,n_2,\dots,n_k}, P^{m}_{n_1,n_2,\dots,n_k})$. Then there exists an equivalent system (A_1,B_1,C_1,D) that is LQG-block-balanced σ-input-normal (Lyapunov-block-balanced σ-input-normal, bounded-real-block-balanced σ-input-normal, positive-real-block-balanced σ-input-normal) with respect to $n_1,n_2,\dots,n_k,\sigma > 0$. All equivalent LQG-block-balanced σ-input-normal (Lyapunov-block-balanced σ-input-normal, bounded-real block-balanced σ-input-normal, positive-real block-balanced σ-input-normal) systems with respect to $n_1,n_2,\dots,n_k,\sigma > 0$, are given by*

$(QA_1Q^*, QB_1, C_1Q^*, D)$,
$Q = \mathrm{diag}(Q_1, Q_i, \dots, Q_k), Q_i$, *unitary*, $i = 1, 2, \dots, k$.

Proof. Let $(A, B, C, D) \in L^{p,m}_{n_1,n_2,\dots,n_k}$. There is an equivalent LQG-balanced realization of the system (cf. [24]) that can be brought into balanced σ-input-normal form by a diagonal transformation. Due to the structure of the collection of singular values, the resulting realization is then also in LQG block-balanced σ-input-normal form with respect to $n_1, n_2, \dots, n_k$. The structure of the unitary matrix Q follows from the requirement that the LQG-gramian Y has to be block-diagonal and the fact that the spectra of the blocks Y_i, $i = 1, \dots, k$, are strictly ordered. A similar reasoning can be given for the other classes of systems. □

In each of the various classes, we now define a suitable canonical form for block-balanced σ-input-normal systems.

Definition 5.3. Let $k \in \{1, \dots, n\}$ and let $n_1, \dots, n_k$ be such that $\sum_{i=1}^k n_i = n$. For each $i = 1, 2, \dots, k$ let $d^i = (d^i_1, \dots, d^i_m)$ be a sequence of (n_i, m)-dynamical indices. Denote by D_k these k sequences $d^i = (d^i_1, \dots, d^i_m)$ of (n_i, m)-dynamical indices, $i = 1, \dots, k$.

1. The set of all systems $(A, B, C, D) \in L^{p,m}_{n_1,\dots,n_k}$ $(S^{p,m}_{n_1,n_2,\dots,n_k}, B^{p,m}_{n_1,n_2,\dots,n_k}, P^{m}_{n_1,n_2,\dots,n_k})$ such that for any (and therefore for each) equivalent block-balanced σ-input-normal system (A_1, B_1, C_1, D), with $\sigma > 0$, the subsystem $(A_1(i,i), B_1(i), C_1(i), D)$ has the property that $N(A_1(i,i), B_1(i); d^i)$ is nonsingular, i.e. is in $L^{p,m}_{n_i;d^i}(S^{p,m}_{n_i;d^i}, B^{p,m}_{n_i;d^i}, P^{m}_{n_i;d^i})$, $i = 1, \dots, k$, is denoted by $L^{p,m}_{n_1,\dots,n_k;D_k}$ $(S^{p,m}_{n_1,n_2,\dots,n_k;D_k}, B^{p,m}_{n_1,n_2,\dots,n_k;D_k}; P^{m}_{n_1,n_2,\dots,n_k;D_k})$.
2. Let $\sigma > 0$. *A system* $(A, B, C, D) \in L^{p,m}_{n_1,\dots,n_k;D_k}$ $(S^{p,m}_{n_1,n_2,\dots,n_k;D_k}, B^{p,m}_{n_1,n_2,\dots,n_k;D_k}; P^{m}_{n_1,n_2,\dots,n_k;D_k})$ *is said to be in block-balanced* σ-input-normal LQG (Lyapunov, bounded-real, positive-real) canonical form corresponding to the family of dynamical indices D_k if
   (a) it is LQG-block-balanced σ-input-normal (Lyapunov-block-balanced σ-input-normal, bounded-real block-balanced σ-input-normal, positive-real block-balanced σ-input-normal);
   (b) the square matrix $N(A(i,i), B(i); d^i)$ is positive upper triangular for each $i = 1, \dots, k$.
   The set of all such system is denoted by σ-$IL^{p,m}_{n_1,\dots,n_k;D_k}$ $(\sigma$-$IS^{p,m}_{n_1,\dots,n_k;D_k}$, σ-$IB^{p,m}_{n_1,\dots,n_k;D_k}$, σ-$IP^{m}_{n_1,\dots,n_k;D_k})$.

Note that the sets of systems $L^{p,m}_{n_1,\dots,n_k;D_k}$, $S^{p,m}_{n_1,n_2,\dots,n_k;D_k}$, $B^{p,m}_{n_1,n_2,\dots,n_k;D_k}$ and $P^{m}_{n_1,n_2,\dots,n_k;D_k}$ are independent of the choice of $\sigma > 0$. In Lemma 5.2 it will be shown that these sets are relatively open.

