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On the role of reachability in the analysis of NMR experiments

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Abstract

It is shown that the system theoretic concepts of reachability and observability are relevant in the analysis of NMR experiments. Both one-dimensional and two-dimensional NMR experiments are considered.

1 Introduction

In the paper [4] the following system theoretic setup was introduced to describe one- and multi-dimensional NMR experiments. (For a general introduction to NMR experimentation see e.g. [1]). The basic relationship between *inputs* u to the system, i.e. the excitation signals or in particular the radio frequency pulses and the measured *output* y , i.e. the measured induced magnetization, is described by a bilinear system,

$$\dot{x}(t) = Ax(t) + u_1(t)N_1x(t) + u_2(t)N_2x(t) + b_1u_1(t) + b_2u_2(t),$$

$$y(t) = cx(t),$$

$x(t_0) = x_0$, where x is a *state vector*, A , N_1 , N_2 are square matrices, b is a column vector and c is a row vector. The state space X is an n -dimensional Euclidean space, i.e. $X = \mathbb{C}^n$, for some $n \geq 1$.

In the case of no relaxation A is usually skew-hermitian. In this special case it is often more convenient to work with the equivalent bilinear system ([4])

$$\dot{z}(t) = Az(t) + u_1(t)N_1z(t) + u_2(t)N_2z(t)$$

$$y(t) = cz(t),$$

where $z(t) = x(t) + v_{eq}$, with v_{eq} the vector representation of the equilibrium density matrix.

The purpose of this paper is to illustrate that basic notions of systems and control theory such as the classification of the reachable states of the system that describes the NMR experiments turn out to also be of fundamental importance in the description of NMR experiments. In fact it is shown that the reachable states of an NMR system classify the set of all NMR spectra that can be achieved. It should be pointed out that in an earlier paper ([5]) a similar question was considered. The set up in that paper was, however, different in that we also incorporated into the treatment what we called addition schemes that include phase cycling. The inclusion of these addition schemes had as a result that quite different mathematical techniques could be used to those being discussed here. In this paper we will mainly use techniques from differential geometry to analyze the problems at hand.

A fundamental problem in NMR spectroscopy is that of the design of experiments. While we do not propose any constructive methods for new experiments here, we believe that the current investigation will provide a first and necessary step to develop system theoretic methods for the design of NMR experiments. In the context of laser spectroscopy this general approach lead to the design of experiments ([6]).

2 Reachability and NMR experiments

As a means of introduction to this section we will first consider the case of one-dimensional experiments.

It was shown in [4] that the free induction decay (fid) of a typical one dimensional NMR pulse experiment is given by

$$s(t) = ce^{(t-t_m)A}x_0, \quad t \geq t_m,$$

where x_0 is the state of the system at time t_m , the time when the measurements start. The obtained spectrum (ignoring sampling effects etc), i.e. the Fourier transform

of the fid is then given by

$$G(\omega) := c(2\pi i\omega I - A)^{-1}x_0, \quad \omega \in \mathbb{R}.$$

It is therefore clear that the only influence that an experimenter has on the outcome of the experiment is through the vector x_0 . It should be emphasized that here we do not take into consideration phase cycling etc. We are therefore interested in the set of all states that the system can attain. But this is what is known as the reachability or controllability problem in linear and non-linear system theory (see e.g. [3]).

Before proceeding we need to review the definition of reachability. In order to do this we need to be concerned with the class of input functions. Our definition of reachability will be based on the largest class of inputs for which a solution for the system can be obtained. This class of admissible inputs U_{ad} consists of those inputs for which the conditions of Caratheodory's theorem are satisfied for the existence and uniqueness of the corresponding initial value problem. In Section 3 the consequence of this choice of inputs will be further discussed.

A state x_2 of the system is said to be reachable from the state x_1 if there exists an admissible input u , i.e. $u \in U_{ad}$ with $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $t_1 \leq t \leq t_2$, such that if the system is in state x_1 at time t_1 , the input u will drive the system to state x_2 at time t_2 . The set of all states which are reachable from the state x_1 is denoted by $\mathcal{R}(x_1)$. In the study of NMR systems it is usually assumed (although not always satisfied in practice) that the system is in equilibrium i.e. in state 0, when the experiment is started. We are therefore particularly interested in $\mathcal{R}(0)$.

