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Balanced realizations: canonical form, parametrization, model reduction

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Balanced realizations are used to construct a canonical form for the set of minimal and asymptotically stable MIMO systems of a given order. A parametrization result is derived that allows us to show a truncation property of this canonical form.

Notation

$\mathbb{R}^n$, $\mathbb{C}^n$	n-dimensional real and complex euclidean spaces
$\mathrm{Re}(z)$, $\bar{z}$, $\lvert z \rvert$	real part, complex conjugate and modulus of $z \in \mathbb{C}$
$\mathbb{C}^{n \times m}$ ($\mathbb{R}^{n \times m}$)	space of $n \times m$ complex (real) matrices
I_n	$n \times n$ identity matrix
A^{T}	transpose of $A \in \mathbb{R}^{n \times n}$
$\sigma(A)$	set of eigenvalues of the matrix A
$\lambda_{\max}(A)$ ($\lambda_{\min}(A)$)	eigenvalue of $A \in \mathbb{C}^{n \times n}$ with maximal (minimal) modulus
$\mathrm{diag}(A_1, A_2, \ldots, A_m)$	block diagonal matrix with $A_i \in \mathbb{C}^{n_i \times n_i}$
$\hat{I}_n$	$\hat{I}_n = \mathrm{diag}(+1, -1, +1, -1, \ldots)$
$A > 0$	for positive definite symmetric matrix
A skew-symmetric	$A = -A^{\mathrm{T}}$
$Q \in \mathbb{R}^{n \times n}$ orthogonal	$QQ^{\mathrm{T}} = I_n$
$C_n^{p,m}$ ($C_n^{p,m,b}$)	see § 2
$D_n^{p,m}$ ($D_n^{p,m,b}$)	see § 2

1. Introduction

Recently, balanced realizations as defined by Moore (1981) have obtained a great deal of attention. This is mainly due to their interesting properties with respect to model reduction (Pernebo and Silverman 1982, Glover 1984).

Maciejowski (1985) has indicated the potential usefulness of balanced realizations for system identification. In setting up an identification algorithm canonical forms are of great importance in order to avoid identifiability problems. In § 6 a canonical form based on balanced realizations is constructed for the set of minimal and asymptotically stable continuous-time systems. This is an extension to MIMO systems of the canonical form for SISO systems as derived by Ober (1985). A canonical form for the case of distinct singular values was given by Kabamba (1985).

Since the canonical form derived in this paper exhibits interesting structural properties, it is hoped that it will lead to a deeper insight into the structure of the set of minimal and asymptotically stable systems of a given order. Indeed, it allows us to determine the number of connected components of this set (Ober 1987 a).

Section 2 contains definitions and preliminary results concerning balanced realizations. The concept of a canonical form is recalled in § 3.

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A transformation defined in § 4, which is inspired by a bilinear transformation, allows us to carry over results from continuous-time systems to discrete-time systems.

In § 5, a canonical form is derived for the important case of systems with identical singular values. This canonical form is then used in § 6 to derive a canonical form for the general case.

The canonical form for asymptotically stable and minimal systems as derived in § 6 exhibits a certain structure. In § 7 the converse problem is considered. It is shown that if a system has this structure, it is necessarily minimal and asymptotically stable.

The parametrization result in § 7 then allows us to prove in § 8 a general truncation property for the canonical form as derived in § 6. Using the transformation from continous-time systems to discrete-time systems as established in § 4, two alternative model reduction procedures are proposed.

2. Balanced realizations

In this section, continuous and discrete-time balanced realizations will be defined, as introduced by Moore (1981). For easy reference, some properties of these realizations will be stated at the same time.

2.1. *Continuous-time balanced realizations*

Let $C_n^{p,m} = \{(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} | (A,B,C)$ is minimal and asymptotically stable continuous-time system$\}$.

Definition 2.1 (Moore 1981)

Let $(A, B, C) \in C_n^{p,m}$. (A, B, C) is called balanced if for

$$W_c = \int_0^\infty \exp(At)\, BB^T \exp(tA^T)\, dt$$

$$W_0 = \int_0^\infty \exp(A^T t)\, C^T C \exp(tA)\, dt$$

we have $W_c = W_0 =: \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$.

The positive numbers $\sigma_1, \ldots, \sigma_n$ are called the singular values of the system (A, B, C). Denote by $C_n^{p,m,b} \subseteq C_n^{p,m}$ the subset of all balanced systems. □

To talk about balanced realizations is justified by the following theorem.

Theorem 2.2 (Moore 1981)

Let $(A, B, C) \in C_n^{p,m}$, then there exists $T \in GL(n)$ such that (TAT^{-1}, TB, CT^{-1}) is balanced. □

An equivalent characterization of a system to be balanced can be given in terms of Lyapunov equations. These are a major tool in working with balanced realizations.

Theorem 2.3 (Moore 1981)

Let $(A, B, C) \in C_n^{p,m}$, then (A, B, C) is balanced iff there exists a diagonal matrix $\Sigma > 0$ such that

$$A\Sigma + \Sigma A^T = -BB^T$$

and

$$A^{\mathrm{T}}\Sigma + \Sigma A = -C^{\mathrm{T}}C$$

In this case $\Sigma = W_0 = W_c$. □

The question of uniqueness of a balanced realization is answered by the following theorem.

Theorem 2.4 (Moore 1981)

Let $(A, B, C) \in C_n^{p,m,b}$ with $\Sigma = \mathrm{diag}\,(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \ldots, \sigma_k I_{n(k)})$, $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ and $\sum_{j=1}^{k} n(j) = n$.

Then (A, B, C) is unique up to an orthogonal state-space transformation of the form

$$Q = \mathrm{diag}\,(Q_1, Q_2, \ldots, Q_k)$$

with orthogonal $Q_i \in \mathbb{R}^{n(i) \times n(i)}$ for all $1 \leqslant i \leqslant k$. □

The following truncation property makes balanced realizations particularly interesting for model reduction.

Theorem 2.5 (Pernebo and Silverman 1982)

Let $(A, B, C) \in C_n^{p,m,b}$ be conformally partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = (C_1 \quad C_2)$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, 2$$

If Σ_1 and Σ_2 have no eigenvalues in common, then we have $(A_{ii}, B_i, C_i) \in C_{n_i}^{p,m,b}$ for $i = 1, 2$. □

A generalization of this result will be given in § 8.

The following lemma will be useful later.

Lemma 2.6

Suppose that the system (A, B) satisfies the Lyapunov equation $A\Sigma + \Sigma A^{\mathrm{T}} = -BB^{\mathrm{T}}$ for some diagonal $\Sigma > 0$. Then

(i) (A, B) is asymptotically stable if and only if (A, B) is controllable.

(ii) $\mathrm{Re}\,(\lambda) \leqslant 0$ for all $\lambda \in \sigma(A)$.

Proof

See Pernebo and Silverman (1982). □

2.2. *Discrete-time balanced realizations*

Let $D_n^{p,m} = \{(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} | (A, B, C)$ is minimal and asymptotically stable discrete-time system$\}$.

As in the continuous-time case, the definition of a balanced realization is based on the equality of the observability and the controllability gramian.

Definition 2.7 (Moore 1981)

Let $(A, B, C) \in D_n^{p,m}$. (A, B, C) is called balanced if for

$$W_c = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k$$

$$W_0 = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

we have $W_c = W_0 =: \Sigma =: \mathrm{diag}\,(\sigma_1, \dots, \sigma_n)$.

The positive numbers $\sigma_1, \dots, \sigma_n$ are called the singular values of the system (A, B, C). Denote by $D_n^{p,m,b} \subseteq D_n^{p,m}$ the subset of all balanced systems. □

For easy reference we mention the results corresponding to those of the continuous-time case.

Theorem 2.8 (Moore 1981)

Let $(A, B, C) \in D_n^{p,m}$, then there exists $T \in GL(n)$ such that (TAT^{-1}, TB, CT^{-1}) is balanced. □

Theorem 2.9 (Moore 1981)

Let $(A, B, C) \in D_n^{p,m}$, then (A, B, C) is balanced iff there exists a diagonal matrix $\Sigma > 0$ such that

$$A\Sigma A^T - \Sigma = -BB^T$$

$$A^T \Sigma A - \Sigma = -C^T C$$

In this case $\Sigma = W_0 = W_c$. □

Theorem 2.10 (Moore 1981)

Let $(A, B, C) \in D_n^{p,m,b}$ with $\Sigma = \mathrm{diag}\,(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \dots, \sigma_k I_{n(k)})$, where $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$ and $\sum_{j=1}^{k} n(j) = n$.

Then (A, B, C) is unique up to an orthogonal state-space transformation of the form

$$Q = \mathrm{diag}\,(Q_1, \dots, Q_k)$$

with orthogonal $Q_i \in \mathbb{R}^{n(i)\times n(i)}$ for all $1 \leqslant i \leqslant k$. □

Theorem 2.11 (Pernebo and Silverman 1982)

Let $(A, B, C) \in D_n^{p,m,b}$ be conformally partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = (C_1 \quad C_2)$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_i \in \mathbb{R}^{n_i \times n_i}, \quad i = 1, 2$$

If $\lambda_{\min}(\Sigma_1) > \lambda_{\max}(\Sigma_2)$, then the subsystem (A_{11}, B_1, C_1) is asymptotically stable and minimal. □

3. Canonical forms

In system identification, we have to find a system describing a given input–output sequence. It is well known that from an input–output point of view two nth-order minimal systems can only be distinguished if they are not equivalent as defined below.

