THE FISHER INFORMATION MATRIX FOR TWO-DIMENSIONAL SEPARABLE-DENOMINATOR CONTINUOUS SYSTEMS

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ABSTRACT

In this paper, the Fisher information matrix for the noisy output data generated by a two-dimensional separable-denominator continuous system is derived in an explicit form in terms of system parameters. For uniformly sampled data it is shown through a simplified case that how the Fisher information matrix can be expressed through the solutions of Lyapunov equations.

1. INTRODUCTION

Data that can be considered to be generated by a two-dimensional (2D) separable-denominator continuous system appears in many areas of applications, e.g. nuclear magnetic resonance (NMR) spectroscopy [1]. Estimation of system parameters from measured data set is a frequently encountered problem in those applications. The question therefore naturally arises as to the accuracy with which these parameters can be estimated. The Cramer Rao lower bound (CRLB) gives a lower bound for the covariance of the parameter estimates of an unbiased estimation procedure for a given data set [2, 3]. The CRLB is in fact typically calculated as the inverse of a matrix called the Fisher information matrix. The relevance of this result is not only to evaluate a particular estimation procedure but can also give guidance for an appropriate design of an experiment to collect data [4].

Various methods have been suggested for the computation of the CRLB for undamped 2D exponential signals with additive noise [5]. However, to our best knowledge, a closed form expression for the CRLB for the parameter estimation problem of a 2D dynamic system is not available in the literature. Recently, a systematic investigation of the CRLB or Fisher information matrix for the case of one-dimensional (1D) deterministic dynamic systems corrupted by measurement noise is presented in [6]. In this paper, we generalize the results of [6] to 2D separable-denominator systems. This generalization is, however, not a straightforward extension of the results in [6] due to the significantly more intricate structure of 2D systems.

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We consider a 2D complex single-input single-output continuous system with a separable denominator using Roesser's model (RM)

$$\begin{bmatrix} \frac{\partial}{\partial t_1} x_\theta^h(t_1,t_2) \\ \frac{\partial}{\partial t_2} x_\theta^v(t_1,t_2) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_\theta^h(t_1,t_2) \\ x_\theta^v(t_1,t_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1,t_2), \quad (1)$$

$$y_\theta(t_1,t_2) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_\theta^h(t_1,t_2) \\ x_\theta^v(t_1,t_2) \end{bmatrix}, \quad (2)$$

where $t_1 \geq 0$, $t_2 \geq 0$, A_{11}, A_{12}, A_{22}, B_1, B_2, C_1 and C_2 are complex matrices of appropriate dimensions depending on the unknown parameter vector $\Theta := \begin{bmatrix} \theta_1 & \theta_2 & \ldots & \theta_K \end{bmatrix}^T$, $x_\theta^h(t_1,t_2)$, and $x_\theta^v(t_1,t_2)$ are horizontal and vertical state vectors respectively, $u(t_1,t_2)$ is the input, and $\frac{\partial}{\partial t_j}$ denotes partial derivative with respect to t_j $(j = 1,2)$. The boundary conditions are given by $x_\theta^h(0,t_2)$, $x_\theta^v(t_1,0)$, $t_1 \geq 0$, $t_2 \geq 0$. In the following lemma we characterize the input-output description of such a system. See [7] for a proof.

Lemma 1.1 *The output of the above 2D separable-denominator continuous system is given by*

$$y_\theta(t_1,t_2) = v_\theta(t_1,t_2) + q_\theta(t_1,t_2), \;\; t_1 \geq 0, \;\; t_2 \geq 0.$$

Here $v_\theta(t_1,t_2)$ is the system response due to non-zero boundary conditions and $q_\theta(t_1,t_2)$ is the system response due to system input, which are given by

$$v_\theta(t_1,t_2) = C_1 e^{A_{11}t_1} x_\theta^h(0,t_2) + C_2 e^{A_{22}t_2} x_\theta^v(t_1,0) + \int_0^{t_1} C_1 e^{A_{11}(t_1-\tau_1)} A_{12} e^{A_{22}t_2} x_\theta^v(\tau_1,0) d\tau_1,$$

and

$$q_\theta(t_1,t_2) = \int_0^{t_1} C_1 e^{A_{11}(t_1-\tau_1)} B_1 u(\tau_1,t_2) d\tau_1 + \int_0^{t_2} C_2 \cdot e^{A_{22}(t_2-\tau_2)} B_2 u(t_1,\tau_2) d\tau_2 + \int_0^{t_1}\int_0^{t_2} C_1 \cdot e^{A_{11}(t_1-\tau_1)} A_{12} e^{A_{22}(t_2-\tau_2)} B_2 u(\tau_1,\tau_2) d\tau_1 d\tau_2.$$

