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Topology of the set of asymptotically stable minimal systems

RAIMUND J. OBER†

It is shown that the set of asymptotically stable and minimal single-input–single-output systems of order n has $(n+1)$ connected components. In the case of asymptotically stable and minimal multiple-input–multiple-output systems, however, there is only one connected component.

1. Introduction

Glover (1975) and Brockett (1976) pointed out the importance of the topological and geometric structure of the set of linear systems for system identification. Brockett (1976) showed that the set of minimal nth-order single-input–single-output (SISO) systems has $(n+1)$ pathwise connected components. Glover (1975) had proved that the set of multiple-input–multiple-output (MIMO) systems, however, has only one pathwise connected component. Using the canonical form derived by Ober (1987), it is shown that the same results are true if we restrict ourselves to asymptotically stable systems. The reason for considering the set of asymptotically stable systems is that in system identification the model set is normally restricted to this class of systems.

Let $L_n^{p,m}$ denote the set of all n-dimensional minimal state-space systems with p-dimensional output space and m-dimensional input space and let $C_n^{p,m}$ be the subset of all continuous-time asymptotically stable systems. $D_n^{p,m}$ denotes the subset of all discrete-time asymptotically stable systems.

As is well known, two systems (A_1, B_1, C_1) and (A_2, B_2, C_2) in $L_n^{p,m}$ realize the same transfer function if and only if (A_1, B_1, C_1) and (A_2, B_2, C_2) are equivalent with respect to system equivalence (write $(A_1, B_1, C_1) \sim (A_2, B_2, C_2)$), i.e. if and only if there exists $T \in GI(n)$ such that $A_1 = TA_2T^{-1}$, $B_1 = TB_2$ and $C_1 = C_2T^{-1}$. We therefore consider the quotient space $L_n^{p,m}/\sim$ of $L_n^{p,m}$ with respect to system equivalence. Similarly, we shall be considering $C_n^{p,m}/\sim$ and $D_n^{p,m}/\sim$.

The sets $L_n^{p,m}/\sim$, $C_n^{p,m}/\sim$ and $D_n^{p,m}/\sim$ will be topologized as follows:

Embed $L_n^{p,m}$ in $\mathbb{R}^{n^2+nm+np}$ by

$$(A, B, C) \to (a^1, a^2, \ldots, a^n, b^1, \ldots, b^n, (c^1), \ldots, (c^n))$$

where a^i, b^i, c^i are the ith rows of A, B, C^T respectively.

If $L_n^{p,m} \subseteq \mathbb{R}^{n^2+nm+np}$ is endowed with the subspace topology, we assume $L_n^{p,m}/\sim$ to have the quotient topology. Denote by $\pi: L_n^{p,m} \to L_n^{p,m}/\sim$ the canonical projection. Similar definitions apply to $C_n^{p,m}/\sim$ and $D_n^{p,m}/\sim$.

Section 2 contains background definitions and results on balanced realizations, including a characterization of all asymptotically stable and minimal systems of a given order. The case of SISO systems is treated in § 3, and § 4 contains the results concerning MIMO systems.

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† Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, England.

2. Balanced realizations

In this section continuous- and discrete-time balanced realizations will be defined as they were introduced by Moore (1981).

2.1. *Continuous-time balanced realizations*

The definition of a balanced realization for continuous-time systems is as follows.

Definition 2.1 (Moore 1981)

Let $(A, B, C) \in C_n^{p,m}$. (A, B, C) is balanced if for

$$W_c = \int_0^\infty \exp(At)BB^T \exp(tA^T)\,dt$$

$$W_0 = \int_0^\infty \exp(A^T t)C^T C \exp(tA)\,dt$$

we have $W_c = W_0 =: \Sigma =: \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$.

The positive numbers $\sigma_1, \ldots, \sigma_n$ are called the singular values of the system (A, B, C). Denote by $C_n^{p,m,b} \subseteq C_n^{p,m}$ the subset of all balanced systems.

Analogously to the continuous-time case, minimal and asymptotically stable discrete-time systems are said to be balanced if the controllability and observability gramians are identical and diagonal.

Definition 2.2 (Moore 1981)

Let $(A, B, C) \in D_n^{p,m}$. (A, B, C) is balanced if for

$$W_c = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k$$

$$W_0 = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

we have $W_c = W_0 =: \Sigma =: \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$.

The positive numbers $\sigma_1, \ldots, \sigma_n$ are called the singular values of the system (A, B, C). Denote by $D_n^{p,m,b} \subseteq D_n^{p,m}$ the subset of all balanced systems.

The following proposition shows that topological results on continuous-time systems can be carried over to discrete-time systems.

Proposition 2.3

There exists a homeomorphism

$$T_n : C_n^{p,m} \to D_n^{p,m}$$

which preserves system equivalence and maps continuous-time balanced systems to discrete-time balanced systems. □

Proof

Follows from Proposition 4.1 of Ober (1987).

The following characterization and parametrization result (Ober 1987) in terms of a canonical form is central for the remainder of the paper.

Theorem 2.4

The following two statements are equivalent.

(I) $$(\tilde{A}, \tilde{B}, \tilde{C}) \in C_n^{p,m}$$

with $\Sigma = \operatorname{diag}(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \ldots, \sigma_k I_{n(k)})$

such that $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ and $\sum_{j=1}^{k} n(j) = n$.

