Infinite Dimensional Balanced Realizations and their Approximation

Raimund Ober

Department of Engineering, University of Cambridge,
Trumpington Street, Cambridge, CB2 1PZ, England.

Abstract

The problem of generalizing the notion of a balanced realization to infinite dimensional systems is considered. The approach taken is based on the generalization of a parametrezation of finite dimensional balanced realizations. Using semigroup theory, an existence theorem and approximation results are derived.

I. Introduction

Balanced realizations for finite dimensional linear systems as introduced by B.C. Moore ([5]) have attracted a great deal of attention, mainly due to their interesting properties with respect to model reduction. These properties suggest that balanced realizations could be useful to consider the important problem of the approximation of infinite dimensional systems.

While Curtain and Glover ([1]) consider a realization problem, the approach taken in this paper to the generalization of balanced realizations to the infinite dimensional setting is motivated by structural aspects as they appear in the parametrezation of finite dimensional balanced systems. It is hoped that this approach leads to some insight in the underlying principles of model reduction of balanced realizations.

Existence and characterization problems will be considered in Section II. The main emphasis will be placed on the problem of finite dimensional approximations of infinite dimensional balanced realizations as discussed in Section III.

l^2 denotes the Hilbert space of square summable sequences, whereas $L^2([0,\infty[)$ is the Hilbert space of square integrable functions on $[0,\infty[$. A strongly continuous semigroup with generator $(A, D(A))$ will be denoted by $(e^{tA})_{t\geq 0}$. For standard techniques in semigroup theory I refer to Pazy ([7]) and for Hilbert space arguments see Weidmann ([8]).

For the proofs of the results presented here see ([6]).

II. Existence and Characterization Results

B.C. Moore ([5]) gave the following definition:

Definition II.1

Let (A, b, c) be a minimal realization of an n-dimensional asymptotically stable SISO system. Then (A, b, c) is balanced if

$$\int_0^\infty e^{tA} bb^T e^{tA^T} dt = \int_0^\infty e^{tA^T} c^T c\, e^{tA} dt = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

The positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ are called the singular values of the system (A, b, c). □

For the case of distinct singular values the following characterization can be given.

Theorem II.2.([1], [4], [6])

Let $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$.

Then:

(A, b, c) is a minimal balanced realization of an asymptotically stable system with distinct singular values $\sigma_1, \sigma_2, \ldots, \sigma_n \to 0$

if and only if

$$b = (b_1, b_2, \ldots, b_n)^T, \quad b_i \neq 0 \quad \text{for} \quad 1 \leqslant i \leqslant n$$
$$c = (s_1 b_1, s_2 b_2, \ldots, s_n b_n), \quad s_i = \pm 1 \quad \text{for} \quad 1 \leqslant i \leqslant n$$
$$A = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1\leqslant i,\, j\leqslant n} \quad \square$$

This charactization is taken as a starting point for the generalization of the concept of balanced realizations to infinite dimensional systems. This approach is further motivated by a result by Curtain and Glover ([2]) in connection with the derivation of a balanced realization for a certain class of systems. It follows easily from their result that a balanced realization (A, b, c) of such a system can be parametrized as:

$$b = (b_1, b_2, \ldots)^T, \quad b_i \neq 0 \quad \text{for} \quad 1 \leqslant i < \infty$$
$$c = (s_1 b_1, s_2 b_2, \ldots), \quad s_i = \pm 1 \quad \text{for} \quad 1 \leqslant i < \infty$$

and the generator A has a matrix representation given by:

$$A = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1\leqslant i,\, j<\infty}$$

Moreover, the domain of definition $D(A)$ of the generator A contains the finite sequences.

The main theorem of this section shows that an infinite dimensional matrix parametrized in this way is the generator of a semigroup of contractions.

Theorem II.3

Let $A = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1\leqslant i,\, j<\infty}$,

where $b = (b_1, b_2, \ldots)^T \in l^2$, $b_i \neq 0$, for $1 \leqslant i < \infty$

$\sigma = (\sigma_1, \sigma_2, \ldots)$ s.t. $0 < \sigma_i < M$, $M \in \mathbb{R}$,

$\sigma_i \neq \sigma_j$, for all $i \neq j$

$s_i = \pm 1$ for $1 \leqslant i < \infty$ s.t. $\lim_{i\to\infty} s_i$ exists.

Then there exists a domain of definition $D(A) \subseteq l^2$ containing the finite sequences such that $(A, D(A))$ generates a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ of contractions. □

Thus it makes sense to say that (A, b, c) is a balanced realization if (A, b, c) is given as in the previous theorem.

For the case of a selfadjoint generator we can now formulate a characterization result analagous to the finite dimensional case.

Theorem II.4

Let $(A, D(A))$ be the generator of a strongly continuous semigroup on l^2 such that $D(A)$ contains the finite sequences. Consider the system $((A, D(A)), b)$, $b \in l^2$. The following two statements are equivalent:

(i) A has a matrix representation

$$A = \left(\frac{-b_i b_j}{\sigma_i + \sigma_j}\right)_{1\leqslant i,\, j<\infty}, \quad b_i \neq 0 \quad \text{for} \quad 1 \leqslant i < \infty$$

with $0 < \sigma_i < M < \infty$ for all $1 \leqslant i < \infty$

and $\sigma_i \neq \sigma_j$ for all $i \neq j$.