Definition 3.1

Two minimal systems (A_1, B_1, C_1) and (A_2, B_2, C_2) are called equivalent (write $(A_1, B_1, C_1) \sim (A_2, B_2, C_2)$) if there exists $T \in GL(n)$ such that

$$A_1 = TA_2T^{-1}, \quad B_1 = TB_2 \quad \text{and} \quad C_1 = C_2T^{-1}$$ □

It is clear that system equivalence is an equivalence relation on $C_n^{p,m}$ and $D_n^{p,m}$.

A unique representation of a linear system can be obtained by deriving a canonical form.

Definition 3.2

A canonical form for an equivalence relation '$\sim$' on a set X is a map $\Gamma: X \to X$ which satisfies for all $x, y \in X$:

(i) $\Gamma(x) \sim x$

(ii) $x \sim y \Leftrightarrow \Gamma(x) = \Gamma(y)$ □

In the derivation of canonical forms, so-called invariants often play an important role.

Definition 3.3

Let S be a set. An invariant for an equivalence relation on a set X is a map $I: X \to S$ which satisfies for all $x, y \in X$:

$$x \sim y \Rightarrow I(x) = I(y)$$ □

Example 3.4

The map $\Sigma: C_n^{p,m} \to \mathbb{R}^n$

$$(A, B, C) \to (\sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_n > 0$$

which assigns each system (A, B, C) its set of singular values, is an invariant for system equivalence.

4. Transformation $T_n: C_n^{p,m} \to D_n^{p,m}$

A standard method to relate results for continuous-time systems to discrete-time systems is to use a matrix transformation induced by a bilinear transformation of transfer functions (e.g. see Glover 1984).

The following proposition contains a collection of results which will be used in the sequel.

Proposition 4.1

(i) The map $T_n: C_n^{p,m} \to D_n^{p,m}$

$$(A, B, C) \to ((I-A)^{-1}(I+A), \sqrt{2}(I-A)^{-1}B, \sqrt{2}C(I-A)^{-1})$$

is a bijection with inverse

$$T_n^{-1}: D_n^{p,m} \to C_n^{p,m}$$

$$(A, B, C) \to ((I+A)^{-1}(A-I), \sqrt{2}(I+A)^{-1}B, \sqrt{2}C(I+A)^{-1})$$

Moreover, the observability and controllability gramians are invariant under T_n.

(ii) T_n and T_n^{-1} preserve system equivalence, i.e.

$$(A_1, B_1, C_1) \sim (A_2, B_2, C_2), \quad (A_i, B_i, C_i) \in C_n^{p,m}, \quad i = 1, 2$$

$$\Leftrightarrow T_n((A_1, B_1, C_1)) \sim T_n(A_2, B_2, C_2))$$

(iii) T_n maps continuous-time balanced systems to discrete-time balanced systems, i.e.

$$T_n(C_n^{p,m,b}) = D_n^{p,m,b}$$

Proof

We shall first show that $T_n((A, B, C)) = (\tilde{A}, \tilde{B}, \tilde{C})$ is asymptotically stable for $(A, B, C) \in C_n^{p,m}$. Let $(A, B, C) \in C_n^{p,m}$. Note that $(I-A)^{-1}$ exists. Now, assume there exists

$$x \in \mathbb{C}^n,\ x \neq 0,\ \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda| \geqslant 1$$

such that

$$\tilde{A}x = \lambda x \Rightarrow (I-A)^{-1}(I+A)x = \lambda x \Rightarrow (I+A)x = (I-A)\lambda x$$

$$\Rightarrow A(1+\lambda)x = (\lambda - 1)x$$

Since $x \neq 0$ we have $\lambda \neq -1$

$$\Rightarrow Ax = \frac{\lambda - 1}{\lambda + 1} x$$

But $\mathrm{Re}\left[\dfrac{\lambda-1}{\lambda+1}\right] \geqslant 0$, which is a contradiction to the asymptotic stability of A.

It can easily be verified (see, for example, Glover 1984) that

$$\tilde{A}P\tilde{A}^T - P = -\tilde{B}\tilde{B}^T$$

$$\tilde{A}^T Q \tilde{A} - Q = -\tilde{C}^T\tilde{C}$$

where P is the controllability gramian and Q the observability gramian of (A, B, C). Since $\tilde{A}$ is asymptotically stable this shows that P is the controllability gramian and Q the observability gramian of $(\tilde{A}, \tilde{B}, \tilde{C})$. Thus $(\tilde{A}, \tilde{B}, \tilde{C})$ is minimal and hence $T_n(C_n^{p,m}) \subseteq D_n^{p,m}$. Similarly, it follows that $T_n^{-1}(D_n^{p,m}) \subseteq C_n^{p,m}$.

Straightforward calculations show that T_n is a bijection with inverse T_n^{-1} and that T_n as well as T_n^{-1} preserve system equivalence, which shows (i) and (ii).

(iii) follows from the fact that T_n leaves the controllability and observability gramians invariant. □

The importance of this transformation in our context lies in the fact that it carries over canonical forms.

Proposition 4.2

Let $\Gamma: C_n^{p,m} \to C_n^{p,m,b}$ be a canonical form for system equivalence, then $\tilde{\Gamma}: D_n^{p,m} \to D_n^{p,m,b}$ defined by $\tilde{\Gamma} = T_n \Gamma T_n^{-1}$ is a canonical form for system equivalence on $D_n^{p,m}$.

Proof

Let $S_1, S_2 \in D_n^{p,m}$

(i) We have

$$T_n^{-1}\tilde{\Gamma}(S_1) = \Gamma T_n^{-1}(S_1) \sim T_n^{-1}(S_1) \Rightarrow \tilde{\Gamma}(S_1) \sim S_1$$

(ii)

$$S_1 \sim S_2 \Rightarrow T_n^{-1}(S_1) \sim T_n^{-1}(S_2) \Rightarrow \Gamma(T_n^{-1}(S_1)) = \Gamma(T_n^{-1}(S_2)) \Rightarrow \tilde{\Gamma}(S_1) = \tilde{\Gamma}(S_2)$$

$$\tilde{\Gamma}(S_1) = \tilde{\Gamma}(S_2) \Rightarrow \Gamma(T_n^{-1}(S_1)) = \Gamma(T_n^{-1}(S_2)) \Rightarrow T_n^{-1}(S_1) \sim T_n^{-1}(S_2) \Rightarrow S_1 \sim S_2$$

Thus $\tilde{\Gamma}$ is a canonical form.

For $S \in D_n^{p,m}$ we have $T_n^{-1}(S) \in C_n^{p,m}$

$$\Rightarrow \Gamma(T_n^{-1}(S)) \in C_n^{p,m,b} \Rightarrow \tilde{\Gamma}(S) \in D_n^{p,m,b}$$ □

5. Canonical form for $C_n^{p,m}$: the case of identical singular values

In this section, a canonical form is derived for the set of asymptotically stable minimal continuous-time systems with identical singular values. This set of systems is interesting in its own right since the strictly proper part of an asymptotically stable all-pass transfer function can be characterized by a system with identical singular values (Glover 1984, Ober 1987 b).

To make the derivation of the canonical form clearer, it is split into three steps.

Step 1

In the first step an arbitrary system $(A, B, C) \in C_n^{p,m}$ with identical singular values is brought to a balanced form which has interesting symmetry properties.

Proposition 5.1

Let $(\tilde{A}, \tilde{B}, \tilde{C}) \in C_n^{p,m}$ with identical singular values $\sigma_1 = \sigma_2 = \dots = \sigma_n := \sigma, \sigma > 0$. Then there exists an equivalent balanced system (A, B, C) such that

(i)
$$BB^{\mathrm{T}} = C^{\mathrm{T}}C = \mathrm{diag}\,(\lambda_1 I_{r(1)}, \dots, \lambda_l I_{r(l)}, 0, \dots, 0)$$

with

$$\lambda_1 > \lambda_2 > \dots > \lambda_l > 0, r(i) \in \mathbb{N} \quad \text{for} \quad 1 \leqslant i \leqslant l$$

and

$$\sum_{i=1}^{l} r(i) = \mathrm{rank}\,(BB^{\mathrm{T}}) =: r_0$$

(ii) We have

$$A = \frac{-1}{2\sigma}\,\mathrm{diag}\,(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}, 0, \ldots, 0) + \mathring{A}$$

where $\mathring{A}$ is skew-symmetric.

(iii) The numbers $l, r(1), \ldots, r(l)$ are invariants for system equivalence.