Lemma 5.2. *Let $k \in \{1, \dots, n\}$. Let $n_1, n_2, \dots, n_k$ be such that $\sum_{i=1}^k n_i = n$. Let $\mathscr{D}(n_1, \dots, n_k; m)$ be the collection of all k-tuples of $(n_i; m)$-dynamical indices $d^i = (d_1^i, d_2^i, \dots, d_m^i)$, $i = 1, \dots, k$. Let $D_k \in \mathscr{D}(n_1, n_2, \dots, n_k; m)$. The set $L^{p,m}_{n_1,\dots,n_k;D_k}(S^{p,m}_{n_1,\dots,n_k;D_k},\ B^{p,m}_{n_1,\dots,n_k;D_k},\ P^{m}_{n_1,\dots,n_k;D_k})$ is relatively open in $L^{p,m}_n(S^{p,m}_n,\ B^{p,m}_n,\ P^m_n)$.*

In the proof of this and the following lemma use will be made of some new notation. In Section 3 the matrix $N(A, B; d)$ is defined where $d = (d_1, d_2, \dots, d_m)$ is a sequence of (n, m)-dynamical indices and $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$. If the pair (A, B) is replaced by a pair $(\tilde{A}, \tilde{B}) \in \mathbb{K}^{\tilde{n}\times\tilde{n}} \times \mathbb{K}^{\tilde{n}\times m}$, where $\tilde{n} \geqslant n$ is a positive integer, the definition still makes perfect sense and the result will be denoted by $\tilde{N}(\tilde{A}, \tilde{B}; d)$. If $\tilde{n} > n$, then $\tilde{N}(\tilde{A}, \tilde{B}; d)$ will not be a square matrix but a 'tall' rectangular $\tilde{n} \times n$ matrix. The precise definition is as follows. Let $\tilde{n} \geqslant n$ be a positive integer. Let $(\tilde{A}, \tilde{B}) \in \mathbb{K}^{\tilde{n}\times\tilde{n}} \times \mathbb{K}^{\tilde{n}\times m}$ and let $d = (d_1, d_2, \dots, d_m)$ be a sequence of (n, m)-dynamical indices. Let $(s_1, \dots, s_l)$ be the corresponding sequence of step sizes. Let $(r_1, r_2, \dots, r_l)$ be the associated sequence of reversed step sizes with multiplicities $\rho_1, \dots, \rho_{h'}$ and let $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_{h'}$ be the same sequence but ignoring multiplicities. Let π be the ordering permutation of the sequence of dynamical indices $(d_1, \dots, d_m)$. Let $(\delta_1, \dots, \delta_h)$ be the corresponding multiplicities and let $(\tilde{d}_1, \dots, \tilde{d}_h)$ be the corresponding sequence of ordered dynamical indices ignoring the multiplicities. For $j = 1, \dots, l$ let

$$\tilde{N}_{l+1-j} := \left[\tilde{A}^{d_{\pi(1)}-j}\tilde{B}e_{\pi(1)}, \tilde{A}^{d_{\pi(2)}-j}\tilde{B}e_{\pi(2)}, \dots, \tilde{A}^{d_{\pi(s_j)}-j}\tilde{B}e_{\pi(s_j)}\right]$$

and let

$$\tilde{N}(\tilde{A}, \tilde{B}; d) := [\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_l].$$

Proof of Lemma 5.2. First note that $L^{p,m}_{n_1,\dots,n_k}(S^{p,m}_{n_1,\dots,n_k},\ B^{p,m}_{n_1,\dots,n_k},\ P^{m}_{n_1,\dots,n_k})$ is relatively open in $L^{p,m}_n\ (S^{p,m}_n,\ B^{p,m}_n,\ P^m_n)$. This follows from the continuity of the LQG-(Lyapunov, bounded-real, positive-real) singular values as functions of $(A, B, C, D) \in L^{p,m}_n(S^{p,m}_n,\ B^{p,m}_n,\ P^m_n)$. Here one uses the fact that the hermitian stabilizing solutions of the various Riccati equations involved, as well as the solutions of the Lyapunov equations involved, depend continuously on the system matrices (cf. [2,18]).

If we now show that $L^{p,m}_{n_1,\dots,n_k;D_k}(S^{p,m}_{n_1,\dots,n_k;D_k},\ B^{p,m}_{n_1,\dots,n_k;D_k},\ P^{m}_{n_1,\dots,n_k;D_k})$ is relatively open in $L^{p,m}_{n_1,\dots,n_k}(S^{p,m}_{n_1,\dots,n_k},\ B^{p,m}_{n_1,\dots,n_k},\ P^{m}_{n_1,\dots,n_k})$ then the statement in the lemma follows. Let us give the proof for $S^{p,m}_{n_1,\dots,n_k;D_k}$. Consider a mapping of state space systems $(A, B, C, D) \in S^{p,m}_{n_1,\dots,n_k}$ which is constructed as follows. Let Z be the controllability gramian and $Z^{1/2}$ its positive definite hermitian square root. This is a continuous function of (A,B,C,D). Form the equivalent σ-input-normal realization $(A_0, B_0, C_0, D_0) := (Z^{-1/2}AZ^{1/2},\ Z^{-1/2}B,\ CZ^{1/2},\ D)$, with $\sigma = 1$. The observability gramian of this system has the squares of the singular values as its eigenvalues. Due to the separation properties of the singular values given by the inequalities (6) the orthogonal projection operator $\Pi_1 : \mathbb{K}^n \to \mathbb{K}^n$ onto the combined eigenspaces of the first n_1 eigenvalues $\{\sigma_1^2, \dots, \sigma_{n_1}^2\}$, multiplicities included, depends continuously on the observability gramian (cf. [17], Theorem 5.1, p. 107) and therefore on the quadruple of matrices defining the system. Similarly one can define orthogonal projection operators $\Pi_i : \mathbb{K}^n \to \mathbb{K}^n$ for $i = 2, \dots, k$ corresponding to the other $k-1$ groups of eigenvalues of the observability gramian in an obvious way. Now consider the matrices $\tilde{N}(\Pi_1 A_0 \Pi_1, \Pi_1 B_0; d^1)$, $\tilde{N}(\Pi_2 A_0 \Pi_2, \Pi_2 B_0; d^2), \dots, \tilde{N}(\Pi_k A_0 \Pi_k, \Pi_k B_0; d^k)$. Note that these are $n \times n_1, n \times n_2, \dots, n \times n_k$ matrices respectively. The subset of state space systems $(A, B, C, D) \in S^{p,m}_{n_1,\dots,n_k}$ for which these matrices have full column rank is clearly open. We will now show that this subset is in fact the set $S^{p,m}_{n_1,\dots,n_k;D_k}$. Indeed choose an orthonormal basis for the n_1-dimensional image space of Π_1, an orthonormal basis for the n_2-dimensional image space of Π_2, etc, ending with an orthonormal basis for the n_k-dimensional image space of Π_k. Combining these bases one obtains an orthonormal basis for the n-dimensional state space. Let (A_1, B_1, C_1, D) be the state space system that results after the transformation to the new basis. Then, by construction, (A_1, B_1, C_1, D) has block-diagonal observability gramian $Y = \mathrm{diag}(Y_1, \dots, Y_i, \dots, Y_k)$, with $Y_i \in \mathbb{K}^{n_i \times n_i}$, $i = 1, \dots, k$ such that