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Assume that we have acquired noise corrupted samples $s_\theta(n,m)$ $(n = 0, 1, \dots, N-1;\ m = 0, 1, \dots, M-1)$ of the measured output of a 2D separable-denominator continuous system at various points (t_{1n}, t_{2m}), i.e.

$$s_{\theta}(n,m) = y_\theta(t_{1n}, t_{2m}) + w(n,m),$$

where $y_\theta(t_{1n}, t_{2m})$ is the noise free data acquired at the sampling point (t_{1n}, t_{2m}) and $w(n,m)$ is the measurement noise component assumed to be complex Gaussian with zero mean. The real and imaginary parts of $w(n,m)$ are assumed to have variance $\sigma^2_{n,m}$, and to be independent/uncorrelated, i.e. $\mathrm{var}(\mathrm{Re}\{w(n,m)\}) = \mathrm{var}(\mathrm{Im}\{w(n,m)\}) = \sigma^2_{n,m}$ and $E(\mathrm{Re}\{w(n,m)\}\mathrm{Im}\{w(n,m)\}) = 0$.

By the Cramer Rao Lower bound, any unbiased estimate $\hat{\Theta}$ of Θ has a variance (provided certain regularity conditions hold) such that

$$\mathrm{var}(\hat{\Theta}) \geq I^{-1}(\Theta),$$

where $\mathrm{var}(\hat{\Theta}) \geq I^{-1}(\Theta)$ is interpreted as meaning that the matrix $(\mathrm{var}(\hat{\Theta}) - I^{-1}(\Theta))$ is positive semidefinite [2]. Here $I(\Theta)$ is the Fisher information matrix given by

$$[I(\Theta)]_{st} = -E\left(\frac{\partial^2 \ln(p(S;\Theta))}{\partial\theta_s \partial\theta_t}\right), \quad 1 \leq s,t \leq K,$$

where Θ is the unknown parameter vector, S is the measured data set, $p(S;\Theta)$ is the probability density function of the measurements and $E(\cdot)$ is the expected value with respect to the underlying probability measure.

Section 2 discusses the derivation of the Fisher information matrix for the 2D data set given by the system defined in (1) and (2). For the special but important case of uniformly sampled data we show by a simplified case in Section 3 that the computation of the Fisher information matrix can be reduced to the computation of solutions to certain Lyapunov equations. Proofs can be found in [7].

We denote by $\mathrm{diag}\,(M_1, M_2, \dots, M_r)$ the block diagonal matrix whose diagonal block entries are $M_1, M_2, \dots, M_r$, and all other block entries are zero matrices. We denote by $\mathrm{col}\,(M_1, M_2, \dots, M_r)$ the block column matrix as

$$\mathrm{col}\,(M_1, M_2, \dots, M_r) = \begin{bmatrix} M_1 & M_2 & \dots & M_r \end{bmatrix}^T.$$

2. FISHER INFORMATION MATRIX

With the Gaussian noise model discussed in Section 1 the probability density function is given by

$$p(S;\Theta) = \Pi_{n=0}^{N-1}\Pi_{m=0}^{M-1}\frac{1}{\sqrt{2\pi\sigma^2_{n,m}}} \cdot e^{-\frac{1}{2\sigma^2_{n,m}}[\mathrm{Re}\{s_\theta(n,m)\} - \mathrm{Re}\{y_\theta(t_{1n},t_{2m})\}]^2} \cdot e^{-\frac{1}{2\sigma^2_{n,m}}[\mathrm{Im}\{s_\theta(n,m)\} - \mathrm{Im}\{y_\theta(t_{1n},t_{2m})\}]^2}.$$

In the following lemma we are going to collect some basic results on the Fisher information matrix adapted to the particular data model that we consider [2].