(iv) The systems which are equivalent to (A, B, C) and have the structure given in (i) and (ii) are precisely those systems that can be obtained from (A, B, C) by an orthogonal state-space transformation of the form

$$Q = \mathrm{diag}\,(Q_1, Q_2, \ldots, Q_{l+1})$$

with orthogonal $Q_i \in \mathbb{R}^{r(i)\times r(i)}$ for all $1 \leqslant i \leqslant l+1$, where we set $r(l+1) := n - r_0$.

Proof

Assume without loss of generality that $(\tilde{A}, \tilde{B}, \tilde{C})$ is balanced.

(i) Since $\Sigma = \sigma I_n$, it follows from the Lyapunov equations that $\tilde{B}\tilde{B}^{\mathrm{T}} = \tilde{C}^{\mathrm{T}}\tilde{C}$.

By Theorem 2.4 we have the freedom to perform an arbitrary orthogonal state-space transformation. As $\tilde{B}\tilde{B}^{\mathrm{T}} = \tilde{C}^{\mathrm{T}}\tilde{C}$ is symmetric and positive semi-definite, there exists an orthogonal $\tilde{Q} \in \mathbb{R}^{n\times n}$ such that

$$\tilde{Q}\tilde{B}\tilde{B}^{\mathrm{T}}\tilde{Q}^{\mathrm{T}} = \tilde{Q}\tilde{C}^{\mathrm{T}}\tilde{C}\tilde{Q}^{\mathrm{T}} =: \mathrm{diag}\,(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}, 0, \ldots, 0)$$

with $\lambda_1 > \lambda_2 > \ldots > \lambda_l > 0$ and $\sum_{i=1}^{l} r(i) = \mathrm{rank}\,(\tilde{B}\tilde{B}^{\mathrm{T}})$. Thus take $(A, B, C) := (\tilde{Q}\tilde{A}\tilde{Q}^{\mathrm{T}}, \tilde{Q}\tilde{B}, \tilde{C}\tilde{Q}^{\mathrm{T}})$.

(ii) If we consider the Lyapunov equation

$$A\Sigma + \Sigma A^{\mathrm{T}} = -BB^{\mathrm{T}}$$

componentwise, we obtain $\sigma_i a_{ji} + \sigma_j a_{ij} = -b_j b_i^{\mathrm{T}}$ where b_i is the ith row of B and $A = (a_{ij})_{1\leqslant i,j\leqslant n}$.

Since by (i) $b_j b_i^{\mathrm{T}} = 0$ for $j \neq i$ and by assumption $\sigma_i = \sigma_j = \sigma$, we have

$$\sigma(a_{ji} + a_{ij}) = 0 \Rightarrow a_{ij} = -a_{ji} \quad \text{for all } i \neq j$$

Again from this Lyapunov equation we have for the diagonal of A that

$$(a_{11}, a_{22}, \ldots, a_{nn}) = -\frac{1}{2\sigma}(b_1 b_1^{\mathrm{T}}, b_2 b_2^{\mathrm{T}}, \ldots, b_n b_n^{\mathrm{T}})$$

$$= -\frac{1}{2\sigma}\,\mathrm{diag}\,(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}, 0, \ldots, 0)$$

which implies (ii).

(iii) Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two equivalent systems such that $B_1 B_1^{\mathrm{T}}$ and $B_2 B_2^{\mathrm{T}}$ have the form given in (i). By Theorem 2.4 there exists an orthogonal $\bar{Q} \in \mathbb{R}^{n\times n}$ such that $B_1 = \bar{Q}B_2$. Thus

$$B_1 B_1^{\mathrm{T}} = \bar{Q}B_2 B_2^{\mathrm{T}}\bar{Q}^{\mathrm{T}} = \mathrm{diag}\,(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}, 0, \ldots, 0)$$

By the uniqueness of the eigenvalue problem and the ordering of the diagonal entries of $B_1 B_1^T$ and $B_2 B_2^T$, we have $B_1 B_1^T = B_2 B_2^T$. This shows the invariance of $l, r(1), \ldots, r(l)$.

(iv) Let $Q \in \mathbb{R}^{n \times n}$ such that

$$B_1 B_1^T = Q B_2 B_2^T Q^T = Q B_1 B_1^T Q^T \Leftrightarrow B_1 B_1^T Q = Q B_1 B_1^T$$

$\Leftrightarrow Q = \text{diag}(Q_1, \ldots, Q_{l+1})$ with orthogonal $Q_i \in \mathbb{R}^{r(i) \times r(i)}$ for $1 \leqslant i \leqslant l+1$. □

For the case of SISO systems, we obtain the following corollary.

Corollary 5.2

Let $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n^{1,1}$ with identical singular values $\sigma_1 = \sigma_2 = \ldots = \sigma_n := \sigma, \sigma > 0$. Then there exists an equivalent balanced system (A, b, c) such that

(i) $bb^T = c^T c = \text{diag}(\lambda_1, 0, \ldots, 0)$, with $\lambda_1 > 0$

(ii) we have

$$A = \frac{-1}{2\sigma} \text{diag}(\lambda_1, 0, \ldots, 0) + \mathring{A}$$

where $\mathring{A}$ is skew symmetric.

(iii) the systems equivalent to (A, b, c) and having the structure given in (i) and (ii) are precisely those systems that can be obtained from (A, b, c) by an orthogonal state-space transformation of the form

$$Q = \text{diag}(s_1, \tilde{Q}), \quad s_1 = \pm 1$$

with orthogonal $\tilde{Q} \in \mathbb{R}^{(n-1) \times (n-1)}$.

Proof

The proof follows immediately, since rank $(bb^T) = 1$. □

Step 2

In Step 2, the B-matrix is transformed to its final form. This is achieved by applying a certain QR-factorization, which will be given in Lemma 5.4.

Lemma 5.3 will frequently be needed.

Lemma 5.3

Let $a \in \mathbb{R}^n$, $a \neq 0$, then there exists orthogonal $\bar{Q} \in \mathbb{R}^{n \times n}$ such that $a^T \bar{Q}^T = (a_1, 0, \ldots, 0)$, $a_1 > 0$.

The orthogonal matrices Q preserving the structure of $(a_1, 0, \ldots, 0)$, i.e. $(a_1, 0, \ldots, 0)Q^T = (b, 0, \ldots, 0)$ with $b > 0$, are of the form

$$Q = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \tilde{Q} & \\ 0 & & & \end{bmatrix}$$

where $\tilde{Q} \in \mathbb{R}^{(n-1) \times (n-1)}$ is orthogonal.

Proof

The first statement is clear. To show that the class of all orthogonal Q such that

$$(a_1, \ldots, 0)Q^{\mathrm{T}} = (b, 0, \ldots, 0), \; b > 0$$

has the described form, first note that necessarily $b = a_1$. Write $Q^{\mathrm{T}} = (q_{ij})_{1 \leqslant i,j \leqslant n}$. Then

$$(a_1, 0, \ldots, 0)Q^{\mathrm{T}} = (q_{11}a_1, q_{12}a_1, \ldots, q_{1n}a_1)$$
$$\Rightarrow q_{11} = 1 \quad \text{and} \quad q_{12} = \ldots = q_{1n} = 0$$

From the orthogonality of Q^{T} we have

$$q_{11}q_{i1} = 0 \quad \text{for} \quad 2 \leqslant i \leqslant n \quad \text{and hence } q_{i1} = 0 \quad \text{for } 2 \leqslant i \leqslant n$$

This shows the postulated structure of Q. □

Lemma 5.4

Let $M \in \mathbb{R}^{n \times l}$ with rank $M = n$. Then there exists an orthogonal matrix $Q_0 \in \mathbb{R}^{n \times n}$ and a set of natural numbers $1 \leqslant i_1 < i_2 < \ldots < i_n \leqslant l$ such that

$$M_0 := Q_0 M = \begin{bmatrix} 0 & \ldots & 0 & m_{1i_1} & x & \ldots & x & x & x & \ldots & & & \\ 0 & \ldots & & 0 & 0 & \ldots & 0 & m_{2i_2} & x & & & & \\ \vdots & & & \vdots & \vdots & & & & & & & & \\ 0 & \ldots & & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & m_{ni_n} & x & \ldots \end{bmatrix}$$

with $m_{ji_j} > 0$ for all $1 \leqslant j \leqslant n$.

M_0 is unique, i.e. the only orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that QM_0 has the same structure as M_0 is $Q = I_n$.

Proof

Write $M = (m_1, m_2, \ldots, m_l)$, $m_j \in \mathbb{R}^n$ for all $1 \leqslant j \leqslant l$. Let i_1 be such that $m_{i_1} \neq 0$ and $m_j = 0$ for all $1 \leqslant j < i_1$. Choose $Q_1 \in \mathbb{R}^{n \times n}$ such that

$$Q_1 m_{i_j} =: \begin{bmatrix} m_{1i_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{with } m_{1i_1} > 0$$

So

$$M_1 := Q_1 M = \left[\begin{array}{cccc|ccc} 0 & \ldots & 0 & m_{1i_1} & x & \ldots & x \\ 0 & \ldots & & 0 & & & \\ \vdots & & & \vdots & & M_2 & \\ 0 & \ldots & & 0 & & & \end{array}\right]$$

with rank $(M_2) = n - 1$.