$$\mathrm{spec}(Y_1) \succ \mathrm{spec}(Y_2) \succ \cdots \succ \mathrm{spec}(Y_k).$$

So (A_1, B_1, C_1, D) is block-balanced σ-input-normal. Performing the state space basis transformation the matrix $\tilde{N}(\Pi_1 A_0 \Pi_1, \Pi_1 B_0; d^1)$ is transformed to the $n \times n_1$ matrix

$$\begin{bmatrix} N(A_1(1,1), B_1(1); d^1) \\ 0 \end{bmatrix},$$

and therefore $\mathrm{rank}(\tilde{N}(\Pi_1 A_0 \Pi_1, \Pi_1 B_0; d^1)) = \mathrm{rank}(N(A_1(1,1), B_1(1); d^1))$. Note that the matrix $N(A_1(1,1), B_1(1); d^1))$ is an $n_1 \times n_1$ matrix. In a similar fashion one finds that $\mathrm{rank}(\tilde{N}(\Pi_i A_0 \Pi_i, \Pi_i B_0; d^i)) = \mathrm{rank}(N(A_1(i,i), B_1(i); d^i)), i = 2, \dots, k$. It follows that the matrices $\tilde{N}(\Pi_1 A_0 \Pi_1, \Pi_1 B_0; d^1), \tilde{N}(\Pi_2 A_0 \Pi_2, \Pi_2 B_0; d^2), \dots, \tilde{N}(\Pi_k A_0 \Pi_k, \Pi_k B_0; d^k)$ all have full (column) rank if and only if all the (square) matrices $N(A_1(1,1), B_1(1); d^1), N(A_1(2,2), B_1(2); d^2), \dots, N(A_1(k,k), B_1(k); d^k)$ have full rank. It follows that $S^{p,m}_{n_1,\dots,n_k;D_k}$ is indeed open. A similar proof works for the other classes. □

The following lemma shows that the term canonical form was justified in the previous definition.

Lemma 5.3. *Let $k \in \{1, \dots, n\}$. Let $n_1, \dots, n_k$,* be such that $\sum_{i=1}^k K\ n_i = n$ *and let $\sigma > 0$. Denote by D_k the k sequences $d^i = (d_1^i, \dots, d_m^i), i = 1, \dots, k$ of (n_i, m)-dynamical indices. Let $(A, B, C, D) \in L^{p,m}_{n_1,\dots,n_k;D_k}$ $(S^{p,m}_{n_1,\dots,n_k;D_k}, B^{p,m}_{n_1,\dots,n_k;D_k}, P^{p,m}_{n_1,\dots,n_k;D_k})$.*

There exists a unique $(A_1, B_1, C_1, D) \in \sigma\text{-}IL^{p,m}_{n_1,\dots,n_k;D_k}(\sigma\text{-}IS^{p,m}_{n_1,\dots,n_k;D_k}, \sigma\text{-}IB^{p,m}_{n_1,\dots,n_k;D_k}, \sigma\text{-}IP^{p,m}_{n_1,\dots,n_k;D_k})$ *that is equivalent to* (A, B, C, D). *The induced map* $\Gamma^L_{\sigma,D_k}(\Gamma^S_{\sigma,D_k}, \Gamma^B_{\sigma,D_k}, \Gamma^P_{\sigma,D_k})$ *that maps the system* $A, B, C, D)$ *to the system* (A_1, B_1, C_1, D) *is a canonical form.*

In the real case the canonical form $\Gamma^L_{\sigma,D_k}(\Gamma^S_{\sigma,D_k}; \Gamma^B_{\sigma,D_k}; \Gamma^P_{\sigma,D_k})$ *is real analytic and hence continuous. In the complex case the canonical form is real analytic, and hence continuous, in the real parameters that are obtained by taking real and imaginary parts of the quantities involved.*

Proof. The canonical form is defined by demanding that each of the block diagonal subsystems be in the input-normal canonical form of Section 4. This determines completely the choice of $Q_1, \dots, Q_k$, as mentioned in Lemma 5.1.