(II) There exists a unique $T \in GI(n)$ such that for

$$(A, B, C) := \Gamma((\tilde{A}, \tilde{B}, \tilde{C})) := (T\tilde{A}T^{-1}, T\tilde{B}, \tilde{C}T^{-1})$$

we have:

B-matrix

(1) Partition $B = \begin{bmatrix} B^1 \\ \vdots \\ B^k \end{bmatrix}$ with $B^j \in \mathbb{R}^{n(j) \times m}$

then for $1 \leqslant j \leqslant k$,

$$B^j (B^j)^{\mathrm{T}} = \operatorname{diag}(\lambda_1^{(j)} I_{r(j,1)}, \lambda_2^{(j)} I_{r(j;2)}, \ldots, \lambda_{l(j)}^{(j)} I_{r(j;l(j))}, 0, \ldots, 0)$$

such that

$$\lambda_1^{(j)} > \lambda_2^{(j)} > \ldots > \lambda_{l(j)}^{(j)} > 0$$

and

$$\sum_{i=1}^{l(j)} r(j;i) =: r_0(j) \leqslant \min(p, m)$$

(2) For each $1 \leqslant j \leqslant k$, B^j has the following structure:

$$B^j = \begin{bmatrix} B(j;1) \\ \vdots \\ B(j;l(j)) \\ \mathbf{0} \end{bmatrix} \quad \text{with } B(j;i) \in \mathbb{R}^{r(j;i) \times m} \quad \text{for } 1 \leqslant i \leqslant l(j)$$

The precise structure of $B(j;i) =: (b(j;i)_{st})_{\substack{l \leqslant s \leqslant r(j;i) \\ 1 \leqslant t \leqslant m}}$ is given by the indices

$$1 \leqslant t(j;i,1) < t(j;i,2) < \ldots < t(j;i,r(j;i)) \leqslant m \quad \text{for } 1 \leqslant i \leqslant l(j)$$

We have

$$b(j;i)_{st(j;i,s)} > 0 \quad \text{for all } 1 \leqslant s \leqslant r(j;i)$$

$$b(j;i)_{st} = 0 \quad \text{for all } 1 \leqslant t < t(j;i,s) \quad \text{and} \quad 1 \leqslant s \leqslant r(j;i)$$

i.e.

$$B(j;i)=\begin{bmatrix} 0\dots 0 & b(j;i)_{1t(j;i,1)} & & & & & \\ 0\dots 0 & 0 & \dots & 0 & b(j;i)_{2t(j;i,2)} & & b(j;i)_{st} \\ \vdots & \vdots & & \vdots & 0 & \dots & \\ 0\dots 0 & 0 & \dots & 0 & 0 & \dots & 0 \quad b(j;i)_{r(j;i)t(j,r(j;i))} \end{bmatrix}$$

C-matrix

C admits the representation

$$C=(C^1 \quad C^2 \quad \dots \quad C^k), \quad C^j \in \mathbb{R}^{p\times n(j)}$$

with

$$C^j=(U^j \quad \mathbf{0})\,\mathrm{diag}\,((\lambda_1^{(j)})^{1/2} I_{r(j,1)}, \dots, (\lambda_{l(j)}^{(j)})^{1/2} I_{r(j,l(j))}, 0, \dots, 0)$$

for unique $U^j \in \mathbb{R}^{p\times r_0(j)}$ such that $(U^j)^{\mathrm{T}} U^j = I_{r_0(j)}$ for $1 \leqslant j \leqslant k$.

A-matrix

A admits a partitioning $A=(A(i,j))_{1\leqslant i,j\leqslant k}$ with $A(i,j)\in\mathbb{R}^{n(i)\times n(j)}$ for all $1\leqslant i,j\leqslant k$, with the following properties.

(i) *Block-diagonal entries* $A(j,j)$

$A(j,j)$, $1\leqslant j\leqslant k$, can be partitioned as

$$A(j,j)=\begin{bmatrix} A(j,j)_{11} & A(j,j)_{12} \\ A(j,j)_{21} & A(j,j)_{22}\end{bmatrix}, \quad A(j,j)_{11}\in\mathbb{R}^{r_0(j)\times r_0(j)}$$

with

(1) $A(j,j)_{11}=\dfrac{-1}{2\sigma_j}\mathrm{diag}\,(\lambda_1^{(j)} I_{r(j;1)}, \dots, \lambda_{l(j)}^{(j)} I_{r(j;l(j))}) + \tilde{A}(j,j)_{11}$

where $\tilde{A}(j,j)_{11}$ is skew-symmetric.

(2) There exists $q(j)\in\mathbb{N}$, $q(j)\geqslant 1$ and a set of indices

$$(g(j;1),h(j;1)),\dots,(g(j;q(j)),h(j;q(j)))\in\mathbb{N}\times\mathbb{N}$$

with

$$1=h(j;1)<\dots<h(j;i)<h(j;i+1)<\dots\leqslant n-r_0(j)$$
$$1\leqslant g(j;q(j))<\dots<g(j;i+1)<g(j;i)<\dots\leqslant r_0(j)$$

such that for $A(j,j)_{12}=:(a(j)_{st})_{\substack{1\leqslant s\leqslant r_0(j)\\ 1\leqslant t\leqslant n(j)-r_0(j)}}$ we have

$$a(j)_{g(j;i)h(j;i)}>0 \quad \text{for } 1\leqslant i\leqslant q(j)$$
$$a(j)_{g(j;i)t}=0 \quad \text{for } t>h(j;i), \text{ where } 1\leqslant i\leqslant q(j)$$
$$a(j)_{st}=0 \quad \text{for } t\geqslant h(j;i) \text{ and } s>g(j;i), \text{ where } 1\leqslant i\leqslant q(j)$$

and all other entries of $A(j,j)_{12}$ are unspecified.

i.e.

$$A(j,j)_{12} = \begin{bmatrix} \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots \\ \times & \times & \dots & \times & a(j)_{g(j,2)h(j,2)} & 0 & . \\ \times & \times & & \times & 0 & 0 & . \\ . & . & & . & 0 & . & . \\ \times & \times & \dots & \times & 0 & & \\ a(j)_{g(j,1)h(j,1)} & 0 & \dots & 0 & 0 & & \\ 0 & 0 & \dots & 0 & & & \\ . & . & \dots & . & & & \end{bmatrix}$$

(3) $A(j,j)_{21} = -A(j,j)_{12}^{\mathrm{T}}$.