By Lemma 5.3, the orthogonal Q such that

$$QM_1 = \left[\begin{array}{cccc|ccc} 0 & \dots & 0 & \bar{m}_{1i_1} & x & \dots & x \\ 0 & \dots & & 0 & & & \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & & 0 & x & \dots & x \end{array}\right] \quad \text{for } \bar{m}_{1i_1} > 0$$

have the form

$$Q = \operatorname{diag}(1, \bar{Q}) \quad \text{with orthogonal } \bar{Q} \in \mathbb{R}^{(n-1)\times(n-1)}$$

Now proceeding in the same way, we can first bring M_2 to the desired form. The fact that rank $M = n$ implies that this process can be continued inductively until the postulated structure is attained. □

We can now prove the following proposition, which states the final form of the B- and C-matrix.

Proposition 5.5

Let $(\bar{A}, \bar{B}, \bar{C})$ be given in the form derived in Proposition 5.1. Then there exists

$$\tilde{Q} = \operatorname{diag}(Q_1, \dots, Q_l, I_{r(l+1)})$$

with orthogonal $Q_i \in \mathbb{R}^{r(i)\times r(i)}$ for all $1 \leqslant i \leqslant l$ and a set of integer-valued indices

$$1 \leqslant t(i,1) < t(i,2) < \dots < t(i, r(i)) \leqslant m \quad \text{for} \quad 1 \leqslant i \leqslant l$$

such that for $(A, B, C) := (\tilde{Q}\bar{A}\tilde{Q}^{\mathrm{T}}, \tilde{Q}\bar{B}, \bar{C}\tilde{Q}^{\mathrm{T}})$, B and C have the following structure.

B-matrix

For

$$B = \begin{bmatrix} B(1) \\ \vdots \\ B(l) \\ O \end{bmatrix}, \quad B(i) = (b(i)_{st})_{\substack{1 \leqslant s \leqslant r(i) \\ 1 \leqslant t \leqslant m}} \in \mathbb{R}^{r(i)\times m} \quad \text{for } 1 \leqslant i \leqslant l$$

we have

$$b(i)_{st(i,s)} > 0 \quad \text{for all } 1 \leqslant s \leqslant r(i)$$
$$b(i)_{st} = 0 \quad \text{for all } 1 \leqslant t < t(i,s) \text{ and } 1 \leqslant s \leqslant r(i)$$

i.e.

$$B(i) = \begin{bmatrix} 0 & \dots & 0 & b(i)_{1t(i,1)} & & & & & & \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & b(i)_{2t(i,2)} & \dots & b(i)_{st} \\ \cdot & & \cdot & \cdot & \cdot & & 0 & 0 & & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \\ 0 & \dots & & 0 & 0 & \dots & 0 & 0 & \dots & b(i)_{r(i)t(i,r(i))} \ \dots \end{bmatrix}$$

C-matrix

$$C=(U \quad O)\operatorname{diag}(\lambda_1^{1/2} I_{r(1)}, \ldots, \lambda_l^{1/2} I_{r(l)}, 0, \ldots, 0)$$

for some unique $U \in \mathbb{R}^{p \times r_0}$ such that $U^T U = I_{r_0}$.

The indices $\{t(i,1), \ldots, t(i, r(i))\}$, $1 \leqslant i \leqslant l$, are invariants for system equivalence. The systems equivalent to (A, B, C) having the derived B-matrix structure are precisely those systems that can be obtained from (A, B, C) by an orthogonal state-space transformation of the form $Q = \operatorname{diag}(I_{r_0}, Q_{l+1})$ with orthogonal $Q_{l+1} \in \mathbb{R}^{r(l+1) \times r(l+1)}$.

Proof

Structure of the B-matrix

Write

$$\tilde{B} = \begin{bmatrix} \tilde{B}(1) \\ \vdots \\ \tilde{B}(l) \\ O \end{bmatrix}$$

where $\tilde{B}(i) \in \mathbb{R}^{r(i) \times m}$ for $1 \leqslant i \leqslant l$.

Then $\operatorname{rank}(\tilde{B}(i)) = r(i)$ since $\tilde{B}(i)\tilde{B}^T(i) = \lambda_i I_{r(i)}$, $\lambda_i > 0$.

Now apply Lemma 5.4 to show that for all $1 \leqslant i \leqslant l$ there exists an orthogonal $Q_i \in \mathbb{R}^{r(i) \times r(i)}$ such that $B(i) := Q_i \tilde{B}(i)$ has the desired structure. The invariance of the indices $\{t(i,1), \ldots, t(i, r(i))\}$, $1 \leqslant i \leqslant l$, and the last statement also follow from Lemma 5.4.

Structure of the C-matrix

Since $C^T C = \operatorname{diag}(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}, 0, \ldots, 0)$, we can write

$$C =: (\tilde{C} \quad O), \quad \tilde{C} \in \mathbb{R}^{p \times r_0}$$

with

$$\tilde{C}^T \tilde{C} = \operatorname{diag}(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)})$$

Normalizing the columns of $\tilde{C}$ we set:

$$U := \tilde{C} \operatorname{diag}(\lambda_1^{-1/2} I_{r(1)}, \ldots, \lambda_l^{-1/2} I_{r(l)})$$

Then $U^T U = I_{r_0}$ and

$$C = (\tilde{C} \quad O) = (U \quad O)\operatorname{diag}(\lambda_1^{1/2} I_{r(1)}, \ldots, \lambda_l^{1/2} I_{r(l)}, 0, \ldots, 0)$$

The uniqueness of U follows in a straightforward way. □

Specializing to the SISO case we have the following corollary.

Corollary 5.6

Let $(\tilde{A}, \tilde{b}, \tilde{c})$ be given in the form derived in Corollary 5.2. Then there exists

$$\tilde{Q} = \operatorname{diag}(s, I_{n-1}), \quad s = \pm 1$$

such that for $(A, b, c) := (\tilde{Q}\tilde{A}\tilde{Q}^T, \tilde{Q}\tilde{b}, \tilde{c}\tilde{Q}^T)$ we have

$$b = (b_1, 0, \ldots, 0)^T, \quad b_1 > 0$$
$$c = (s_1 b_1, 0, \ldots, 0)$$

The systems equivalent to (A, b, c) having the derived b-vector structure are precisely those systems that can be obtained from (A, b, c) by an orthogonal state-space transformation of the form $Q = \operatorname{diag}(1, Q_0)$ with orthogonal $Q_0 \in \mathbb{R}^{(n-1)\times(n-1)}$. □

Step 3

Here the final structure of the A-matrix is derived. We need the following little lemma.

Lemma 5.7

Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

with arbitrary $A_1 \in \mathbb{R}^{n\times n}$ and skew-symmetric A_2, then A has an eigenvalue on the imaginary axis.

Proof

Since A_2 is skew-symmetric, there exists $w \in \mathbb{R}$ and $x \in \mathbb{C}^m$, $x \neq 0$, such that $A_2 x = iwx$. Setting

$$\tilde{x} = \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathbb{C}^{n+m}$$

we have $A\tilde{x} = iw\tilde{x}$. □

Proposition 5.8

Let $(\tilde{A}, \tilde{B}, \tilde{C})$ be given in the form derived in Proposition 5.5. Then there exists $Q = \operatorname{diag}(I_{r_0}, Q_{l+1})$ such that we have for

$$A := Q\tilde{A}Q^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{r_0 \times r_0}$$

(1) $$A_{11} = \frac{-1}{2\sigma} \operatorname{diag}(\lambda_1 I_{r(1)}, \ldots, \lambda_l I_{r(l)}) + \mathring{A}_{11}$$

where $\mathring{A}_{11}$ is skew-symmetric,

(2) there exists $q \in \mathbb{N}$, $q \geq 1$ and a set of indices

$$(g(1), h(1)), \ldots, (g(q), h(q)) \in \mathbb{N} \times \mathbb{N}$$

with

$$1 = h(1) < \ldots < h(i) < h(i+1) < \ldots \leq n - r_0$$
$$1 \leq g(q) < \ldots < g(i+1) < g(i) < \ldots \leq r_0$$

such that for

$$A_{12} =: (a_{st})_{\substack{1 \leqslant s \leqslant r_0 \\ 1 \leqslant t \leqslant n-r_0}}$$

we have

$$\begin{aligned} a_{g(i)h(i)} &> 0 \quad \text{for } 1 \leqslant i \leqslant q \\ a_{g(i)t} &= 0 \quad \text{for } t > h(i) \text{ where } 1 \leqslant i \leqslant q \\ a_{st} &= 0 \quad \text{for } t \geqslant h(i) \text{ and } s > g(i) \text{ where } 1 \leqslant i \leqslant q \end{aligned}$$

i.e.

$$A_{12} = \begin{bmatrix} x & x & \dots & x & x & x & \dots \\ x & x & \dots & x & a_{g(2)h(2)} & 0 & \dots \\ x & x & & x & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & \vdots & & \\ x & x & \dots & x & 0 & & \\ a_{g(1)h(1)} & 0 & \dots & 0 & & & \\ 0 & 0 & & & & \mathbf{0} & \\ \vdots & \vdots & & & & & \end{bmatrix}$$

(3) $A_{21} = -A_{12}^{\mathrm{T}}$

(4)

$$A_{22} = \begin{bmatrix} 0 & \alpha_2 & & & \\ -\alpha_2 & 0 & \alpha_3 & \mathbf{0} & \\ & -\alpha_3 & 0 & & \\ & \mathbf{0} & & & \alpha_{n-r_0} \\ & & & -\alpha_{n-r_0} & 0 \end{bmatrix}$$

with

$$\alpha_i \begin{cases} = 0 & \text{if } i = h(s) \text{ for some } 1 \leqslant s \leqslant q \\ > 0 & \text{otherwise} \end{cases}$$

The indices q, $(g(1), h(1)), \dots, (g(q), h(q))$ are invariants for system equivalence.