The proof of the real analyticity of the canonical form can be given along the same lines as the proof of Lemma 5.2. Here we need first the result that $Z^{1/2}$ depends real analytically on the real and imaginary parts of the entries of the system matrices, which was shown in the proof of Lemma 4.3. Secondly, we need the result that the orthogonal projection operators $\Pi_1, \Pi_2, \dots, \Pi_k$ are not only continuous, but in fact depend real analytically on the real and imaginary parts of the entries of (A, B, C, D). This follows from the results in [17], Ch. 2, Section 1.4. Starting from the σ-input-normal realization (A_0, B_0, C_0, D_0), the canonical form prescribes the choice of an orthonormal basis for the state space. The orthonormal bases for the image spaces of $\Pi_1, \Pi_2, \dots, \Pi_k$, which together form an orthonormal basis for the state space, are in this case obtained by Gram *N*Schmidt orthonormalization of the column vectors of the matrices $\tilde{N}(\Pi_1 A_0 \Pi_1, \Pi_1 B_0; d^1), \tilde{N}(\Pi_2 A_0 \Pi_2, \Pi_2 B_0; d^2), \dots, \tilde{N}(\Pi_k A_0 \Pi_k, \Pi_k B_0; d^k)$. (Compare the proof of Lemma 4.3.) So this results in a real analytic state space basis transformation. Therefore the canonical form is real analytic. □

We now investigate the structure of this canonical form.

Proposition 5.2. *Let* $k \in \{1, \dots, n\}$ *and let* $n_1, \dots, n_k$ *be such that* $\sum_{i=1}^k n_i = n$. *Denote by* D_k *the k sequences* $d^i = (d^i_1, \dots, d^i_m)$, *of* (n_i, m)*-dynamical indices,* $i = 1, \dots, k$. *Let* $\sigma > 0$ *and let* $(A, B, C, D) \in \sigma\text{-}IL^{p,m}_{n_1,\dots,n_k;D_k}(\sigma\text{-}IS^{p,m}_{n_1,\dots,n_k;D_k}, \sigma\text{-}IB^{p,m}_{n_1,\dots,n_k;D_k}, \sigma\text{-}IP^{p,m}_{n_1,\dots,n_k;D_k})$. *Let* (A, B, C, D) *be block-partitioned according to* $n_1, \dots, n_k$ *with*

$$A = \begin{pmatrix} A(1,1) & \cdots & A(1,i) & \cdots & A(1,k) \\ \vdots & & & & \vdots \\ A(j,1) & \cdots & A(j,i) & \cdots & A(j,k) \\ \vdots & & & & \vdots \\ A(k,1) & \cdots & A(k,i) & \cdots & A(k,k) \end{pmatrix},$$

$$B = \left(B^{\mathrm{T}}(1)\ \ B^{\mathrm{T}}(2)\ \ \cdots\ \ B^{\mathrm{T}}(k)\right)^{\mathrm{T}},$$

and

$$C = (C(1)\ \ C(2)\ \ \cdots\ \ C(k)).$$

Then

1. *for each* $i = 1, \ldots, k$, $M(A(i,i), B(i); d^i)$ *is a simple positive upper triangular matrix. In particular, for each* $i = 1, \ldots, k$, $(A(i,i), B(i), C(i), D)$ *is given as in Lemma 4.4*;
2. *the* $A(i,j)$*-entries,* $1 \leqslant i,j \leqslant k, i \neq j$, *of* A *are given*

   (a) *in the* σ-$IL^{p,m}_{n_1,\ldots,n_k}$*-case by the solutions to the linear equations*:

$$\begin{aligned} A(i,j) + A(j,i)^* &= B(i)R_L^{-1}D^*C(j) + C(i)^*DR_L^{-1}B(j)^* \\ &\quad + \sigma C(i)^*S_L^{-1}C(j) - \frac{1}{\sigma}B(i)R_L^{-1}B(j)^*, \\ A(j,i)^*Y_j + Y_iA(i,j) &= Y_iB(i)R_L^{-1}D^*C(j) + C(i)^*DR_L^{-1}B(j)^*Y_j \\ &\quad - C(i)^*S_L^{-1}C(j) + Y_iB(i)R_L^{-1}B(j)^*Y_j, \end{aligned}$$

   where $Y = \mathrm{diag}(Y_1, \ldots, Y_i, \ldots, Y_k)$, $Y_i \in \mathbb{K}^{n_i \times n_i}$, $i = 1, \ldots, k$, *is the stabilizing solution to the Riccati equation* (L1), *and* R_L, S_L *as in* (L3);

   (b) *in the* σ-$IS^{p,m}_{n_1,\ldots,n_k}$*-case by the solutions to the linear equations*:

$$A(i,j) + A(j,i)^* = -\frac{1}{\sigma}B(i)B(j)^*,$$

$$A(j,i)^*Y_j + Y_iA(i,j) = -C(i)^*C(j),$$

   where $Y = \mathrm{diag}(Y_1, \ldots, Y_i, \ldots, Y_k)$, $Y_i \in \mathbb{K}^{n_i \times n_i}$, $i = 1, \ldots, k$, *is the positive definite solution to the Lyapunov equation* (S1).

   (c) *in the* σ-$IB^{p,m}_{n_1,\ldots,n_k}$*-case by the solutions to the linear equations*:

$$\begin{aligned} A(i,j) + A(j,i)^* &= -B(i)R_B^{-1}D^*C(j) - C(i)^*DR_B^{-1}B(j)^* \\ &\quad - \sigma C(i)^*S_B^{-1}C(j) - \frac{1}{\sigma}B(i)R_B^{-1}B(j)^*, \\ A(j,i)^*Y_j + Y_iA(i,j) &= -Y_iB(i)R_B^{-1}D^*C(j) \\ &\quad - C(i)^*DR_B^{-1}B(j)^*Y_j - C(i)^*S_B^{-1}C(j) - Y_iB(i)R_B^{-1}B(j)^*Y_j, \end{aligned}$$

   where $Y = \mathrm{diag}(Y_1, \ldots, Y_i, \ldots, Y_k)$, $Y_i \in \mathbb{K}^{n_i \times n_i}$, $i = 1, \ldots, k$, *is the stabilizing solution to the Riccati equation* (B1), *and* R_B, S_B *as in* (B3);