Lemma 2.1 *1.) For* $1 \leq s,t \leq K$

$$[I(\Theta)]_{st} = -E\left(\frac{\partial^2 \ln(p(S;\Theta))}{\partial\theta_s\partial\theta_t}\right) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\frac{1}{\sigma^2_{n,m}} \cdot \mathrm{Re}\left\{\frac{\partial y_\theta(t_{1n},t_{2m})}{\partial\theta_s}\frac{\partial y^*_\theta(t_{1n},t_{2m})}{\partial\theta_t}\right\},$$

where $(\cdot)^*$ *denotes conjugate.*

2.) Let

$$D_{y_\theta(t_{1n},t_{2m})} := \mathrm{col}\,\left(\frac{\partial y_\theta(t_{1n},t_{2m})}{\partial\theta_1}, \frac{\partial y_\theta(t_{1n},t_{2m})}{\partial\theta_2}, \dots, \frac{\partial y_\theta(t_{1n},t_{2m})}{\partial\theta_K}\right).$$

Then

$$I(\Theta) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\frac{1}{\sigma^2_{n,m}}\mathrm{Re}\left\{D_{y_\theta(t_{1n},t_{2m})}D^H_{y_\theta(t_{1n},t_{2m})}\right\},$$

where $(\cdot)^H$ *denotes complex conjugate transpose.*

In order to calculate the Fisher information matrix it is necessary to compute the derivative $\frac{\partial y_\theta(t_{1n},t_{2m})}{\partial\theta_s}$ of the output with respect to the elements θ_s of the parameter vector Θ, $s = 1, \dots, K$. We first quote Lemma 5.2-30 in [3].

Lemma 2.2 *Let A be any complex square matrix depending on the parameter vector Θ, and denote for $s = 1, \dots, K$, $t \geq 0$,*

$$\partial_s(e^{At}) := \begin{bmatrix} e^{At} & 0 \\ \frac{\partial e^{At}}{\partial\theta_s} & e^{At} \end{bmatrix}, \quad \partial_s A := \begin{bmatrix} A & 0 \\ \frac{\partial A}{\partial\theta_s} & A \end{bmatrix}.$$

We have for $s = 1, \dots, K$ and $t \geq 0$,

$$\partial_s(e^{At}) = e^{\partial_s A t}.$$

In the following lemma, we consider the derivative of a product of matrices depending on the parameter vector Θ.

Lemma 2.3 *Consider the matrix product $H_1H_2\cdots H_l$, where H_1, H_2, $\cdots$, H_l are matrices depending on the parameter vector Θ. Then*

$$\frac{\partial(H_1H_2\cdots H_l)}{\partial\theta_s} = \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix}\begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix}\cdots \times \begin{bmatrix} H_{l-1} & 0 \\ \frac{\partial H_{l-1}}{\partial\theta_s} & H_{l-1} \end{bmatrix}\begin{bmatrix} H_l \\ \frac{\partial H_l}{\partial\theta_s} \end{bmatrix}.$$

Lemma 2.4 *With the notations in Section 1, consider for $t_1 \geq 0$, $t_2 \geq 0$,*

$$y_\theta(t_1,t_2) = v_\theta(t_1,t_2) + q_\theta(t_1,t_2).$$

Then for $1 \leq s \leq K$,

$$\frac{\partial y_\theta(t_1,t_2)}{\partial\theta_s} = \frac{\partial v_\theta(t_1,t_2)}{\partial\theta_s} + \frac{\partial q_\theta(t_1,t_2)}{\partial\theta_s},$$

where

$$\frac{\partial v_\theta(t_1,t_2)}{\partial\theta_s} = \partial_s C_1 e^{\partial_s A_{11} t_1}\partial_s x_\theta^h(0,t_2) + \partial_s C_2 e^{\partial_s A_{22} t_2}$$
$$\cdot\, \partial_s x_\theta^v(t_1,0) + \int_0^{t_1} \partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s A_{12} e^{\partial_s A_{22} t_2}$$
$$\cdot\, \partial_s x_\theta^v(\tau_1,0)d\tau_1,$$

and

1.) for bounded piecewise continuous input $u(t_1,t_2)$

$$\frac{\partial q_\theta(t_1,t_2)}{\partial\theta_s} = \int_0^{t_1} \partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s B_1 u(\tau_1,t_2)d\tau_1$$
$$+ \int_0^{t_2} \partial_s C_2 e^{\partial_s A_{22}(t_2-\tau_2)}\partial_s B_2 u(t_1,\tau_2)d\tau_2$$
$$+ \int_0^{t_1}\int_0^{t_2} \partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s A_{12} e^{\partial_s A_{22}(t_2-\tau_2)}$$
$$\cdot\, \partial_s B_2 u(\tau_1,\tau_2)d\tau_1 d\tau_2 + \partial_s D u(t_1,t_2);$$