(ii) A is selfadjoint and

a.) $\int_0^\infty e^{tA} b b^T e^{tA} dt = diag(\sigma_1, \sigma_2, \ldots)$ with $0 < \sigma_i < M < \infty$ for all $1 \leqslant i < \infty$ and $\sigma_i \neq \sigma_j$ for all $i \neq j$.

b.) $\lim_{t\to\infty} b^T e^{tA} x = 0$ for all $x \in l^2$. □

III. Finite Dimensional Approximations

Let (A, b, c) be a system in balanced form as defined in Section II, i.e. $b = (b_1, b_2, \ldots)^T$, $c = (s_1 b_1, s_2 b_2, \ldots)$

$$A = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1 \leqslant i, j < \infty}$$

with the parameters satisfying the conditions of Theorem II.3. We consider the following truncation $(A(n), b(n), c(n))$, $n \geqslant 1$, of this system, with

$$A(n) = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1 \leqslant i, j \leqslant n}$$
$$b(n) = (b_1, b_2, \ldots, b_n)^T,$$
$$c(n) = (s_1 b_1, s_2 b_2, \ldots, s_n b_n)$$

By Theorem II.2 $(A(n), b(n), c(n))$ is an n-dimensional balanced, minimal and asymptotically stable system. In order to be able to formulate the following theorems we have to interpret $A(n)$ as acting on l^2 by

$$x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)^T \mapsto \left(\left(A(n)\,(x_1, \ldots, x_n)^T\right)^T, 0, 0, \ldots\right)^T$$

Similarly $b(n)$, $c(n)$ are interpreted as operators on l^2.

The following Trotter-Kato type theorem is the key result in this section.

Theorem III.1

Let $(A, D(A))$ be given as in Theorem II.3, then for $(A(n))_{n \geqslant 1}$ defined as above we have for all $x \in l^2$,

$$e^{tA(n)} x \to e^{tA} x \quad \text{as} \quad n \to \infty,\ t \geqslant 0$$

The limit is uniform in t for t in bounded intervals. □

This allows us to show the following convergence result for observability and controllability operators. For a system (A, b, c) the observability operator ϑ is defined by

$$\vartheta \ : \ l^2 \to L^2([o, \infty[)$$
$$x \mapsto c e^{tA} x.$$

The reachability operator is given as

$$\mathcal{R} \ : \ L^2([0, \infty[) \to l^2$$
$$u(t) \mapsto \int_0^\infty e^{tA} b u(t) dt.$$

Theorem III.2

Let (A, b, c) be defined as above. Then ϑ and $\mathcal{R}$ are bounded. Moreover,

1.) If $(\vartheta_n)_{n \geqslant 1}$, $(\mathcal{R}_n)_{n \geqslant 1}$ are the observability operators of the approximating systems $\left((A(n), b(n), c(n)\right)_{n \geqslant 1}$

Then $\vartheta_n \to \vartheta$ weakly
$\mathcal{R}_n \to \mathcal{R}$ weakly.

2.) If we have

$$\int_0^\infty e^{tA} b b^T e^{tA^T} dt = \int_0^\infty e^{tA^T} c^T c\, e^{tA} dt = diag(\sigma_1, \sigma_2, \ldots)$$

then

$\vartheta_n \to \vartheta$ strongly
$\mathcal{R}_n \to \mathcal{R}$ strongly □

Corollary III.3

For the impulse responses $h_n(t) = c(n) e^{tA(n)} b(n)$, $n \geqslant 1$, of the approximating systems we have

$$h_n(t) \to h(t) \quad \text{weakly in} \quad L^2([0, \infty[);$$

If moreover

$$\int_0^\infty e^{tA} b b^T e^{tA^T} dt = \int_0^\infty e^{tA^T} c^T c\, e^{tA} dt = diag(\sigma_1, \sigma_2, \ldots)$$

then

$$h_n(t) \to h(t) \quad \text{in} \quad L^2([0, \infty[). \quad \square$$

Hankel operators are of great importance in model reduction ([3]). The Hankel operator associated with a linear system (A, b, c) is defined by:

$$H \ : \ L^2([0, \infty[) \to L^2([0, \infty[)$$
$$h(t) \mapsto (H(u))(s) = \int_0^\infty h(t + s)\ u(t)\ dt,$$

where $h(t) = c e^{tA} b$.

Corollary III.4

For the Hankel operators $(H_n)_{n \geqslant 1}$ of the approximating systems we have

$$H_n \to H \quad \text{weakly}$$

If we have moreover

$$\int_0^\infty e^{tA} b b^T e^{tA^T} dt = \int_0^\infty e^{tA^T} c^T c\, e^{tA} dt = diag(\sigma_1, \sigma_2, \ldots)$$

then

$$H_n \to H \quad \text{strongly.} \quad \square$$

Remark 1:

The approximation results presented here can be applied to the balanced realizations considered in ([1]) provided that the conditions in Theorem II.3 are fulfilled.

Remark 2:

It can be shown ([6]) that the results in this paper also hold under the more general conditions

$$b = (b_1, b_2, \ldots)^T \in l^2 \quad , \ b_i \neq 0 \quad \text{for } 1 \leqslant i < \infty$$
$$s_i = \pm 1 \quad \text{and} \quad \sigma_1 > \sigma_2 > \ldots$$

such that $A\Sigma$ is bounded as an operator on l^2 where

$$A = \left(\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1 \leqslant i, j < \infty} \quad \text{and} \quad \Sigma = diag(\sigma_1, \sigma_2, \ldots)$$

References

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