   (d) *in the* σ-$IP^{m}_{n_1,\ldots,n_k}$*-case by the solutions to the linear equations*:

$$\begin{aligned} A(i,j) + A(j,i)^* &= B(i)R_P^{-1}C(j) + C(i)^*R_P^{-1}B(j)^* \\ &\quad - \sigma C(i)^*R_P^{-1}C(j) - \frac{1}{\sigma}B(i)R_P^{-1}B(j)^*, \\ A(j,i)^*Y_j + Y_iA(i,j) &= Y_iB(i)R_P^{-1}C(j) + C(i)^*R_P^{-1}B(j)^*Y_j \\ &\quad - C(i)^*R_P^{-1}C(j) - Y_iB(i)R_P^{-1}B(j)^*Y_j, \end{aligned}$$

where $Y = \mathrm{diag}(Y_1, \dots, Y_i, \dots, Y_k)$, $Y_i \in \mathbb{K}^{n_i \times n_i}$, $i = 1, \dots, k$, *is the stabilizing solution to the Riccati equation* (P1), *and* R_P *as in* (P3).

Proof. The linear equations in (a), as well as in (b)–(d), have a unique solution, because the spectra of Y_i and Y_j, $i \neq j$, have no element in common. One way to show this is by diagonalization of Y_i and Y_j. The argument is analogous to the one given in the proof of Theorem 4.1 of [13]. The other statements follow in a straightforward manner from the definition of the canonical form and the results of Section 4. □

We now come to parametrize block-balanced σ-input-normal systems. To this end we define the following parameter spaces. As before let $n_1, \dots, n_k$, $\sum_{i=1}^k n_i = n$ and let us denote by D_k the k sequences $d^i = (d^i_1, \dots, d^i_m)$ of (n_i, m)-dynamical indices, $i = 1, \dots, k$.

Let $\sigma > 0$ and let $\tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L}$ $(\tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S}, \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B}, \tilde{\Theta}^{m}_{\sigma,n_1,\dots,n_k;D_k,P})$ be the parameter space of block diagonal systems $(\tilde{A}, B, C, D)$ with the following properties: Let (A, B, C, D) be block partitioned according to $n_1, n_2, \dots, n_k$, then

1. $(\tilde{A}(i,i), B(i), C(i), D) \in \tilde{\Theta}^{p,m}_{\sigma,d^i,L}$, $(\tilde{\Theta}^{p,m}_{\sigma,d^i,S}, \tilde{\Theta}^{p,m}_{\sigma,d^i,B}, \tilde{\Theta}^{m}_{\sigma,d^i,P})$, $i = 1, \dots, k$.
2. All off-diagonal blocks of the matrix $\tilde{A}$ are zero, i.e. if $i \neq j$ then $\tilde{A}(i,j) = 0$.
3. If Y_i is the stabilizing solution to the Riccati equation (L1) (Lyapunov equation (S1), Riccati equation (B1), Riccati equation (P1)) for the system

$$(A(i,i), B(i), C(i), D) := \phi_{\sigma,d^i,L}((\tilde{A}(i,i), B(i), C(i), D)),$$
$$(\phi_{\sigma,d^i,S}((\tilde{A}(i,i), B(i), C(i), D)), \phi_{\sigma,d^i,B}((\tilde{A}(i,i), B(i), C(i), D)),$$
$$\phi_{\sigma,d^i,P}((\tilde{A}(i,i), B(i), C(i), D))),$$

$i = 1, \dots, k$, then

$$\mathrm{spec}(Y_1) \succ \mathrm{spec}(Y_2) \succ \cdots \succ \mathrm{spec}(Y_k) \succ 0.$$

In the case of bounded-real and positive-real systems we require moreover that $\sigma^{-1} \succ \mathrm{spec}(Y_1)$.

Analogously to the treatment of the previous section we find it convenient from a notational point of view to consider the parameter space in terms of systems. It is, however, again straightforward to describe this parameter space as a subset of the Euclidean space of appropriate dimension.

From $(\tilde{A}, B, C, D) \in \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L}$ $(\tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S}, \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B}, \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,P})$ we construct a state space system (A,B,C,D), partitioned according to $n_1, n_2, \dots, n_k$ by taking $A(i,i)$, $i = 1, \dots, k$, as above, with corresponding observability gramian Y_i, $i = 1, \dots, k$ and by taking the off-diagonal blocks of A to be the solutions of the linear equations (2)(a) ((2)(b)–(d)) of Proposition 5.2. Denote by $\Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L}$ $(\Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S}, \Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B}, \Theta^{m}_{\sigma,n_1,\dots,n_k;D_k,P})$ the set of all so constructed systems (A, B, C, D) and denote by

$$\phi_{\sigma,n_1,\dots,n_k;D_k,L} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L} \to \Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L}$$

$$\Big(\phi_{\sigma,n_1,\dots,n_k;D_k,S} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S} \to \Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S},$$

$$\phi_{\sigma,n_1,\dots,n_k;D_k,B} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B} \to \Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B},$$

$$\phi_{\sigma,n_1,\dots,n_k;D_k,P} : \tilde{\Theta}^{m}_{\sigma,n_1,\dots,n_k;D_k,P} \to \Theta^{m}_{\sigma,n_1,\dots,n_k;D_k,P}\Big)$$

the corresponding parametrization map.