2.) for impulse response input $u(t_1,t_2)=\delta(t_1,t_2)$, *with* $B_1=0$, $C_2=0$ *and* $D=0$

$$\frac{\partial q_\theta(t_1,t_2)}{\partial\theta_s} = \partial_s C_1 e^{\partial_s A_{11} t_1}\partial_s A_{12} e^{\partial_s A_{22} t_2}\partial_s B_2.$$

Here

$$\partial_s A_{11} := \begin{bmatrix} A_{11} & 0 \\ \frac{\partial A_{11}}{\partial\theta_s} & A_{11}\end{bmatrix},\ \partial_s A_{12} := \begin{bmatrix} A_{12} & 0 \\ \frac{\partial A_{12}}{\partial\theta_s} & A_{12}\end{bmatrix},$$
$$\partial_s A_{22} := \begin{bmatrix} A_{22} & 0 \\ \frac{\partial A_{22}}{\partial\theta_s} & A_{22}\end{bmatrix},\ \partial_s B_1 := \begin{bmatrix} B_1 \\ \frac{\partial B_1}{\partial\theta_s}\end{bmatrix},$$
$$\partial_s B_2 := \begin{bmatrix} B_2 \\ \frac{\partial B_2}{\partial\theta_s}\end{bmatrix},\ \partial_s C_1 := \begin{bmatrix} \frac{\partial C_1}{\partial\theta_s} & C_1\end{bmatrix},$$
$$\partial_s C_2 := \begin{bmatrix} \frac{\partial C_2}{\partial\theta_s} & C_2\end{bmatrix},\ \partial_s D := \frac{\partial D}{\partial\theta_s},$$
$$\partial_s x_\theta^v(t_1,0) := \begin{bmatrix} x_\theta^v(t_1,0) \\ \frac{\partial x_\theta^v(t_1,0)}{\partial\theta_s}\end{bmatrix},\ \partial_s x_\theta^h(0,t_2) := \begin{bmatrix} x_\theta^h(0,t_2) \\ \frac{\partial x_\theta^h(0,t_2)}{\partial\theta_s}\end{bmatrix}.$$

In the following theorem we summarize the previous results and state the general expression for the Fisher information matrix for the data set corresponding to the output of a 2D separable-denominator continuous system.

Theorem 2.1 *Consider the augmented derivative system given by*

$$D_{A_{11}} := \text{diag}\,(\partial_1 A_{11}, \partial_2 A_{11}, \ldots, \partial_K A_{11}),$$
$$D_{A_{22}} := \text{diag}\,(\partial_1 A_{22}, \partial_2 A_{22}, \ldots, \partial_K A_{22}),$$
$$D_{A_{12}} := \text{diag}\,(\partial_1 A_{12}, \partial_2 A_{12}, \ldots, \partial_K A_{12}),$$
$$D_{C_1} := \text{diag}\,(\partial_1 C_1, \partial_2 C_1, \ldots, \partial_K C_1),$$
$$D_{C_2} := \text{diag}\,(\partial_1 C_2, \partial_2 C_2, \ldots, \partial_K C_2),$$
$$D_{B_1} := \text{col}\,(\partial_1 B_1, \partial_2 B_1, \ldots, \partial_K B_1),$$
$$D_{B_2} := \text{col}\,(\partial_1 B_2, \partial_2 B_2, \ldots, \partial_K B_2),$$
$$D_D := \text{col}\,(\partial_1 D, \partial_2 D, \ldots, \partial_K D),$$
$$D_{x_\theta^h(0,t_2)} := \text{col}\,(\partial_1 x_\theta^h(0,t_2), \partial_2 x_\theta^h(0,t_2), \ldots, \partial_K x_\theta^h(0,t_2)),$$
$$D_{x_\theta^v(t_1,0)} := \text{col}\,(\partial_1 x_\theta^v(t_1,0), \partial_2 x_\theta^v(t_1,0), \ldots, \partial_K x_\theta^v(t_1,0)).$$