The representation $(A, B, C) := (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$ is unique, i.e. if (A_1, B_1, C_1) is equivalent to (A, B, C) and is given in the form derived in Steps 1–3, then $(A_1, B_1, C_1) = (A, B, C)$.

Proof

Write

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in \mathbb{R}^{r_0 \times r_0}$$

To derive the desired structure for $\tilde{A}_{12}$ and $\tilde{A}_{22}$, i.e. the indices $(g(1), h(1)), \ldots, (g(q), h(q))$, we proceed stepwise at the same time reducing the freedom to perform an orthogonal state-space transformation.

(i) Reduction of the freedom to perform a state-space transformation from $Q_1 = \operatorname{diag}(I_{r_0}, \tilde{Q}_1)$, $\tilde{Q}_1 \in \mathbb{R}^{(n-r_0)\times(n-r_0)}$ orthogonal, to $Q_2 = \operatorname{diag}(I_{r_0+1}, \tilde{Q}_2)$, $\tilde{Q}_2 \in \mathbb{R}^{(n-r_0-1)\times(n-r_0-1)}$ orthogonal.

From Proposition 5.5 we know that all systems equivalent to $(\tilde{A}, \tilde{B}, \tilde{C})$ having the structure stated there can be obtained from $(\tilde{A}, \tilde{B}, \tilde{C})$ by an orthogonal state-space transformation of the form $Q_1 = \operatorname{diag}(I_{r_0}, \tilde{Q}_1)$, $\tilde{Q}_1 \in \mathbb{R}^{(n-r_0)\times(n-r_0)}$ orthogonal.

Consider

$$\tilde{A}_{12} =: \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{r_0} \end{bmatrix}$$

where $\tilde{a}_i$ is the ith row of $\tilde{A}_{12}$.

We first have to show that $\tilde{A}_{12} \neq 0$. To do this, note that $\tilde{A}_{22}$ is skew-symmetric (Proposition 5.1). Thus by Lemma 5.7 $\tilde{A}_{12} = 0$ would contradict the asymptotic stability of $\tilde{A}$. So we can find $g(1)$ such that $\tilde{a}_{g(1)} \neq 0$ and $\tilde{a}_j = 0$ for $j > g(1)$.

By Lemma 5.3, there exists an orthogonal $\bar{Q}_1 \in \mathbb{R}^{(n-r_0)\times(n-r_0)}$ such that $\tilde{a}_{g(1)}\bar{Q}_1^{\mathrm{T}} = (a_{g(1)1}, 0, \ldots, 0)$ for $a_{g(1)1} > 0$. Let $(A, B, C) := (Q_1 \tilde{A} Q_1^{\mathrm{T}}, Q_1 \tilde{B}, \tilde{C} Q_1^{\mathrm{T}})$ where $Q_1 = \operatorname{diag}(I_{r_0}, \bar{Q}_1)$. Writing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{r_0 \times r_0}$$

we have

$$A_{12} = \tilde{A}_{12}\bar{Q}_1^{\mathrm{T}} = \begin{bmatrix} x & x & \ldots & x \\ \vdots & \vdots & & \vdots \\ x & x & \ldots & x \\ a_{g(1)1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

From Lemma 5.3 and Proposition 5.5, it follows that all systems that are equivalent to (A, B, C) and have the same structure as (A, B, C) can be obtained from (A, B, C) by a state-space transformation of the form

$$Q_2 = \operatorname{diag}(I_{r_0+1}, \tilde{Q}_2), \quad \tilde{Q}_2 \in \mathbb{R}^{(n-r_0-1)\times(n-r_0-1)} \text{ orthogonal}$$

This also shows the invariance of the index $(g(1), h(1)) := (g(1), 1)$.

(ii) Reduction of the degree of freedom to perform a state-space transformation from

$$Q_2 = \operatorname{diag}(I_{r_0+1}, \tilde{Q}_2), \quad \tilde{Q}_2 \in \mathbb{R}^{(n-r_0-1)\times(n-r_0-1)} \text{ orthogonal}$$

to

$$Q_3 = \operatorname{diag}(I_{r_0+2}, \tilde{Q}_3), \quad \tilde{Q}_3 \in \mathbb{R}^{(n-r_0-2)\times(n-r_0-2)} \text{ orthogonal}$$

Assume that $(\tilde{A}, \tilde{B}, \tilde{C})$ is given in the form derived in (i). Write

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \text{with } \tilde{A}_{22} = \begin{bmatrix} 0 & a \\ -a^{\mathrm{T}} & \bar{A}_{22} \end{bmatrix} \in \mathbb{R}^{(n-r_0)\times(n-r_0)}$$

For $Q_2 = \operatorname{diag}(I_{r_0+1}, \tilde{Q}_2)$, $\tilde{Q}_2$ orthogonal, we have

$$Q_2 \tilde{A} Q_2^{\mathrm{T}} = \left[\begin{array}{c|cc} \tilde{A}_{11} & \multicolumn{2}{c}{\tilde{A}_{12}\operatorname{diag}(1, \tilde{Q}_2^{\mathrm{T}})} \\ \hline & 0 & a\tilde{Q}_2^{\mathrm{T}} \\ \operatorname{diag}(1, \tilde{Q}_2)\tilde{A}_{21} & & \\ & -\tilde{Q}_2 a^{\mathrm{T}} & \tilde{Q}_2 \bar{A}_{22} \tilde{Q}_2^{\mathrm{T}} \end{array}\right]$$

Case 1: $a \neq 0$. By Lemma 5.3, there exists an orthogonal $\bar{Q}_2$ such that $a\bar{Q}_2^{\mathrm{T}} = (\alpha_2, 0, \ldots, 0)$ for $\alpha_2 > 0$. All orthogonal $\tilde{Q}_2$ such that $(\alpha_2, 0, \ldots, 0)\tilde{Q}_2^{\mathrm{T}} = (\alpha_2^1, 0, \ldots, 0)$ with $\alpha_2^1 > 0$ are of the form $\tilde{Q}_2 = \operatorname{diag}(1, \tilde{Q}_3)$ with orthogonal $\tilde{Q}_3 \in \mathbb{R}^{(n-r_0-2)\times(n-r_0-2)}$.

Let $(A, B, C) := (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$ where $Q = \operatorname{diag}(I_{r_0+1}, \bar{Q}_2)$. It thus follows that all systems equivalent to (A, B, C) having the same structure as (A, B, C) can be obtained from (A, B, C) by an orthogonal state-space transformation with $Q_3 = \operatorname{diag}(I_{r_0+2}, \tilde{Q}_3)$, $\tilde{Q}_3 \in \mathbb{R}^{(n-r_0-2)\times(n-r_0-2)}$.

Case 2: $a = 0$. This implies $\alpha_2 = 0$. Now write $\tilde{A}_{12} =: (A_{12}^1 | A_{12}^2)$, where $A_{12}^1 \in \mathbb{R}^{r_0}$. We have to show that $A_{12}^2 \neq 0$. If we assume $A_{12}^2 = 0$, then $\tilde{A}$ can be written as $\tilde{A} = \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix}$ with $\hat{A}_{22} \in \mathbb{R}^{(n-r_0-1)\times(n-r_0-1)}$ skew-symmetric. Thus by Lemma 5.7 we have a contradiction to the asymptotic stability of $\tilde{A}$. Write

$$A_{12}^2 = \begin{bmatrix} a_1 \\ \vdots \\ a_{r_0} \end{bmatrix}, \quad a_i^{\mathrm{T}} \in \mathbb{R}^{n-r_0-1} \quad \text{for } 1 \leqslant i \leqslant r_0$$

Since $A_{12}^2 \neq 0$, there exists $g(2)$ such that $a_{g(2)} \neq 0$ and $a_j = 0$ for all $j > g(2)$. From (i) we have necessarily that $g(2) < g(1)$. Set $(g(2), h(2)) := (g(2), 2)$.