The following lemma states that we have obtained a parametrization for each of the spaces of state space systems in block-balanced σ-input-normal canonical form, that were introduced in part (b) of Definition 5.3.

Lemma 5.4. *Let $\sigma > 0$. Let $k \in \{1, \dots, n\}$ and let $n_1, \dots, n_k$ be such that $\sum_{i=1}^k n_i = n$. Denote by D_k the k sequences $d^i = (d^i_1, \dots, d^i_m)$, of (n_i, m)-dynamical indices, $i = 1, \dots, k$. Then*

$$\Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L} = \sigma\text{-}IL^{p,m}_{n_1,\dots,n_k;D_k}, \quad \Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,S} = \sigma\text{-}IS^{p,m}_{n_1,\dots,n_k;D_k},$$
$$\Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B} = \sigma\text{-}IB^{p,m}_{n_1,\dots,n_k;D_k}, \quad \Theta^{m}_{\sigma,n_1,\dots,n_k;D_k,P} = \sigma\text{-}IP^{m}_{n_1,\dots,n_k;D_k}.$$

Proof. That $\Theta^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L} \subset \sigma\text{-}IP^{p,m}_{n_1,\dots,n_k;D_k}$, follows by the construction of the state space systems with the parametrization map $\phi_{\sigma,n_1,\dots,n_k;D_k,L}$ together with the augmentation property of Proposition 5.1. The reverse set inclusion is the content of Proposition 5.2. The other three statements follow analogously. □

The following theorem collects some of the main results on the canonical forms and parametrizations that were introduced in this section. If the underlying field is $\mathbb{K} = \mathbb{C}$ then we identify $\mathbb{C}$ with $\mathbb{R}^2$ and interpret real analyticity in this way.

Theorem 5.1. *Let $\sigma > 0$. Denote by $\mathscr{D}(n_1, \dots, n_k; m)$ the collection of all k-tuples of sequences of $(n_i; m)$-dynamical indices $d^i = (d^i_1, d^i_2, \dots, d^i_m), i = 1, \dots, k$.*
1. *The collection*

$$\Big\{\Gamma^L_{\sigma,D_k} : L^{p,m}_{n_1,\dots,n_k;D_k} \to \sigma\text{-}IL^{p,m}_{n_1,\dots,n_k;D_k} \mid k = 1, \dots, n; \sum_{i=1}^k n_i = n;$$
$$D_k \in \mathscr{D}(n_1, \dots, n_k; m)\Big\}$$

forms an overlapping set of (real analytic) canonical forms covering $L^{p,m}_n$. Similarly

$$\{\Gamma^S_{\sigma,D_k} : S^{p,m}_{n_1,\ldots,n_k;D_k} \to \sigma\text{-}IS^{p,m}_{n_1,\ldots,n_k;D_k} \mid k = 1,\ldots,n; \sum_{i=1}^{k} n_i = n;$$
$$D_k \in \mathscr{D}(n_1,\ldots,n_k;m)\},$$

$$\{\Gamma^B_{\sigma,D_k} : B^{p,m}_{n_1,\ldots,n_k;D_k} \to \sigma\text{-}IB^{p,m}_{n_1,\ldots,n_k;D_k} \mid k = 1,\ldots,n; \sum_{i=1}^{k} n_i = n;$$
$$D_k \in \mathscr{D}(n_1,\ldots,n_k;m)\},$$

$$\{\Gamma^P_{\sigma,D_k} : P^{m}_{n_1,\ldots,n_k;D_k} \to \sigma\text{-}IP^{m}_{n_1,\ldots,n_k;D_k} \mid k = 1,\ldots,n; \sum_{i=1}^{k} n_i = n;$$
$$D_k \in \mathscr{D}(n_1,\ldots,n_k;m)\},$$

respectively, form an overlapping set of (real analytic) canonical forms covering $S^{p,m}_n$, $B^{p,m}_n$, P^m_n *respectively.*
2. *The mappings*

$$\phi_{\sigma,n_1,\ldots,n_k;D_k,L} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\ldots,n_k;D_k,L} \to \sigma\text{-}IL^{p,m}_{n_1,\ldots,n_k;D_k};$$
$$\phi_{\sigma,n_1,\ldots,n_k;D_k,S} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\ldots,n_k;D_k,S} \to \sigma\text{-}IS^{p,m}_{n_1,\ldots,n_k;D_k};$$
$$\phi_{\sigma,n_1,\ldots,n_k;D_k,B} : \tilde{\Theta}^{p,m}_{\sigma,n_1,\ldots,n_k;D_k,B} \to \sigma\text{-}IB^{p,m}_{n_1,\ldots,n_k;D_k};$$
$$\phi_{\sigma,n_1,\ldots,n_k;D_k,P} : \tilde{\Theta}^{m}_{\sigma,n_1,\ldots,n_k;D_k,P} \to \sigma\text{-}IP^{m}_{n_1,\ldots,n_k;D_k}$$

are real analytic diffeomorphisms.
3. *Let*

$$\hat{\phi}_{\sigma,n_1,\ldots,n_k;D_k,L} : \tilde{\Theta}^{p,m}_{\sigma;D_k,L} \to L^{p,m}_{n_1,\ldots,n_k;D_k}/\sim; \quad \theta \mapsto \phi_{\sigma,n_1,\ldots,n_k;D_k,L}(\theta)/\sim$$

denote the mapping from the parameter space into the set of equivalence classes of state space systems. The collection of mappings