(4)

$$A(j,j)_{22} = \begin{bmatrix} 0 & \alpha(j)_2 & & & & \\ -\alpha(j)_2 & 0 & \alpha(j)_3 & & \mathbf{0} & \\ & -\alpha(j)_3 & 0 & . & & \\ & & & \dots & . & \\ & \mathbf{0} & & . & 0 & \alpha(j)_{n(j)-r_0(j)} \\ & & & & -\alpha(j)_{n(j)-r_0(j)} & 0 \end{bmatrix}$$

with

$$\alpha(j)_i = \begin{cases} 0 & \text{if } i = h(j,s) \text{ for some } 1 \leqslant s \leqslant q(j) \\ > 0 & \text{otherwise} \end{cases}$$

(ii) *Off-diagonal blocks* $A(i,j)$, $(i \neq j)$

$$A(i,j) = \begin{bmatrix} \tilde{A}(i,j) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with } \tilde{A}(i,j) =: (a(i,j)_{st})_{\substack{1 \leqslant s \leqslant r_0(i) \\ 1 \leqslant t \leqslant r_0(j)}} \in \mathbb{R}^{r_0(i) \times r_0(j)}$$

where

$$a(i,j)_{st} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j b(i)_s b(j)_t^{\mathrm{T}} - \sigma_i c(i)_s^{\mathrm{T}} c(j)_t)$$

where $b(i)_s$ is the sth row of B^i and $c(i)_s$ is the sth column of C^i. □

Note that the map Γ: $C_n^{p,m} \to C_n^{p,m}$

$$(\tilde{A}, \tilde{B}, \tilde{C}) \to (A, B, C) = \Gamma((\tilde{A}, \tilde{B}, \tilde{C}))$$

defines a canonical form on $C_n^{p,m}$ with respect to system equivalence.

Specializing to the SISO case we obtain the following.

Corollary 2.5

The following two statements are equivalent.

(I) $$(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n^{1,1}$$

with

$$\Sigma = \operatorname{diag}(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \ldots, \sigma_k I_{n(k)})$$

such that $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ and $\sum_{j=1}^{k} n(j) = n$.

(II) There exists a unique $T \in GI(n)$ such that for

$$(A, b, c) := \Gamma((\tilde{A}, \tilde{b}, \tilde{c})) := (T\tilde{A}T^{-1}, T\tilde{b}, T^{-1}\tilde{c})$$

we have the following.

(i) $b^{\mathrm{T}} = (\underbrace{b_1, 0, \ldots, 0}_{n(1)}, \underbrace{b_2, 0, \ldots, 0}_{n(2)}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n(k)})$

with $b_j > 0$ for $1 \leqslant j \leqslant k$,

(ii) $c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n(1)}, \underbrace{s_2 b_2, 0, \ldots, 0}_{n(2)}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n(k)})$

where $s_i = \pm 1$ for $1 \leqslant i \leqslant k$.

(iii) For $A = (A(i,j))_{1 \leqslant i,j \leqslant k}$, $A(i,j) \in \mathbb{R}^{n(i) \times n(j)}$, we have the following.

(1) Block diagonal entries $A(j,j)$ for all $1 \leqslant j \leqslant k$

$$A(j,j) = \begin{bmatrix} a(j,j) & \alpha(j)_1 & & \cdot & \cdot \\ -\alpha(j)_1 & 0 & \alpha(j)_2 & \cdot & 0 \\ & -\alpha(j)_2 & 0 & \alpha(j)_3 & \cdot \\ & & \cdot & \cdot & \cdot \\ 0 & & \cdot & 0 & \alpha(j)_{n(j)-1} \\ & & \cdot & -\alpha(j)_{n(j)-1} & 0 \end{bmatrix}$$

with $a(j,j) = -b_j^2/2\sigma_j$,

$$\alpha(j)_i > 0 \quad \text{for all } 1 \leqslant i \leqslant n(j) - 1$$

(2) Off-diagonal block $A(i,j)$ for all $1 \leqslant i, j \leqslant k$, $i \neq j$

$$A(i,j) = \begin{bmatrix} a(i,j) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = s_i s_j A(j,i)^{\mathrm{T}}$$

with

$$a(i,j) = \frac{-1}{s_i s_j \sigma_i + \sigma_j} b_i b_j$$ □

Remark

Note that for (A, b, c) as given above we have

$$A^{\mathrm{T}} = SAS \qquad c^{\mathrm{T}} = Sb$$

where

$$S = \mathrm{diag}\,(s_1 \hat{I}_{n(1)}, s_2 \hat{I}_{n(2)}, \ldots, s_k \hat{I}_{n(k)})$$

and

$$\hat{I}_{n(j)} = \mathrm{diag}\,(+1, -1, +1, \ldots) \in \mathbb{R}^{n(j) \times n(j)}$$

This 'sign symmetry' of (A, b, c) will be important in the characterization of the connected components for SISO systems.

3. SISO systems

The Cauchy index of the transfer function of a system will be a major tool in the treatment of the connectivity problem for SISO systems.

Definition 3.1

Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients. The Cauchy index $\mathrm{CI}(g(x))$ of $g(x) = p(x)/q(x)$ is defined as the number of jumps from $-\infty$ to $+\infty$ less the number of jumps from $+\infty$ to $-\infty$ of $g(x)$ when x varies from $-\infty$ to $+\infty$.