Now choose $\bar{Q}_2$ orthogonal such that

$$a_{g(2)}\bar{Q}_2^{\mathrm{T}} = (a_{g(2)h(2)}, 0, \ldots, 0), \quad a_{g(2)h(2)} > 0$$

and let $(A, B, C) := (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$ where $Q = \operatorname{diag}(I_{r_0+1}, \bar{Q}_2)$. Then

$$A_{12} = \tilde{A}_{12}\operatorname{diag}(1, \bar{Q}_2^{\mathrm{T}}) = \begin{bmatrix} x & x & x & \cdots & x \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & x \\ x & a_{g(2)2} & 0 & \cdots & 0 \\ x & 0 & & & \\ \vdots & \vdots & & & \\ x & & & \mathbf{0} & \\ a_{g(1)1} & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

By the standard argument, we obtain the desired reduction of the freedom to perform a state-space transformation and the invariance of the index $(g(2), 2)$.

(iii) Starting with $Q_3 = \operatorname{diag}(I_{r_0+2}, \tilde{Q}_3)$, $\tilde{Q}_3 \in \mathbb{R}^{(n-r_0-2)\times(n-r_0-2)}$ orthogonal, the same procedure as in (ii) is repeated inductively until the only freedom to perform a state-space transformation is given by matrices of the form $Q = \operatorname{diag}(I_{n-1}, \pm 1)$. Thus assume that $(\tilde{A}, \tilde{B}, \tilde{C})$ has the corresponding form. Write

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad \text{with} \quad \tilde{A}_{22} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

and $\tilde{A}_{12} = (\tilde{A}_{12}^1 \mid \tilde{A}_{12}^2)$, $\tilde{A}_{12}^2 \in \mathbb{R}^{n-2}$.

Assume the last index derived is $(g(\tilde{m}), h(\tilde{m}))$.

Case 1: $a \neq 0$. Then $q = \tilde{m}$. Let $(A, B, C) = (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$ with $Q = \operatorname{diag}(I_{n-1}, \operatorname{sign}(a))$. Then (A, B, C) has the postulated structure with $\alpha_{n-r_0} := |a|$. It follows that the only orthogonal $\tilde{Q} \in \mathbb{R}^{n\times n}$ preserving the desired structure is $\tilde{Q} = I_n$.

Case 2: $a = 0$. Then $q = \tilde{m} + 1$ and $\alpha_{n-r_0} = 0$. Consider $\tilde{A}_{12}^2$ as defined above. The usual argument leads to $\tilde{A}_{12}^2$ being non-zero. Write

$$\tilde{A}_{12}^2 = \begin{bmatrix} a_1 \\ \vdots \\ a_{r_0} \end{bmatrix}, \quad a_i \in \mathbb{R} \quad \text{for } 1 \leqslant i \leqslant r_0$$

Let $g(\tilde{m}+1) := g(q)$ be such that $a(q) \neq 0$ and $a_j = 0$ for $j > g(q)$. The structure of $\tilde{A}_{12}$ implies that $1 \leqslant g(q) < g(q-1)$. So $(g(q), h(q)) := (g(q), n - r_0)$ with $n - r_0 = h(q) > h(q-1) = h(\tilde{m})$.

Now let $Q = \operatorname{diag}(I_{n-1}, \operatorname{sign}(a_{g(q)}))$. Then $(A, B, C) := (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$ has the desired form with $a_{g(q)h(q)} := |a_{g(q)}|$. By the usual argument, we have the uniqueness of A which implies the uniqueness of (A, B, C) and the invariance of q and $(g(q), h(q))$.

This completes the derivation of the structure of A_{11} and A_{22} as given in the statement of the proposition. The properties of A_{11} and A_{21} follow from Proposition 5.1. □

In the SISO case, we have the following corollary.

Corollary 5.9

Let $(\tilde{A}, \tilde{b}, \tilde{c})$ be given in the form derived in Corollary 5.6. Then there exists $Q = \operatorname{diag}(1, Q_0)$, $Q_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ orthogonal, such that we have

$$A = Q\tilde{A}Q^{\mathrm{T}} = \begin{bmatrix} a_{11} & \alpha_1 & & & & \\ -\alpha_1 & 0 & \alpha_2 & & \mathbf{0} & \\ & -\alpha_2 & 0 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & 0 & \alpha_{n-1} \\ & & & & -\alpha_{n-1} & 0 \end{bmatrix}$$

with $a_{11} = -b_1^2/2\sigma$, $\alpha_i > 0$ for $1 \leqslant i \leqslant n-1$. □

Rephrasing the results of Steps 1–3 we arrive at the main result of this section.

Theorem 5.10

The representation constructed in Propositions 5.1, 5.5 and 5.8 defines a canonical form for the set of minimal and asymptotically stable continuous-time systems with identical singular values.

6. A canonical form for $C_n^{p,m}$: the general case

In this section, a canonical form $\Gamma_n : C_n^{p,m} \to C_n^{p,m,b}$ will be derived. This is done by considering for a system $(A, B, C) \in C_n^{p,m}$ the subsystems corresponding to identical singular values. It is then possible to bring each of these subsystems to the canonical form derived in § 5. This already defines a canonical form for (A, B, C).

6.1. *MIMO systems*

In the following theorem, the canonical form will be stated for arbitrary dimensions of the input and output spaces.

Theorem 6.1

Let $(\tilde{A}, \tilde{B}, \tilde{C}) \in C_n^{p,m}$ have singular values $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$, with σ_j having multiplicity $n(j)$, for $1 \leqslant j \leqslant k$, $\sum_{j=1}^{k} n(j) = n$. Then there exists an equivalent system $(A, B, C) =: \Gamma((\tilde{A}, \tilde{B}, \tilde{C})) \in C_n^{p,m,b}$ with the following properties.

B-matrix

(1) Partition

$$B = \begin{bmatrix} B^1 \\ \vdots \\ B^k \end{bmatrix}$$

with $B^j \in \mathbb{R}^{n(j) \times m}$ then for $1 \leqslant j \leqslant k$,

$$B^j (B^j)^{\mathrm{T}} = \operatorname{diag}\left(\lambda_1^{(j)} I_{r(j;1)}, \lambda_2^{(j)} I_{r(j;2)}, \dots, \lambda_{l(j)}^{(j)} I_{r(j;l(j))}, 0, \dots, 0\right)$$

such that $\lambda_1^{(j)} > \lambda_2^{(j)} > \dots > \lambda_{l(j)}^{(j)} > 0$ and

$$\sum_{i=1}^{l(j)} r(j;i) =: r_0(j) \leqslant \min(p, m)$$

(2) For each $1 \leqslant j \leqslant k$, B^j has the following structure:

$$B^j = \begin{bmatrix} B(j;1) \\ \vdots \\ B(j;l(j)) \\ 0 \end{bmatrix}$$

with $B(j;i) \in \mathbb{R}^{r(j;i)\times m}$ for $1 \leqslant i \leqslant l(j)$. The precise structure of

$$B(j;i) =: (b(j;i)_{st})_{\substack{1\leqslant s\leqslant r(j;i)\\ 1\leqslant t\leqslant m}}$$

is given by the indices:

$$1 \leqslant t(j;i,1) < t(j;i,2) < \dots < t(j;i,r(j;i)) \leqslant m \quad \text{for } 1 \leqslant i \leqslant l(j)$$

We have

$$b(j;i)_{st(j;i,s)} > 0 \quad \text{for all } 1 \leqslant s \leqslant r(j;i)$$
$$b(j;i)_{st} = 0 \quad \text{for all } 1 \leqslant t < t(j;i,s) \text{ and } 1 \leqslant s \leqslant r(j;i)$$

i.e.

$$B(j;i) = \begin{bmatrix} 0 & \dots & 0 & b(j;i)_{1t(j;i,1)} & & & & & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & b(j;i)_{2t(j;i,2)} & \dots & b(j;i)_{st} & \\ & & & & & & 0 & & & \\ \vdots & & & & & \vdots & & & & \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & b(j;i)_{r(j;i)t(j,r(j;i))} \end{bmatrix}$$

C-matrix

C admits the representation

$$C = (C^1 \quad C^2 \quad \dots \quad C^k), \quad C^j \in \mathbb{R}^{p\times n(j)}$$

with

$$C^j = (U^j \quad O)\operatorname{diag}((\lambda_1^{(j)})^{1/2} I_{r(j;1)}, \dots, (\lambda_{l(j)}^{(j)})^{1/2} I_{r(j;l(j))}, 0, \dots, 0)$$

for unique $U^j \in \mathbb{R}^{p\times ro(j)}$ such that $(U^j)^{\mathrm{T}} U^j = I_{ro(j)}$ for $1 \leqslant j \leqslant k$.

A-matrix

A admits a partitioning $A = (A(i,j))_{1\leqslant i,j\leqslant k}$ with $A(i,j) \in \mathbb{R}^{n(i)\times n(j)}$ for all $1 \leqslant i,j \leqslant k$, with the following properties.

(i) Block diagram entries $A(j,j)$:

$A(j,j)$, $1 \leqslant j \leqslant k$, can be partitioned as

$$A(j,j) = \begin{bmatrix} A(j,j)_{11} & A(j,j)_{12} \\ A(j,j)_{21} & A(j,j)_{22} \end{bmatrix}, \quad A(j,j)_{11} \in \mathbb{R}^{ro(j)\times ro(j)}$$

with

(1) $$A(j,j)_{11} = \frac{-1}{2\sigma_j}\operatorname{diag}(\lambda_1^{(j)} I_{r(j;1)}, \dots, \lambda_{l(j)}^{(j)} I_{r(j;l(j))}) + \mathring{A}(j,j)_{11}$$

where $\mathring{A}(j,j)_{11}$ is skew-symmetric.