We can describe the set of all possible one-dimensional NMR spectra (within our framework and without phase cycling) by

$$\{c(2\pi i\omega I - A)^{-1}x \mid x \in \mathcal{R}(0)\}.$$

In the above discussion we considered admissible input functions or excitation signals. In practice, however, excitation signals are radio frequency pulses, which after a coordinate transformation are translated to constant inputs (see e.g. [1],[4]). Constant or piecewise constant inputs are also of great importance from a theoretical point of view since the bilinear system of equations has an analytical solution in the case of constant/piecewise constant inputs. In many situations there is no loss of generality in restricting the inputs to piecewise constant ones for the purposes of calculating the reachable sets. A precise statement appears in the next section.

If a constant input $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is applied to the bilinear system for $t_0 \leq t \leq t_0 + \Delta t$, it was shown in [4] that the solution to the bilinear system is given by

$$x(t) = e^{\Delta t A_p} x(t_0) + (e^{\Delta t A_p} - I)A_p^{-1}b_p$$

for $t_0 \leq t \leq t_0 + \Delta t$, where $A_p := A + u_1 N_1 + u_2 N_2$ and $b_p := b_1 u_1 + b_2 u_2$. Here, as in the remainder of the paper, we assume that A_p is invertible whenever we write A_p^{-1} .

If a system is in state $x(t_0)$ at time t_0 when the 'pulse' input is started, the pulse will move the system to the state

$$x(t_0 + \Delta t) = P x(t_0) + z$$

at time $t_0 + \Delta t$ where $P = e^{\Delta t A_p}$ and $z = (e^{\Delta t A_p} - I)A_p^{-1}b_p$. Note that this representation also includes the case in which no pulse has been applied. In this case $z = 0$ and $P = e^{\Delta t A}$.

This 'affine' structure implies that the effect of a sequence of k pulses is to move an initial state $x(t_0)$ to the state

$$T x(t_0) + e,$$

where $T = P_k P_{k-1} \cdots P_1$ and

$$e = P_k P_{k-1} \cdots P_2 z_1 + P_k P_{k-1} \cdots P_3 z_2 + \cdots + P_k z_{k-1} + z_k,$$

with $P_j = e^{\Delta t_j A_p}$ and $z_j = (e^{\Delta t_j A_p} - I)A_p^{-1}b_p$, $j = 1, \dots, k$.

The three blocks of pulses that often characterize a two-dimensional experiment are therefore determined by three matrices T_1, T_2 and T_3 and three vectors e_1, e_2 and e_3 . Here the notation is such that the pair (T_1, e_1) describes the preparation block of pulses, (T_2, e_2) stands for any possible pulses in the middle of the evolution period and (T_3, e_3) describes the pulses during the mixing period. In this paper we will only consider the case where $T_2 = I$ and $e_2 = 0$, i.e. the case when no pulses are applied in the center of the evolution period. Moreover, we shall assume that before each scan the system is in equilibrium, i.e. $x_0 = 0$. Hence ([4]) the free induction decay of such a system is given by

$$s(t_1, t_2) = c e^{t_2 A} T_3 e^{t_1 A} e_1 + c e^{t_2 A} e_3, \quad t_1, t_2 \geq 0.$$

In the above expression as usual t_1 stands for the measured time and t_2 for the length of the evolution period. The spectrum of a two-dimensional experiment is given by

$$G(\omega_1, \omega_2) =$$

$$c(2\pi i\omega_1 I - A)^{-1} T_3 (2\pi i\omega_2 I - A)^{-1} e_1 + \delta_0(\omega_1) e_3,$$

$\omega_1, \omega_2 \in \mathbb{R}$. The term $\delta_0(\omega_1) e_3$ arises from the term $c e^{t_2 A}$ in the time domain data. Note that since it is independent of t_1 , it in fact shows up as a constant in the t_1 time direction. In any practical situation this term would be removed before Fourier transforming the data, since it is common practice to remove a constant level in a signal before the Fourier transform is carried out. We can therefore assume that the spectrum is given by

$$G(\omega_1, \omega_2) = c(2\pi i\omega_1 I - A)^{-1} T_3 (2\pi i\omega_2 I - A)^{-1} e_1,$$

$\omega_1, \omega_2 \in \mathbb{R}$. As pointed out above, the matrix T_3 which determines the pattern of cross peaks in the spectrum

is given by $T_3 = P_k \cdots P_1$ for some $k \geq 1$, where $P_j = e^{(t_j - t_{j-1})(A + u_1^j N_1 + u_2^j N_2)}$ for $0 < t_0 < t_1 < \cdots < t_k$, and $u_1^j, u_2^j \in \mathbb{R}$, $j = 1, \dots, k$. We denote by \mathcal{RT} the set of all matrices T_3 as defined above.