A theorem by Anderson (1972) gives a useful characterization of the Cauchy index for transfer functions. First we need the following definition.

Definition 3.2

Let A be a real symmetric matrix. The signature of A is the difference between the number of positive and the number of negative eigenvalues.

Theorem 3.3 (Anderson 1972)

Let $(A, b, c) \in L_n^{1,1}$ and consider the transfer function $g(s) = c(sI - A)^{-1} b$. Then there exists a unique symmetric matrix P satisfying

$$PA = A^{\mathrm{T}} P, \quad c^{\mathrm{T}} = Pb$$

and the signature of P is equal to $\mathrm{CI}(g(s))$.

Fernando and Nicholson (1983) have pointed out the applicability of this theorem to balanced realizations. We know that for $(A, b, c) \in \Gamma(C_n^{1,1})$ there exists a sign matrix $S = \mathrm{diag}\,(s_1, s_2, \ldots, s_n)$, $s_i = \pm 1$, for $1 \leqslant i \leqslant n$, such that

$$A^{\mathrm{T}} = SAS, \quad c^{\mathrm{T}} = Sb$$

Thus by Anderson's result we have that

$$\mathrm{CI}(g(s)) = \sum_{i=1}^{n} s_i$$

where $g(s) = c(sI - A)^{-1} b$.

Brockett (1976) determined the number of connected components of $L_n^{1,1}/\sim$ and characterized those in terms of the Cauchy index of the systems.

Theorem 3.4 (Brockett 1976)

$L_n^{1,1}/\sim$ consists of $(n+1)$ pathwise connected components. Each of these components is uniquely determined by the Cauchy index of the corresponding transfer functions, i.e. the ith component C_i, $0 \leqslant i \leqslant n$, of $L_n^{1,1}/\sim$ is given by

$$C_i = \{\pi((A, b, c)) \mid (A, b, c) \in L_n^{1,1}, \mathrm{CI}(c(sI - A)^{-1} b) = n - 2i\}$$

Remark

Theorem 3.4 is a reformulation of Brockett's original result, using the homeomorphism between rat (n) and $L_n^{1,1}/\sim$ as given by Byrnes and Duncan (1982).

We can now state the main result of this section.

Theorem 3.5

$C_n^{1,1}/\sim$ consists of $(n+1)$ pathwise connected components. Each of these components is uniquely determined by the Cauchy index of the corresponding transfer functions, i.e. the ith component $\tilde{C}_i$, $0 \leqslant i \leqslant n$, of $C_n^{1,1}/\sim$ is given by

$$\tilde{C}_i = \{\pi((A, b, c)) \mid (A, b, c) \in C_n^{1,1}, \mathrm{CI}(c(sI - A)^{-1} b) = n - 2i\}$$

Proof

We first have to show that $C_n^{1,1}/\sim$ has at least $(n+1)$ pathwise connected components. We show that each of the $(n+1)$ connected components of $L^{1,1}/\sim$ contains an element of $C_n^{1,1}/\sim$.

Let $0 \leqslant j \leqslant n$, then we can choose $s_i = \pm 1$, $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^{n} s_i = n - 2j$. Now parametrize (A, b, c) by

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \operatorname{diag}(n, n-1, \ldots, 1)$$
$$b = (b_1, b_2, \ldots, b_n) = (1, \ldots, 1)^{\mathrm{T}}$$
$$S = \operatorname{diag}(s_1, s_2, \ldots, s_n)$$

i.e.

$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1 \leqslant i,j \leqslant n}$$
$$c = (s_1 b_1, s_2 b_2, \ldots, s_n b_n)$$

By Corollary 2.5, $(A, b, c) \in C_n^{1,1,b}$. Theorem 3.3 implies that $\mathrm{CI}(g(s)) = n - 2j$, where $g(s) = c(sI - A)^{-1} b$. Since $C_n^{1,1}/\sim \subseteq L_n^{1,1}/\sim$, we have that $C_n^{1,1}/\sim$ has at least $(n+1)$ pathwise connected components.

To show that there are exactly $(n+1)$ components, we need the following two lemmas.

Lemma 3.6

Each system $(\tilde{A}, \tilde{b}, \tilde{c}) \in \Gamma(C_n^{1,1})$ with

$$\tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1 I_{n(1)}, \tilde{\sigma}_2 I_{n(2)}, \ldots, \tilde{\sigma}_k I_{n(k)}),\ \tilde{\sigma}_1 > \tilde{\sigma}_2 > \ldots > \tilde{\sigma}_k > 0, \quad \sum_{j=1}^{k} n(j) = n$$

and

$$\tilde{S} = \operatorname{diag}(s_1, s_2, \ldots, s_n) = \operatorname{diag}(\tilde{s}_1 \hat{I}_{n(1)}, \hat{s}_2 I_{n(2)}, \ldots, \tilde{s}_k \hat{I}_{n(k)})$$

where $\tilde{s}_i = \pm 1$, and

$$\hat{I}_{n(i)} := \operatorname{diag}(+1, -1, +1, -1, \ldots) \in \mathbb{R}^{n(i) \times n(i)}, \quad 1 \leqslant i \leqslant k,$$

can be pathwise connected in $C_n^{1,1}$ with the system (A, b, c), which is parametrized by

$$\Sigma = \operatorname{diag}(n, n-1, \ldots, 1)$$
$$S = \tilde{S}$$
$$b = (b_1, \ldots, b_n)^{\mathrm{T}} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$$