(2) There exists $q(j) \in \mathbb{N}$, $q(j) \geqslant 1$ and a set of indices

$$(g(j;1), h(j;1)), \dots, (g(j;q(j)), h(j;q(j))) \in \mathbb{N} \times \mathbb{N}$$

with

$$1 = h(j;1) < \dots < h(j;i) < h(j;i+1) < \dots \leqslant n - r_0(j)$$
$$1 \leqslant g(j;q(j)) < \dots < g(j;i+1) < g(j;i) < \dots \leqslant r_0(j)$$

such that for

$$A(j,j)_{12} =: (a(j)_{st})_{\substack{1 \leqslant s \leqslant r_0(j) \\ 1 \leqslant t \leqslant n(j) - r_0(j)}}$$

we have

$$\begin{aligned} a(j)_{g(j;i)h(j;i)} &> 0 \quad \text{for } 1 \leqslant i \leqslant q(j) \\ a(j)_{g(j;i)t} &= 0 \quad \text{for } t > h(j;i), \text{ where } 1 \leqslant i \leqslant q(j) \\ a(j)_{st} &= 0 \quad \text{for } t \geqslant h(j;i) \text{ and } s > g(j;i), \text{ where } 1 \leqslant i \leqslant q(j) \end{aligned}$$

i.e.

$$A(j,j)_{12} = \begin{bmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots & \\ x & x & \dots & x & a(j)_{g(j;2)h(j;2)} & 0 & \dots \\ x & x & \dots & x & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & \vdots & 0 & \dots \\ x & x & \dots & x & 0 & & \\ a(j)_{g(j;1)h(j;1)} & 0 & \dots & 0 & 0 & & \\ 0 & 0 & \dots & 0 & & & \\ \vdots & \vdots & & \vdots & & & \end{bmatrix}$$

(3) $A(j,j)_{21} = -A(j,j)_{12}^{\mathrm{T}}$

(4)

$$A(j,j)_{22} = \begin{bmatrix} 0 & \alpha(j)_2 & & & & \\ -\alpha(j)_2 & 0 & \alpha(j)_3 & \dots & & 0 \\ & -\alpha(j)_3 & 0 & \dots & & \\ & & & \dots & 0 & \alpha(j)_{n(j)-r_0(j)} \\ & 0 & & & -\alpha(j)_{n(j)-r_0(j)} & 0 \end{bmatrix}$$

with

$$\alpha(j)_i \begin{cases} = 0 & \text{if } i = h(j;s) \text{ for some } 1 \leqslant s \leqslant q(j) \\ > 0 & \text{otherwise} \end{cases}$$

(ii) Off-diagonal blocks $A(i,j)$, $(i \neq j)$:

$$A(i,j) = \begin{bmatrix} \tilde{A}(i,j) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with } \tilde{A}(i,j) =: (a(i,j)_{st})_{\substack{1 \leqslant s \leqslant r_0(i) \\ 1 \leqslant t \leqslant r_0(j)}} \in \mathbb{R}^{r_0(i) \times r_0(j)}$$

where

$$a(i,j)_{st} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j b(i)_s b(j)_t^{\mathrm{T}} - \sigma_i c(i)_s^{\mathrm{T}} c(j)_t)$$

with $b(i)_s$ the sth row of B^i and $c(i)_s$ the sth column of C^i.

The so-constructed map $\Gamma: C_n^{p,m} \to C_n^{p,m,b}$ defined by $(\tilde{A}, \tilde{B}, \tilde{C}) \to \Gamma((\tilde{A}, \tilde{B}, \tilde{C}))$ is a canonical form for system equivalence on $C_n^{p,m}$.

Proof

We can assume without loss of generality that $(\tilde{A}, \tilde{B}, \tilde{C})$ is balanced with $\Sigma = \operatorname{diag}(\sigma_1 I_{n(1)}, \dots, \sigma_k I_{n(k)})$.

Partition $(\tilde{A}, \tilde{B}, \tilde{C})$ as

$$\tilde{A} = (\tilde{A}(i,j))_{1 \leqslant i,j \leqslant k} \quad \text{where } \tilde{A}(i,j) \in \mathbb{R}^{n(i) \times n(j)} \text{ for } 1 \leqslant i,j \leqslant k$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}^1 \\ \vdots \\ \tilde{B}^k \end{bmatrix} \quad \text{with } \tilde{B}^j \in \mathbb{R}^{n(j) \times m} \text{ for } 1 \leqslant j \leqslant k$$

$$\tilde{C} = (\tilde{C}^1 \dots \tilde{C}^k) \quad \text{with } \tilde{C}^j \in \mathbb{R}^{p \times n(j)} \quad \text{for } 1 \leqslant j \leqslant k$$

Now consider for $1 \leqslant j \leqslant k$ the subsystem $(\tilde{A}(j,j)), \tilde{B}^j, \tilde{C}^j)$ which is balanced with identical singular values σ_j (Theorem 2.5). By Theorem 5.10 there exists $Q^j \in \mathbb{R}^{n(j) \times n(j)}$ such that

$$(A(j,j), B^j, C^j) := (Q^j \tilde{A}(j,j)(Q^j)^{\mathrm{T}}, Q^j \tilde{B}^j, \tilde{C}^j (Q^j)^{\mathrm{T}})$$

has the desired structure.

Let

$$(A, B, C) = (Q\tilde{A}Q^{\mathrm{T}}, Q\tilde{B}, \tilde{C}Q^{\mathrm{T}})$$

where $Q = \operatorname{diag}(Q^1, Q^2, \dots, Q^k)$. The uniqueness of the representation $(A(j,j), B^j, C^j)$ for $1 \leqslant j \leqslant k$, implies that (A, B, C) is uniquely defined.

It remains to be shown that $A(i,j)$, $(i \neq j)$, has the required structure.

Let $A(i,j) =: (a(i,j)_{st})_{\substack{1 \leqslant s \leqslant n(i) \\ 1 \leqslant t \leqslant n(j)}}$.

Considering the Lyapunov equations component-wise and using the notation introduced in the statement of the theorem, we obtain

$$\sigma_j a(i,j)_{st} + \sigma_i a(j,i)_{ts} = -b(i)_s b(j)_t^{\mathrm{T}}$$

$$\sigma_j a(j,i)_{ts} + \sigma_i a(i,j)_{st} = -c(i)_s^{\mathrm{T}} c(j)_t$$

Solving for $\begin{bmatrix} a(i,j)_{st} \\ a(j,i)_{ts} \end{bmatrix}$ we obtain:

$$\begin{bmatrix} a(i,j)_{st} \\ a(j,i)_{ts} \end{bmatrix} = \frac{1}{\sigma_i^2 - \sigma_j^2} \begin{bmatrix} \sigma_j & -\sigma_i \\ -\sigma_i & \sigma_j \end{bmatrix} \begin{bmatrix} b(i)_s b(j)_t^{\mathrm{T}} \\ c(i)_s^{\mathrm{T}} c(j)_t \end{bmatrix}$$

The result follows by noting that

$$b(j)_s = 0 \quad \text{and} \quad c(j)_s = 0 \quad \text{for} \quad r_0(j) < s \leqslant n(j)$$ □

6.2. *SISO systems*

As an illustration, the previous theorem is now specialized to the case of SISO systems, which is interesting in its own right. Here the structure of the canonical form is considerably simpler.

Corollary 6.2

Let $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n^{1,1}$ have singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ with multiplicities $n(1), \ldots, n(k)$, $\sum_{j=1}^{k} n(j) = n$.

Then there exists a canonical form $(A, b, c) := \Gamma((\tilde{A}, \tilde{b}, \tilde{c})) \in C_n^{1,1,b}$ of $(\tilde{A}, \tilde{b}, \tilde{c})$ such that

(i)
$$b^{\mathrm{T}} = (\underbrace{b_1, 0, \ldots, 0}_{n(1)}, \underbrace{b_2, 0, \ldots, 0}_{n(2)}, \ldots, \underbrace{b_k, 0, \ldots, 0}_{n(k)})$$

with $b_j > 0$ for $1 \leqslant j \leqslant k$.

(ii)
$$c = (\underbrace{s_1 b_1, 0, \ldots, 0}_{n(1)}, \underbrace{s_2 b_2, 0, \ldots, 0}_{n(2)}, \ldots, \underbrace{s_k b_k, 0, \ldots, 0}_{n(k)})$$

where $s_i = \pm 1$ for $1 \leqslant i \leqslant k$.

(iii) For $A = (A(i,j))_{1 \leqslant i,j \leqslant k}$, $A(i,j) \in \mathbb{R}^{n(i) \times n(j)}$ we have the following.