It is important to note that \mathcal{RT} can be seen to be the set of reachable states from the identity matrix of the system

$$\dot{U} = AU + u_1 N_1 U + u_2 N_2 U,$$

driven by piecewise constant inputs. The state space here is the space of square matrices $\mathbb{C}^{n \times n}$, i.e. $U(t) \in \mathbb{C}^{n \times n}$ for $t \geq t_0$.

3 Properties of the set of reachable states

In this section we are going to consider the characterization of the set of reachable states. In fact we will not prove any new results and refer to the literature (e.g. [3]) for the proofs. We believe, however, that the quoted results will be important in the discussion of NMR spectra as introduced in the earlier sections.

The main feature of the set of states reachable from a given point x is that they belong to the orbit of the family of vector fields generated by the control system which passes through the point x . To describe what is meant by an orbit, we will first consider a general nonlinear system of the form

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) \quad (1)$$

where the state x is an element of a differentiable manifold M and f and g_i are smooth vector fields on M .

As pointed out already in Section 2 the discussion of which states are reachable from x depends not only on f and the g_i but also on the class of input functions u_i that are applied. In this paper three classes will be considered: i) Admissible inputs, which were introduced in Section 2: these inputs satisfy the conditions of Caratheodory's theorem for existence and uniqueness of solutions to the corresponding initial value problem. Let us denote this class by the symbol U_{ad} ; ii) Piecewise constant inputs: These are inputs which are concatenations of constant inputs. These will be denoted by U_{pc} ; iii) Smooth inputs: These are inputs which are C^∞ , i.e. infinitely often continuously differentiable functions of time. These will be denoted by U_∞ .

Note that the latter two classes are contained in the first class and that the second is dense (see [2] for details) in the first. The first main fact about reachable sets is the following theorem (see e.g. [3]) which shows that the set $\mathcal{R}(x)$ of reachable states from a point x is not merely a subset of the orbit through x but a 'large' subset, since $\mathcal{R}(x)$ contains an open subset of the orbit.

Theorem 1 Consider the system (1) and let $x \in M$. The set $\mathcal{R}(x)$ of states which are reachable from x with

inputs in U_{ad} is contained in the orbit through x . Furthermore, $\mathcal{R}(x)$ contains an open set (in the topology of $O(x)$) of $O(x)$.

Since piecewise constant controls are attractive from many standpoints (especially for the application at hand) and these are dense in U_{ad} , a reasonable question is whether every state that is reachable from x via an admissible control is also reachable from x via a piecewise constant control. It turns out that this is nearly so. More precisely there is the following result (see [2]).

Theorem 2 Assume that x is in the interior (relative to the topology of $O(x)$) of the reachable set $\mathcal{R}(x)$ due to admissible inputs. Then every state which is in $\mathcal{R}(x)$ is also reachable from x via a piecewise constant input.

Finally, one may want to know whether there is a smooth input which enables one to reach y from x . To that end there is the following theorem (see e.g. [3]).

Theorem 3 Let y be a state of the system (1) that is reachable from x via piecewise constant inputs and is in the interior of the set of reachable states from x . Then y is also reachable from x via inputs in U_∞ .

So far the results on reachability have not addressed the following question: What exactly does $\mathcal{R}(x)$ look like as a subset of the slice through x , i.e. can we assert anything beyond that it contains an open subset of this slice? For this question there are answers only when $\mathcal{R}(x)$ actually equals the slice through x . A necessary and sufficient condition for $\mathcal{R}(x)$ to be all of the slice through x can be given in terms of an object called the Lie saturate (see e.g. [3]). Since the calculation of the Lie saturate is difficult in most cases, we will not state this criterion. Instead we will present a few cases where $\mathcal{R}(x)$ equals the slice through x . The cases presented are the ones that are directly relevant to the applications considered in this paper. The three classes of systems considered here are the following:

A.)

$$\dot{x} = Ax + \sum_{i=1}^2 u_i N_i x \quad (2)$$

where A and the N_i are skew-Hermitian matrices of the appropriate sizes, and $x \in S$ (S is the unit sphere in \mathbb{C}^n). This system arises in the study of the zero relaxation case. Its analysis is very similar to that carried out in [7].