Then

1.) for $t_1 \geq 0$, $t_2 \geq 0$,

$$D_{y_\theta(t_1,t_2)} := D_{v_\theta(t_1,t_2)} + D_{q_\theta(t_1,t_2)},$$

where

$$D_{v_\theta(t_1,t_2)} = D_{C_1} e^{D_{A_{11}} t_1} D_{x_\theta^h(0,t_2)} + D_{C_2} e^{D_{A_{22}} t_2} D_{x_\theta^v(t_1,0)}$$
$$+ \int_0^{t_1} D_{C_1} e^{D_{A_{11}}(t_1-\tau_1)} D_{A_{12}} e^{D_{A_{22}} t_2} D_{x_\theta^v(\tau_1,0)} d\tau_1,$$

and

a) for bounded piecewise continuous input $u(t_1,t_2)$

$$D_{q_\theta(t_1,t_2)} = \int_0^{t_1} D_{C_1} e^{D_{A_{11}}(t_1-\tau_1)} D_{B_1} u(\tau_1,t_2)d\tau_1$$
$$+ \int_0^{t_2} D_{C_2} e^{D_{A_{22}}(t_2-\tau_2)} D_{B_2} u(t_1,\tau_2)d\tau_2$$
$$+ \int_0^{t_1}\int_0^{t_2} D_{C_1} e^{D_{A_{11}}(t_1-\tau_1)} D_{A_{12}} e^{D_{A_{22}}(t_2-\tau_2)}$$
$$\cdot\, D_{B_2} u(\tau_1,\tau_2)d\tau_1 d\tau_2 + D_D u(t_1,t_2);$$

b) for impulse response input $u(t_1,t_2)=\delta(t_1,t_2)$, *with* $B_1=0$, $C_2=0$ *and* $D=0$

$$D_{q_\theta(t_1,t_2)} = D_{C_1} e^{D_{A_{11}} t_1} D_{A_{12}} e^{D_{A_{22}} t_2} D_{B_2};$$

2.) for the 2D data set sampled at (t_{1n}, t_{2m}) $(n = 0,1,\ldots,N-1;\ m = 0,1,\ldots,M-1)$, *the Fisher information matrix is*

$$I(\Theta) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \frac{1}{\sigma_{n,m}^2}\text{Re}\left\{D_{y_\theta(t_{1n},t_{2m})} D^H_{y_\theta(t_{1n},t_{2m})}\right\}. \quad (3)$$

3. FISHER INFORMATION MATRIX FOR UNIFORMLY SAMPLED 2D DATA

Although Theorem 2.1 in the previous section is valid for both uniform and nonuniform sampling schemes, it is computationally rather inefficient to directly compute the 2D summations in (3), particularly in the case when the number of samples is large in one or both of the variables. In this section, we develop an efficient method for calculating the Fisher information matrix for 2D data generated by uniformly sampling the output of a 2D separable-denominator continuous system. For illustration purpose, we assume the system output is given by

$$y(t_1,t_2) = C_1 e^{A_{11} t_1} F + C_1 e^{A_{11} t_1} A_{12} e^{A_{22} t_2} B_2,\ t_1, t_2 \geq 0.$$

To this end, it is assumed that all the eigenvalues of $e^{A_{11}T_1}$ and $e^{A_{22}T_2}$ are in the open unit disc or equivalently the eigenvalues of A_{11} and A_{22} are in the open half plane, where T_1 and T_2 are the sampling intervals for the variables t_1 and t_2 respectively. For convenience of exposition, we denote $A_{d1} := e^{D_{A_{11}}T_1}$ and $A_{d2} := e^{D_{A_{22}}T_2}$.