$$\left\{\hat{\phi}^{-1}_{\sigma,n_1,\ldots,n_k;D_k,L} : L^{p,m}_{n_1,\ldots,n_k;D_k}/\sim \to \tilde{\Theta}^{p,m}_{\sigma,n_1,\ldots,n_k;D_k,L} \mid k = 1,\ldots,n;\right.$$
$$\left.\sum_{i=1}^{k} n_i = n; D_k \in \mathscr{D}(n_1,\ldots,n_k;m)\right\}$$

forms a real analytic atlas of the set of input–output systems $L^{p,m}_n/\sim$. *Using the obvious notation*

$$\left\{\hat{\phi}^{-1}_{\sigma,n_1,\ldots,n_k;D_k,S} : S^{p,m}_{n_1,\ldots,n_k;D_k}/\sim \to \tilde{\Theta}^{p,m}_{\sigma,n_1,\ldots,n_k;D_k,S} \mid k = 1,\ldots,n;\right.$$
$$\left.\sum_{i=1}^{k} n_i = n; D_k \in \mathscr{D}(n_1,\ldots,n_k;m)\right\}$$

forms a real analytic atlas of $S^{p,m}_n/\sim$;

$$\Big\{\hat{\phi}^{-1}_{\sigma,n_1,\dots,n_k;D_k,B} : B^{p,m}_{n_1,\dots,n_k;D_k}/\sim \;\longrightarrow \tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,B} \mid k=1,\dots,n;\; \sum_{i=1}^{k} n_i = n; D_k \in \mathscr{D}(n_1,\dots,n_k;m)\Big\}$$

forms a real analytic atlas of $B^{p,m}_n/\sim$;

$$\Big\{\hat{\phi}^{-1}_{\sigma,n_1,\dots,n_k;D_k,P} : P^{m}_{n_1,\dots,n_k;D_k}/\sim \;\longrightarrow \tilde{\Theta}^{m}_{\sigma,n_1,\dots,n_k;D_k,P} \mid k=1,\dots,n;\; \sum_{i=1}^{k} n_i = n; D_k \in \mathscr{D}(n_1,\dots,n_k;m)\Big\}$$

forms a real analytic atlas of $P^{m}_n/\sim$.

Proof. (1) That each of the canonical forms is real analytic was established in Lemma 5.3. That the collection of canonical forms is covering follows from Theorem 4.1 where the covering property was established for a subset of the canonical forms that are considered here. That the covering is an open covering follows from Lemma 5.2.

(2) Note that the image space of $\phi_{n_1,\dots,n_k;D_k,L}$ as given in the statement of the Theorem is correct due to the equalities presented in Lemma 5.4. To show that the mapping $\phi_{\sigma,n_1,\dots,n_k;D_k,L}$ is real analytic consider the state space system (A,B,C,D) which is the image of a parameter vector in $\tilde{\Theta}^{p,m}_{\sigma,n_1,\dots,n_k;D_k,L}$. It follows from Theorem 4.1 that the subsystems $(A(i,i),B(i),C(i),D), i=1,\dots,k$, depend real analytically on the parameter vector. That $A(i,j)$, $i \neq j$, depends real analytically on the parameter vector follows from the fact that the real and imaginary parts of the entries of $A(i,j)$ form the solution to a set of non-singular linear equations, namely those given in (2) of Proposition 5.2, with coefficients depending real-analytically on the parameter vector.

To show that the inverse mapping is real analytic one just needs to consider the subsystems $(A(i,i),B(i),C(i),D), i=1,2,\dots,k$, because they determine completely the corresponding parameter vector given by $(\tilde{A}(i,i),B(i),C(i),\ D), i=1,\dots,k$. The real analyticity of the inverse mapping follows therefore from the results of Theorem 4.1 applied to each of the subsystems.

A similar proof can be given for the other classes.

(3) First note that if $n_1,\dots,n_k$ is such that $\sum_{i=1}^{k} = n_i = n$, and $\bar{n},\dots,\bar{n}_{\bar{k}}$ is such that $\sum_{i=1}^{k} \bar{n}_i = n; D_k$ a set of k sequences of $(n_i;m)$-dynamical indices, $i=1,\dots,k$, and $\bar{D}_{\bar{k}}$ a set of $\bar{k}$ sequences of $(\bar{n}_i;m)$-dynamical indices, $i=1,\dots,\bar{k}$, then from Lemma 5.2 it follows that $\sigma\text{-}IL^{p,m}_{n_1,n_2,\dots,n_k;D_k} \cap L_{\bar{n}_1,\dots,\bar{n}_{\bar{k}};\bar{D}^{p,m}_{\bar{k}}}$ is relatively open in $\sigma\text{-}IL^{p,m}_{n_1,n_2,\dots,n_k;D_k}$. Using this, it follows from (1) and (2) in a way which is completely analogous to the proof of (3) of Theorem 4.1 that the collection of mappings

$$\left\{\hat{\phi}^{-1}_{\sigma,n_1,\dots,n_k;D_k,L} \mid k = 1,\dots,n; \sum_{i=1}^{k} n_i = n; D_k \in \mathscr{D}(n_1,\dots,n_k;m)\right\}$$

forms a real analytic atlas of $L^{p,m}_{n_1,\dots,n_k;D_k}/\sim$. A similar proof can be given for the other classes. □

Remark. If one wants to work with block-balanced forms instead of block-balanced input-normal forms, the same parametrizations can be used. Starting from a block-balanced input-normal from one has to perform a state space transformation corresponding to the positive 4th order root (i.e. the square root of the square root) of the positive definite block-diagonal matrix $\sigma^{-1}Y$ in order to arrive at a corresponding block-balanced form.

In the following lemma we are going to show that if, and only if, the $C(i)$ matrices satisfy a certain constraint, then the resulting system will be balanced σ-input-normal.