Proof

Note that a continuous change in the parameters $\tilde{\Sigma}$, $\tilde{b}$ and $\alpha(j)_i$, $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant n(j)$ implies a continuous change in the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$ as long as $\tilde{\sigma}_1 > \tilde{\sigma}_2 > \ldots > \tilde{\sigma}_k > 0$. By Corollary 2.5 such perturbations do not change the minimality of $(\tilde{A}, \tilde{b}, \tilde{c})$ provided that $\tilde{\sigma}_1 > \tilde{\sigma}_2 > \ldots > \tilde{\sigma}_k > 0$, $\tilde{b}_i > 0$ for $1 \leqslant i \leqslant k$ and $\tilde{\alpha}(j)_i > 0$ for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant n(j)$. So we can assume without loss of generality that $(\tilde{A}, \tilde{b}, \tilde{c})$ is given in the following standardized form parametrized by

$$\tilde{\Sigma} = \operatorname{diag}(nI_{n(1)}, (n - n(1))I_{n(2)}, \ldots, n(k)I_{n(k)})$$
$$\tilde{b}^{\mathrm{T}} = (\underbrace{1, 0, \ldots, 0}_{n(1)}, \underbrace{1, 0, \ldots, 0}_{n(2)}, \ldots, \underbrace{1, 0, \ldots, 0}_{n(k)})$$

and $\tilde{\alpha}(j)_i = 1$ for all $1 \leqslant j \leqslant k$, $1 \leqslant i \leqslant n(j)$.

Instead of presenting a detailed but very cumbersome proof, an example is given which is general enough to indicate clearly all the features of a complete proof.

Consider the system $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_5^{1,1}$ which is parametrized by

$$\tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1 I_3, \tilde{\sigma}_2 I_2) = \operatorname{diag}(5, 5, 5, 2, 2)$$
$$\tilde{S} = \operatorname{diag}(\tilde{s}_1 \hat{I}_3, \tilde{s}_2 \hat{I}_2), \quad \tilde{s}_1, \tilde{s}_2 = \pm 1$$
$$\tilde{b} = (1, 0, 0, 1, 0)$$
$$\tilde{\alpha}(1)_1 = \tilde{\alpha}(1)_2 = \tilde{\alpha}(2)_1 = 1$$

i.e.

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 1 & 0 & \tilde{a}_{12} & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \tilde{a}_{21} & 0 & 0 & \tilde{a}_{22} & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

where

$$\tilde{a}_{ij} = \frac{-1}{\tilde{s}_i \tilde{s}_j \tilde{\sigma}_1 + \tilde{\sigma}_2}, \quad i, j = 1, 2$$

Using Corollary 2.5, we are going to show that there is a continuous path in $C_5^{1,1}$

connecting $(\tilde{A}, \tilde{b}, \tilde{c})$ with (A, b, c) which is parametrized by

$$\Sigma = \operatorname{diag}(5, 4, 3, 2, 1) =: \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_5)$$
$$b = (1, 1, 1, 1, 1) =: (b_1, b_2, \ldots, b_5)$$
$$S = \operatorname{diag}(s_1, s_2, \ldots, s_5) = \tilde{S}$$

We proceed stepwise and first consider the set of singular values of (A, b, c) corresponding to the first block of identical singular values of $(\tilde{A}, \tilde{b}, \tilde{c})$, i.e. $\sigma_1 = 5$, $\sigma_2 = 4$, $\sigma_3 = 3$.

(I) Within this first block of singular values of (A, b, c) we also proceed stepwise and start considering the singular value of least magnitude, i.e. $\sigma_3 = 3$.

(i) We show that (A, b, c) can be connected in $C_5^{1,1}$ with $(A(1), b(1), c(1))$ which is given by

$$\Sigma(1) = \operatorname{diag}(5, 4, 4, 2, 1)$$
$$b(1) = (1, 1, 0, 1, 1)$$
$$S(1) = S$$

and

$$\alpha(2)_1 = 1$$

i.e.

$$A(1) = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ a_{21} & a_{22} & 1 & a_{24} & a_{25} \\ 0 & -1 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix}$$

where

$$a_{ij} = \frac{-1}{s_i s_j \sigma_i + \sigma_j}, \quad i, j = 1, 2, 4, 5$$

This can be achieved by letting $\sigma_3 \to \sigma_2$ and $b_3 \to 0$ in an appropriate way. Direct verification shows a suitable way is to define the path

$$\sigma_3(t) = 3(1-t) + 4t, \quad t \in [0, 1]$$
$$b_3(t) = 1 - t, \qquad t \in [0, 1]$$

and letting $t \to 1$.

(ii) Next we show that $(A(1), b(1), c(1))$ can be continuously connected in $C_5^{1,1}$ with $(A(2), b(2), c(2))$ which is parametrized by

$$\Sigma(2) = \operatorname{diag}(\sigma_1, \sigma_1, \sigma_1, \sigma_4, \sigma_5) = \operatorname{diag}(5, 5, 5, 2, 1)$$
$$b(2) = (1, 0, 0, 1, 1)$$
$$S(2) = \tilde{S}$$
$$\alpha(1)_1 = \alpha(1)_2 = 1$$

i.e.

$$A(2)=\begin{bmatrix} a_{11} & 1 & 0 & a_{14} & a_{15} \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & a_{44} & a_{45} \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

This can again be achieved by appropriately letting $\sigma_2 \to \sigma_1$ and $b_2 \to 0$. Setting

$$\sigma_2(t) = 4(1-t) + 5t, \quad t \in [0, 1]$$
$$b_2(t) = 1 - t, \qquad t \in [0, 1]$$

the result follows by considering the limit as $t \to 1$.

(II) We now consider the next block of singular values, i.e. $\sigma_4 = 2$ and $\sigma_5 = 1$.