B.)

$$\dot{U} = AU + \sum_{i=1}^2 u_i N_i U \quad (3)$$

where A and the N_i are as in (2) and U is a $n \times n$ unitary matrix. This system arises in our considerations in two ways. First, it is of importance in and of itself in the study of two dimensional experiments. As explained in Section 2 the matrix T_3 which determines the pattern of

cross peaks can be viewed as an element of the reachable set of the system (3) from the identity matrix. Second, it arises in the study of controllability of the system (2) itself. This connection will be explained soon.

C.) Finally, we will consider the following system:

$$\dot{x} = Ax + \sum_{i=1}^2 u_i(N_i x + b_i) \quad (4)$$

This is clearly of the form of the system relevant to us in the case of non-zero relaxation.

Let us begin by analyzing the system (2). We will study this in two distinct ways. The first will make no explicit reference to system (3) whereas the second will. The main result in this case is the following theorem.

Theorem 4 Consider system (2). The reachable set from a $x_0 \in S$ equals the slice through x_0 . In particular, if $\dim \Delta(x_0) = 2n - 1$, then $R(x_0) = S$ and thus every state is reachable from x_0 via controls in any of U_{ad} , U_{pc} or U_{∞} . Here Δ is the involutive distribution generated by Ax and $N_i x$.

Note that, in the event that $\dim \Delta(x_0) \neq 2n - 1$, all that we know is that the reachable set equals the slice. We do not, via this theorem, have any other description of the slice. The following considerations, involving the system (3), are better adapted to that purpose.

Indeed, system (3) describes the evolution of the infinitesimal generator corresponding to the system (2). If we can show that the reachable set from the identity matrix equals some group G , then the reachable set from $x_0 \in S$ equals the orbit of the group G through x_0 , i.e., the set of all points $y \in S$ which are of the form gx_0 where $g \in G$ (for the purposes of this paper, there is no loss of generality in supposing that G is some matrix group, and thus the notation gx_0 means the matrix g multiplying the vector x_0). Of course, except for some special cases a simple description of the orbit is not possible. Nevertheless, this represents an improvement over the general situation. In the event that the group G turns out to be a group acting transitively on S , then we can assert that, for any $x_0 \in S$ the reachable set from x_0 is all of S . Examples of subgroups of $U(n)$ which act transitively on S are $U(n)$ and $SU(n)$. A complete list of the classical matrix Lie groups which act transitively on the sphere is known (and is bigger than the list consisting of $U(n)$ and $SU(n)$). However, the main problem is in recognizing from the Lie algebra generated by A and the N_i (which is the information we possess) whether the group G is indeed one belonging to this list.

However, preliminary to that issue is the question of whether the reachable set from the identity of the system (3) is indeed a group. This is not true for all systems evolving on Lie groups. Thus the following result is of interest.

Theorem 5 Consider the system (3). Suppose that the unique connected Lie group, G , having the Lie algebra generated by the matrices A and the N_i is compact. Then the reachable set from the identity of the system (3) is G . In particular, if the dimension of the Lie algebra generated by the matrices A and the N_i is n^2 , then $G = U(n)$ and G is the reachable set from the identity matrix. Furthermore, in this case the reachable set from any $x_0 \in S$ of the system (2) (with A and the N_i 's the same) is all of S .

Finally, we consider the system (4).

Theorem 6 Assume that A is invertible. Suppose that the involutive distribution generated by Ax , $N_i x$ has dimension equal to $2n$ at $x_0 \in C^n$ (the dimension of C^n). Then, the reachable set from x_0 is all of C^n .

It is not known whether the appropriate statement in the event that the distribution has lower dimension, i.e., the statement that the reachable set through x_0 equals the slice through x_0 , is correct. We conjecture that it is. This is because it seems to us that the proof of Jurdjevic ([3]) relies on a calculation of the Lie saturate and this object is designed precisely to make assertions that the reachable set equals the slice.

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