Lemma 5.5. *Let $k \in \{1,\dots,n\}$. Let $n_1,\dots,n_k$, be such that $\sum_{i=1}^{k} n_i = n$. Denote by D_k the k sequences $d^i = (d^i_1,\dots,d^i_m)$, of (n_i,m)-dynamical indices, $i = 1,\dots,k$. Let $\sigma > 0$ and let $(\tilde{A},B,C,D)$ be block-partitioned according to $n_1,\dots,n_k$. Assume that the ith block diagonal subsystem is such that $(\tilde{A}(i,i),B(i),C(i),D) \in \mathrm{Skew}^{p,m}_{n_i;d^i}$ $i = 1,\dots,k$, and that the off-diagonal blocks are zero: $\tilde{A}(i,j) = 0, i \neq j$. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$.*

1. *The matrix $C(i)$ is such that*

$$\lambda_i^{-1}C^*(i)S_L^{-1}C(i) = B(i)R_L^{-1}B^*(i),$$

$i = 1,\dots,k$, where R_L, S_L as in (L3), *if and only if $(\tilde{A},B,C,D) \in \tilde{\Theta}^{p,m}_{\sigma;D_k,L}$ and $(A,B,C,D) = \phi_{\sigma,n_1,\dots,n_k;D_k,L}((\tilde{A},B,C,D))$ has the property that the stabilizing solution Y to the Riccati equation* (L1) *is given by*

$$Y = \sigma\,\mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2},\dots,\lambda_k I_{n_k}).$$

2. *The matrix $C(i)$ is such that*

$$\lambda_i^{-1}C^*(i)C(i) = B(i)B^*(i),$$

$i = 1,\dots,k$, if and only if $(\tilde{A},B,C,D) \in \tilde{\Theta}^{p,m}_{\sigma;D_k,S}$ and

$$(A,B,C,D) = \phi_{\sigma,n_1,\dots,n_k;D_k,S}\Big((\tilde{A},B,C,D)\Big)$$

has the property that the stabilizing solution Y to the Lyapunov equation (S1) *is given by*

$$Y = \sigma\,\mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2},\dots,\lambda_k I_{n_k}).$$

3. *Assume moreover that* $\lambda_1 < \sigma^{-1}$. *The matrix* $C(i)$ *is such that*

$$\lambda_i^{-1} C^*(i) S_B^{-1} C(i) = B(i) R_B^{-1} B^*(i),$$

$i = 1, \dots, k$, *where* R_B, S_B *are as in* (B3), *if and only if* $(\tilde{A}, B, C, D) \in \tilde{\Theta}^{p,m}_{\sigma;D_k,B}$ *and*

$$(A, B, C, D) = \phi_{\sigma,n,\dots,n_k;D_k;B}((\tilde{A}, B, C, D))$$

has the property that the stabilizing solution Y *to the Riccati equation* (B1) *is given by*

$$Y = \sigma \operatorname{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_k I_{n_k}).$$

4. *Assume moreover that* $\lambda_1 < \sigma^{-1}$. *The matrix* $C(i)$ *is such that*

$$\lambda_i^{-1} C^*(i) R_P^{-1} C(i) = B(i) R_P^{-1} B^*(i),$$

$i = 1, \dots, k$, *where* $R_P = D + D^*$, *if and only if* $(\tilde{A}, B, C, D) \in \tilde{\Theta}^{m}_{\sigma:D_k,P}$ *and*

$$(A, B, C, D) = \phi_{\sigma,n_1,\dots,n_k;D_k,P}((\tilde{A}, B, C, D))$$

has the property that the stabilizing solution Y *to the Riccati equation* (P1) *is given by*

$$Y = \sigma \operatorname{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_k I_{n_k}).$$

Proof. The ‘if’ part follows straightforwardly by substituting the known solutions of the associated Riccati or Lyapunov equations and by considering the relevant subsystems (compare Lemma 4.7).

The ‘only if’ part can be shown as follows.

(1): From Lemma 4.7 it follows that the block diagonal subsystems are minimal and that the solutions to the two Riccati equations are $\sigma\lambda_i I_{n_i}$ and $\sigma I_{n_i}, i = 1, \dots, k$. Hence by Proposition 5.1 and the construction of (A, B, C, D) the result follows.

(2)–(4) follow analogously. □

6. Conclusions

In this paper we have shown how one can construct atlasses of various classes of systems which generalize the well-known balanced parametrizations. The method is fairly general and could also be applied to other classes of systems, for example the class of stable minimum phase systems (which are in close connection with positive real systems, see, e.g., [24,20]). The availability of these atlasses opens up the possibility of search algorithms for optimization over these classes of systems, using balanced and block-balanced parametrizations without having to determine a priori the specific cell of the parametrization which contains the point sought for. This is potentially very useful, especially if in the problem at hand truncation of states with small singular values gives a good initial point for the search algorithm. As an intermediate step in our derivation input-normal overlapping canonical forms are constructed in which nice selections play a crucial role. These canonical forms are of interest in their own right. They seem to be especially suited for multivariable stable all-pass systems. For a number of classes of systems that were studied no explicit atlasses or overlapping canonical forms appear to have been known. Implicitly atlasses for these classes were constructed in [9,26,32]. Apart from the practical possibilities opened up by these atlasses and overlapping canonical forms there are also a number of theoretical advantages, such as the possibility to calculate Riemannian metric tensors (including the Fisher information matrix in a stochastic context, in which the singular values are in fact the well-known canonical correlations) and Riemannian gradients at each point of the manifold of systems involved, and at each system in balanced canonical form in particular, and to further the study of the geometry and topology of these spaces.

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