The final result follows from the fact that $(A(2), b(2), c(2))$ can be continuously connected in $C_5^{1,1}$ with $(\tilde{A}, \tilde{b}, \tilde{c})$ by appropriately letting $\sigma_5 \to \sigma_4$ and $b_5 \to 0$. □

Lemma 3.7

Let $(A_1, b_1, c_1) \in \Gamma(C_n^{1,1})$ be parametrized by

$$\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) = \operatorname{diag}(n, n-1, \ldots, 1)$$
$$b_1^{\mathrm{T}} = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$$
$$S_1 = \operatorname{diag}(s_1, s_2, \ldots, s_l, -s_l, \ldots, s_n)$$

for some $1 \leqslant l \leqslant n-1$ and $s_i = \pm 1$, and let $(A_2, b_2, c_2) \in \Gamma(C_n^{1,1})$ be given by

$$\Sigma_2 = \Sigma_1, \quad b_2 = b_1, \quad S_2 = \operatorname{diag}(s_1, s_2, \ldots, -s_l, s_l, \ldots, s_n)$$

then there is a continuous path in $C_n^{1,1}/\sim\, = \pi(C_n^{1,1})$ connecting $\pi((A_1, b_1, c_1))$ with $\pi((A_2, b_2, c_2))$.

Proof

Partition (A_i, b_i, c_i), $i = 1, 2$, conformally according to

$$A_i = \begin{bmatrix} A_{11} & A_{12}^i & A_{13} \\ A_{21}^i & A_{22}^i & A_{23}^i \\ A_{31} & A_{32}^i & A_{33} \end{bmatrix}, \quad \text{where } A_{11} \in \mathbb{R}^{(l-1)\times(l-1)} \quad \text{and} \quad A_{22}^i \in \mathbb{R}^{2\times 2}$$

$$b_i^{\mathrm{T}} = (\ell_1^{\mathrm{T}} \quad (\ell)_2^{i\mathrm{T}} \quad \ell_3^{\mathrm{T}})$$
$$c_i = (c_1 \quad c_2^i \quad c_3)$$

Consider (A_1, b_1, c_1). By Lemma 3.6 there exists a continuous path in $C_n^{1,1}$ connecting

(A_1, b_1, c_1) with $(\tilde{A}_1, \tilde{b}_1, \tilde{c}_1)$, where

$$\tilde{A}_1 = \begin{bmatrix} A_{11} & \tilde{A}^1_{12} & A_{13} \\ \tilde{A}^1_{21} & \tilde{A}^1_{22} & \tilde{A}^1_{23} \\ A_{31} & \tilde{A}^1_{32} & A_{33} \end{bmatrix}$$

$$\tilde{b}_1 = \begin{bmatrix} b_1 \\ \tilde{b}^1_2 \\ b_3 \end{bmatrix}, \quad \tilde{c}_1 = (c_1 \quad \tilde{c}^1_2 \quad c_3)$$

with

$$\tilde{A}^1_{22} = \begin{bmatrix} \frac{-1}{2\sigma_l} & 1 \\ -1 & 0 \end{bmatrix}, \quad \tilde{b}^1_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{c}^1_2 = (s_l \quad 0)$$

$$\tilde{A}^1_{12} = (a_{12} \quad 0), \quad a_{12} \text{ the first column of } A^1_{12}$$

$$\tilde{A}^1_{32} = (a_{32} \quad 0) \quad a_{32} \text{ the first column of } A^1_{32}$$

$$\tilde{A}^1_{21} = \begin{bmatrix} a_{21} \\ 0 \end{bmatrix}, \quad a_{21} \text{ the first row of } A^1_{21}$$

$$\tilde{A}^1_{23} = \begin{bmatrix} a_{23} \\ 0 \end{bmatrix}, \quad a_{23} \text{ the first row of } A^1_{23}$$

Now consider (A_2, b_2, c_2). By a similar argument to the one given in the proof of Lemma 3.6, i.e. by letting σ_{l+1}converge to σ_l and letting b_l converge to zero in an appropriate way, we can find a continuous path in $C_n^{1,1}$ connecting (A_2, b_2, c_2) with $(\tilde{A}_2, \tilde{b}_2, \tilde{c}_2)$, where

$$\tilde{A}_2 = \begin{bmatrix} A_{11} & \tilde{A}^2_{12} & A_{13} \\ \tilde{A}^2_{21} & \tilde{A}^2_{22} & \tilde{A}^2_{23} \\ A_{31} & \tilde{A}^2_{32} & A_{33} \end{bmatrix}$$

$$\tilde{b}_2 = \begin{bmatrix} b_1 \\ \tilde{b}^2_2 \\ b_3 \end{bmatrix}, \quad \tilde{c}_2 = (c_1 \quad \tilde{c}^2_2 \quad c_3)$$

with

$$\tilde{A}^2_{22} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{-1}{2\sigma_l} \end{bmatrix}, \quad \tilde{b}^2_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{c}^2_2 = (0 \quad s_l)$$

and

$$\tilde{A}^2_{12} = (0 \quad a_{12}), \quad \tilde{A}^2_{32} = (0 \quad a_{32}), \quad \tilde{A}^2_{21} = \begin{bmatrix} 0 \\ a_{21} \end{bmatrix}, \quad \tilde{A}^2_{23} = \begin{bmatrix} 0 \\ a_{23} \end{bmatrix}$$