Theorem 3.1 *Consider a 2D separable-denominator continuous system is such that the input $u(t_1, t_2)$ is a 2D unit impulse function, i.e. $u(t_1, t_2) = \delta(t_1, t_2)$. The boundary conditions are given by $x_\theta^v(t_1, 0) = 0$ and $x_\theta^h(0, t_2) = F$, where F may depend on the parameter vector Θ. The matrices B_1, C_2, D are all zero matrices. These assumptions imply that the deterministic part of the measured signal is given by*

$$y(t_{1n}, t_{2m}) = C_1 e^{A_{11}t_{1n}} F + C_1 e^{A_{11}t_{1n}} A_{12} e^{A_{22}t_{2m}} B_2.$$

Assume that the signal is uniformly sampled with sampling interval T_1 for the variable t_1 and T_2 for t_2 respectively, i.e., at $t_{1n} = nT_1$, $n = 0, 1, \ldots, N-1$; $t_{2m} = mT_2$, $m = 0, 1, \ldots, M-1$. Moreover, assume that all the eigenvalues of A_{11} and A_{22} are in the open left half plane. Then the Fisher information matrix for the 2D data set is given by

$$I(\Theta) = \frac{1}{\sigma^2}\mathrm{Re}\left\{D_{C_1}\left[MP_1 + P_2 + P_3\right]D_{C_1}^H\right\}, \quad (4)$$

where P_1, P_2 and P_3 can be obtained as follows.

P_1 is the unique solution to the following Lyapunov equation

$$A_{d1}P_1A_{d1}^H - P_1 = -D_FD_F^H + A_{d1}^N D_F D_F^H (A_{d1}^N)^H.$$

P_2 is the unique solution to the following Lyapunov equation

$$A_{d1}P_2A_{d1}^H - P_2 = -R + A_{d1}^N R (A_{d1}^N)^H,$$

where $R = D_{A_{12}}(I - A_{d2}^M)(I - A_{d2})^{-1}D_{B_2}D_F^H + D_F D_{B_2}^H((I - A_{d2})^{-1})^H(I - A_{d2}^M)^H D_{A_{12}}^H$.

P_4 is the unique solution to the following Lyapunov equation

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2} D_{B_2}^H (A_{d2}^M)^H,$$

and P_3 is the unique solution to the following Lyapunov equation

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H + A_{d1}^N D_{A_{12}} P_4 D_{A_{12}}^H (A_{d1}^N)^H.$$

Here, $\partial_s F := \begin{bmatrix} F \\ \frac{\partial F}{\partial \theta_s} \end{bmatrix}$ *and* $D_F := \mathrm{col}\,(\partial_1 F, \partial_2 F, \ldots, \partial_K F)$.

In the case that there are an infinite number of equidistant samples in the t_1 variable, i.e. $N \to \infty$ in Theorem 3.1, the previous theorem can be simplified.

Corollary 3.1 *Assume that the 2D system is the same as in Theorem 3.1 except that there are an infinite number of equidistant samples in the t_1 variable, i.e., $t_{1n} = nT_1$, $n = 0, 1, \ldots, \infty$. Then the Fisher information matrix is given by*

$$I(\Theta) = \frac{1}{\sigma^2}\mathrm{Re}\left\{D_{C_1}\left[MP_1 + P_2 + P_3\right]D_{C_1}^H\right\}, \quad (5)$$

where P_1 is the unique solution to the following Lyapunov equation

$$A_{d1}P_1A_{d1}^H - P_1 = -D_FD_F^H.$$

P_2 is the unique solution to the following Lyapunov equation

$$A_{d1}P_2A_{d1}^H - P_2 = -\left(D_{A_{12}}(I - A_{d2}^M)(I - A_{d2})^{-1}D_{B_2}D_F^H + D_F D_{B_2}^H((I - A_{d2})^{-1})^H(I - A_{d2}^M)^H D_{A_{12}}^H\right).$$

P_4 is the same as that in Theorem 3.1, and P_3 is the unique solution to the following Lyapunov equation

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H.$$

Note that the expressions for the Fisher information matrix given in Corollary 3.1 are not only simpler than those in Theorem 3.1, but also give the asymptotic Fisher information matrix when an infinite number of samples are available in t_1.

4. CONCLUSIONS

In this paper, we have developed an explicit expression for the calculation of the Cramer Rao lower bound for the output data set generated by 2D separable-denominator continuous systems. For the special but important case of uniform sampling, the Lyapunov approach is exploited which has speeded up considerably for the calculation of the Fisher information matrix. We believe that the presented results will have a significant impact on applications dealing with a large number of data samples and a large number of parameters to be estimated. Further discussion on implementation details and experimental results can be found in [7].

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