It is easily verified that a state space transformation of the form $Q = \text{diag}(I_{l-1}, \tilde{Q}, I_{n-l-1})$, $\tilde{Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, yields the equivalence of the systems $(\tilde{A}_1, \tilde{b}_1, \tilde{c}_1)$ and $(\tilde{A}_2, \tilde{b}_2, \tilde{c}_2)$. Thus $\pi((\tilde{A}_1, \tilde{b}_1, \tilde{c}_1)) = \pi((\tilde{A}_2, \tilde{b}_2, \tilde{c}_2))$. Since $\pi: C_n^{1,1} \to C_n^{1,1}/\sim$ is continuous we have established the existence of a continuous path in $C_n^{1,1}/\sim$ connecting $\pi((A_1, b_1, c_1))$ with $\pi((A_2, b_2, c_2))$. □

We show now that all systems with a given Cauchy index are pathwise connected in $C_n^{1,1}/\sim$. This is done by taking an arbitrary $(\tilde{A}, \tilde{b}, \tilde{c}) \in \Gamma(C_n^{1,1})$ whose transfer function has the Cauchy index $\text{CI} = p - q$, $p, q \in \mathbb{N}$, $p + q = n$, and connecting it to the standard system (A_0, b_0, c_0), which is given by the parameters

$$\Sigma_0 = \text{diag}(n, n-1, \ldots, 1)$$
$$b_0 = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$$
$$S_0 = (I_p, -I_q)$$

By Lemma 3.6, $(\tilde{A}, \tilde{b}, \tilde{c})$ can be continuously connected to a system (A, b, c) which is parametrized by

$$\Sigma = \text{diag}(n, n-1, \ldots, 1)$$
$$b^{\mathrm{T}} = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$$

and

$$S = \tilde{S}$$

where $\tilde{S}$ is the sign matrix associated with $(\tilde{A}, \tilde{b}, \tilde{c})$.

Lemma 3.7 allows us to swap signs in S such that we can assume $S = S_0$. But this shows that $\pi((\tilde{A}, \tilde{b}, \tilde{c}))$ and $\pi((A_0, b_0, c_0))$ can be pathwise connected. □

Corollary 3.8

$D_n^{1,1}/\sim$ has $n + 1$ connected components each of them being determined by the Cauchy index of the corresponding transfer functions.

Proof

$T_n: C_n^{1,1} \to D_n^{1,1}$ is a homeomorphism. But then

$$\pi T_n \pi^{-1}: C_n^{1,1}/\sim \;\to D_n^{1,1}/\sim$$

is a homeomorphism.

Furthermore T_n preserves the sign matrix, i.e. if $(A, b, c) \in C_n^{1,1}$ has sign matrix S, then $T_n((A, b, c))$ has the sign symmetry property with sign matrix S. □

4. MIMO systems

Glover (1975) showed that for $\max(p, m) \geq 2$, $L_n^{p,m}/\sim$ is pathwise connected. In the following theorem the same result is established for $C_n^{p,m}/\sim$.

Theorem 4.1

If $\max(p, m) \geq 2$ then $C_n^{p,m}/\sim$ is pathwise connected.

Proof

We need the following three lemmas.

Lemma 4.2

Let $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma(C_n^{p,m})$ with $\tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1 I_{n(1)}, \ldots, \tilde{\sigma}_k I_{n(k)})$

$$\tilde{B} = \begin{bmatrix} \tilde{B}^1 \\ \vdots \\ \tilde{B}^k \end{bmatrix}, \quad \tilde{B}^j \in \mathbb{R}^{n(j) \times m}, \quad \operatorname{rank}(\tilde{B}^j(\tilde{B}^j)^{\mathrm{T}}) = r_0(j), \quad \text{for } 1 \leqslant j \leqslant k$$

$$\tilde{C} = (\tilde{C}^1, \ldots, \tilde{C}^k)$$

with $\tilde{C}^j = (U^j \quad \mathbf{0})(\tilde{B}^j(\tilde{B}^j)^{\mathrm{T}})^{1/2}$, where $(U^j)^{\mathrm{T}} U^j = I_{r_0(j)}$ for $1 \leqslant j \leqslant k$.

Then $(\tilde{A}, \tilde{B}, \tilde{C})$ can be pathwise connected in $C_n^{p,m}$ with $(A, B, C) \in \Gamma(C_n^{p,m})$, given by

$$\Sigma = \operatorname{diag}(nI_{n(1)}, (n - n(1))I_{n(2)}, \ldots, n(k)I_{n(k)})$$

$$B = \begin{bmatrix} B^1 \\ \vdots \\ B^k \end{bmatrix}, \quad \text{where } B^j = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{n(j) \times m} \quad \text{for } 1 \leqslant j \leqslant k$$

$$C = (C^1, \ldots, C^k)$$

with

$$C^j = \begin{bmatrix} s_j & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{p \times n(j)}, \quad s_j = \pm 1 \text{ for } 1 \leqslant j \leqslant k$$

For $1 \leqslant j \leqslant k$ the jth diagonal block of A is given by

$$A(j,j) = \begin{bmatrix} a(j,j) & 1 & & & & \\ -1 & 0 & 1 & & \mathbf{0} & \\ & -1 & 0 & \cdot & & \\ & & -1 & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{n(j) \times n(j)}$$

Proof

We can assume without loss of generality that $\tilde{\sigma}_1 = n, \tilde{\sigma}_2 = n - n(1), \ldots, \tilde{\sigma}_k = n(k)$. We use the notation of Theorem 2.4. For $1 \leqslant j \leqslant n$ consider

$$A(j,j) = \begin{bmatrix} A(j,j)_{11} & A(j,j)_{12} \\ A(j,j)_{21} & A(j,j)_{22} \end{bmatrix}$$

where

$$A(j,j)_{11} = \frac{-1}{2\sigma_j} \operatorname{diag}(\lambda_1^j, \ldots, \lambda_{r(j)}^j) + \tilde{A}(j,j)_{11}$$

with skew-symmetric $\tilde{A}(j,j)$.

Now perturb

$$\tilde{A}(j,j)_{11} \quad \text{to} \quad \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & \mathbf{0} & \\ & -1 & 0 & \cdot & & \\ & & -1 & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & -1 & 0 \end{bmatrix}$$

$$A(j,j)_{22} \quad \text{to} \quad \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & \mathbf{0} & \\ & -1 & 0 & \cdot & & \\ & & -1 & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & -1 & 0 \end{bmatrix}$$

and

$$A(j,j)_{12} \quad \text{to} \quad \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{bmatrix}$$

such that $A(j,j)_{12}$ is in canonical form all through the perturbation process.

By Theorem 2.4, these perturbations, if performed in the given order, define a continuous path in $C_n^{p,m}$. So assume that $A(j,j)$ is given by

$$A(j,j) = \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & \mathbf{0} & \\ & -1 & 0 & \cdot & & \\ & & -1 & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & -1 & 0 \end{bmatrix} + \frac{-1}{2\sigma_j} \operatorname{diag}(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{ro(j)}^{(j)}, 0, \ldots, 0)$$

Now we can continuously perturb B^j to

$$B^j = \begin{bmatrix} b_1^j \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{where } b_1^j \text{ is the first row of } B^j$$

without changing the orthogonality of the rows of B^j.

This implies a perturbation of $C^j = (U^j \quad \mathbf{0})(B^j(B^j)^{\mathrm{T}})^{1/2}$ to $C^j = (u_1^j, 0, \ldots, 0)(b_1^j(b_1^j)^{\mathrm{T}})^{1/2}$ where u_1^j is the first column of U^j.

It is now possible to perturb b_1^j to $(1, 0, \ldots, 0)$ without becoming zero. There exists $s_j = \pm 1$ such that u_1^j can be continuously connected with $(s_j, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^p$ preserving the norm of u_1^j.

The perturbations of B^i and C^i imply a perturbation of $A(j,j)$ to

$$A(j,j) = \begin{bmatrix} a(j,j) & 1 & & & & \\ -1 & 0 & 1 & & \mathbf{0} & \\ & -1 & 0 & \cdot & & \\ & & -1 & \cdot & \cdot & \\ & \mathbf{0} & & \cdot & \cdot & 1 \\ & & & & -1 & 0 \end{bmatrix}$$

□

Lemma 4.3

Let (A, B, C) be given as in Lemma 4.2. Then (A, B, C) can be pathwise connected with the system (A_0, B_0, C_0) which is parametrized by

$$\Sigma_0 = \operatorname{diag}(n, n-1, \ldots, 1)$$

$$B_0 = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$C_0 = \begin{bmatrix} S^{\mathrm{T}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{p \times n}$$

where

$$S^{\mathrm{T}} = (1, \ldots, 1) \operatorname{diag}(s_1 \hat{I}_{n(1)}, \ldots, s_k \hat{I}_{n(k)}), \quad (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$$

Proof

The proof is analogous to the proof of Lemma 3.6. □

Lemma 4.4

Let (A_0, B_0, C_0) be given as in Lemma 4.3, then (A_0, B_0, C_0) can be pathwise connected in $C_n^{p,m}$ with the system $(A_1, B_1, C_1) \in \Gamma(C_n^{p,m})$, which is parametrized by

$$\Sigma_1 = \text{diag}\,(n, n-1, \dots, 1)$$

$$B_1 = B_0$$

$$C_1 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Proof

We have to consider two cases.

Case 1: $p \geqslant 2$. Consider

$$C = (u^1, u^2, \dots, u^n)(\text{diag}\,(\lambda_1^{(1)}, \lambda_1^{(2)}, \dots, \lambda_1^{(n)}))^{1/2} = (u^1, u^2, \dots, u^n)$$

where

$$u^i = \begin{bmatrix} \tilde{s}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^p \quad \text{for some } \tilde{s}_i = \pm 1$$

We can perturb u^i to $u^i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $u^i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ can be perturbed to $u^i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

This can be done without changing the unit length of u^i.

Case 2: $p = 1$. Then $C = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n)$ for some $\tilde{s}_i = \pm 1$, $1 \leqslant i \leqslant n$.

Now perform a state-space transformation with

$$Q = \text{diag}\,(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n)$$

Then $CQ^{\mathrm{T}} = (1, 1, \dots, 1)$ and

$$QB = \begin{bmatrix} \tilde{s}_1 & 0 & \dots & 0 \\ \tilde{s}_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \tilde{s}_n & 0 & \dots & 0 \end{bmatrix}$$

Since $\max(p, m) \geqslant 2$ we have $m \geqslant 2$. Thus by a similar argument to the one given in Case 1, we can perturb QB to

$$QB = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

Theorem 2.4 guarantees that in both cases we do not hit non-minimal systems. □

Combining the previous lemmas, we can find for each system $(A, B, C) \in \Gamma(C_n^{p,m})$ a continuous path in $C_n^{p,m}$ connecting it with the system (A_1, B_1, C_1).

Continuity of the projection $\pi: C_n^{p,m} \to C_n^{p,m}/\sim$ implies that $C_n^{p,m}/\sim$ is pathwise connected. □

Corollary 4.5

If $\max(p, m) \geqslant 2$ then $D_n^{p,m}/\sim$ is pathwise connected.

Proof

$T_n: C_n^{p,m} \to D_n^{p,m}$ is a homeomorphism. □

5. Conclusions

The number of connected components is determined for the set of asymptotically stable, minimal systems of given order. This is done by using a canonical form and a parametrization result for balanced realizations. It is believed that this canonical form can be used to examine further the topology of the set of asymptotically stable systems.

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