(1) *Block-diagonal entries* $A(j,j)$:

for all $1 \leqslant j \leqslant k$

$$A(j,j) = \begin{bmatrix} a(j,j) & \alpha(j)_1 & & & & \\ -\alpha(j)_1 & 0 & \alpha(j)_2 & & \mathbf{0} & \\ & -\alpha(j)_2 & 0 & \alpha(j)_3 & & \\ & & \vdots & \vdots & \vdots & \\ & \mathbf{0} & & & 0 & \alpha(j)_{n(j)-1} \\ & & & & -\alpha(j)_{n(j)-1} & 0 \end{bmatrix}$$

with $a(j,j) = \dfrac{-b_j^2}{2\sigma_j}$

$$\alpha(j)_i > 0 \quad \text{for all} \quad 1 \leqslant i \leqslant n(j) - 1$$

(2) *Off-diagonal block* $A(i,j)$:

for all $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A(i,j) = \begin{bmatrix} a(i,j) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = s_i s_j A(j,i)^{\mathrm{T}}$$

with $a(i,j) = \dfrac{-1}{s_i s_j \sigma_i + \sigma_j} b_i b_j$. □

Wilson and Kumar (1983) have shown that each system $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n^{1,1}$ is equivalent to a system $(A, b, c) \in C_n^{1,1,b}$ having a particular sign symmetry, i.e. there exists $S = \mathrm{diag}\,(s_1, s_2, \ldots, s_n)$, $s_i = \pm 1$ for $1 \leqslant i \leqslant n$, such that

$$A^{\mathrm{T}} = SAS$$

$$c^{\mathrm{T}} = Sb$$

The following proposition shows that the canonical form derived for SISO systems also has this sign symmetry.

Proposition 6.3

Let $(A, b, c) \in \Gamma_n(C_n^{1,1})$, then (A, b, c) has the sign symmetry with the sign matrix $S = \mathrm{diag}\,(s_1 \hat{I}_{n(1)}, s_2 \hat{I}_{n(2)}, \ldots, s_k \hat{I}_{n(k)})$ where $s_1, \ldots, s_k$ are the signs appearing in the parametrization of c, and $\hat{I}_k = \mathrm{diag}\,(1, -1, 1, \ldots) \in \mathbb{R}^{k \times k}$.

7. Parametrization

Theorem 6.2 shows that each system in $C_n^{p,m}$ can be represented in a particular form. The following theorem shows that the converse is also true, i.e. each system which has such a form is automatically balanced, minimal and asymptotically stable. In fact, the theorem can be formulated in a slightly more general form.

Theorem 7.1

Let $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ be parametrized in the following way.

Let $\Sigma = \mathrm{diag}\,(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \ldots, \sigma_k I_{n(k)})$ be such that $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ and $\sum_{j=1}^{k} n(j) = n$.

B-matrix

For all $1 \leqslant j \leqslant k$, $B^j \in \mathbb{R}^{n(j) \times m}$ is constructed such that

$$B^j(B^j)^{\mathrm{T}} = \mathrm{diag}\,(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{r_0(j)}^{(j)}, 0, \ldots, 0)$$

with $\lambda_i^{(j)} > 0$ for all $1 \leqslant i \leqslant r_0(j) \leqslant n(j)$.

Set $B := \begin{bmatrix} B^1 \\ \vdots \\ B^k \end{bmatrix}$.

C-matrix

For all $1 \leqslant j \leqslant k$ let $C^j \in \mathbb{R}^{p \times n(j)}$ be such that

$$C^j = (U^j \quad O)(B^j(B^j)^{\mathrm{T}})^{1/2}$$

where $U^j \in \mathbb{R}^{p \times r_0(j)}$ such that $(U^j)^{\mathrm{T}} U^j = I_{r_0(j)}$.

Set $C = (C^1 \quad C^2 \quad \ldots \quad C^k)$.

A-matrix

A admits a partitioning $A=(A(i,j))_{1\leqslant i,j\leqslant k}$ with

$$A(i,j)\in\mathbb{R}^{n(i)\times n(j)} \quad \text{for all} \quad 1\leqslant i,j\leqslant k$$

where the blocks are parametrized as shown below.

(i) Block-diagonal entries $A(j,j)$:

$A(j,j)$ can be partitioned as

$$A(j,j)=\begin{bmatrix} A(j,j)_{11} & A(j,j)_{12} \\ A(j,j)_{21} & A(j,j)_{22} \end{bmatrix}, \quad \text{with } A(j,j)_{11}\in\mathbb{R}^{r_0(j)\times r_0(j)}$$

(1) Set $A(j,j)_{11}=\dfrac{-1}{2\sigma_j}\operatorname{diag}(\lambda_1^j,\lambda_2^j,\ldots,\lambda_{r_0(j)}^j)+\mathring{A}(j,j)_{11}$. $\mathring{A}(j,j)_{11}$ an arbitrarily parametrized skew-symmetric matrix.

(2) For $1\leqslant q(j)\leqslant\min(n-r_0(j),r_0(j))$, $q(j)\in\mathbb{N}$, choose a set of indices

$$(g(j;1),h(j;1)),\ldots,(g(j;q(j)),h(j;q(j)))\in\mathbb{N}\times\mathbb{N}$$

such that

$$1=h(j;1)<\ldots<h(j;i)<h(j;i+1)<\ldots\leqslant n-r_0(j)$$
$$1\leqslant g(j;q(j))<\ldots<g(j;i+1)<g(j;i)<\ldots\leqslant r_0(j)$$

Let $A(j,j)_{12}=:(a(j)_{st})_{\substack{1\leqslant s\leqslant r_0(j)\\ 1\leqslant t\leqslant n(j)-r_0(j)}}$ be such that

$$a(j)_{g(j;i)h(j;i)}\neq 0 \quad \text{for } 1\leqslant i\leqslant q(j)$$
$$a(j)_{g(j;i)t}=0 \quad \text{for } t>h(j;i) \text{ where } 1\leqslant i\leqslant q(j)$$
$$a(j)_{st}=0 \quad \text{for } t\geqslant h(j;i) \text{ and } s>g(j;i), \text{ where } 1\leqslant i\leqslant q(j)$$

(3) Set $A(j,j)_{21}:=-A^{\mathrm{T}}(j,j)_{12}$.

(4) Set

$$A(j,j)_{22}:=\begin{bmatrix} 0 & \alpha(j)_2 & & & & \\ -\alpha(j)_2 & 0 & \alpha(j)_3 & \cdots & \mathbf{0} & \\ & -\alpha(j)_3 & 0 & \cdots & & \\ & & & & \vdots & \\ & \mathbf{0} & & \cdots & 0 & \alpha(j)_{n(j)-r_0(j)} \\ & & & & -\alpha(j)_{n(j)-r_0(j)} & 0 \end{bmatrix}$$

with

$$\alpha(j)_i\begin{cases}\neq 0 & \text{if } i\neq h(j;s) \text{ for all } 1\leqslant s\leqslant q(j)\\ \in\mathbb{R} & \text{otherwise}\end{cases}$$

(ii) Off-diagonal blocks $A(i,j)$:

Set

$$A(i,j):=(a(i,j)_{st}):_{\substack{1\leqslant s\leqslant n(i)\\ 1\leqslant t\leqslant n(j)}}, \quad 1\leqslant i,j\leqslant k,\ i\neq j$$

with

$$a(i,j)_{st} = \frac{1}{\sigma_i^2 - \sigma_j^2}(\sigma_j b(i)_s b(j)_t^{\mathrm{T}} - \sigma_i c(i)_s^{\mathrm{T}} c(j)_t)$$

where $b(j)_s$ is the sth row of B^j and $c(j)_s$ is the sth column of C^j.

Then the so-defined system (A, B, C) is in $C_n^{p,m,b}$ with observability and controllability gramian Σ.

Proof

It is straightforward to check that (A, B, C) and Σ satisfy the Lyapunov equations as given in Theorem 2.3.

To show the asymptotic stability of A, we show first that $A(j,j)$ is asymptotically stable for all $1 \leqslant j \leqslant k$. Since

$$\sigma_j A(j,j) + \sigma_j A^{\mathrm{T}}(j,j) = -B^j(B^j)^{\mathrm{T}} \qquad (*)$$

it follows from Lemma 2.6 that $\mathrm{Re}(\lambda) \leqslant 0$ for all $\lambda \in \sigma(A)$.

By constructing a contradiction, we show that $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A(j,j))$. Assume that there exists $\lambda = iw$, $w \in \mathbb{R}$, and $x \in \mathbb{C}^{n(j)}$, $x \neq 0$ such that $A(j,j)x = \lambda x = iwx$. From $(*)$ we obtain

$$-\bar{x}^{\mathrm{T}} B^j(B^j)^{\mathrm{T}} x = \sigma_j(\bar{x}^{\mathrm{T}} A(j,j)x + \bar{x}^{\mathrm{T}} A(j,j)^{\mathrm{T}} x) = \sigma_j(iw\bar{x}^{\mathrm{T}} x + \overline{iw}\bar{x}^{\mathrm{T}} x) = 0$$

$$\Rightarrow (B^j)^{\mathrm{T}} x = 0$$

$$\Rightarrow B^j(B^j)^{\mathrm{T}} x = 0$$

But

$$B^j(B^j)^{\mathrm{T}} = \mathrm{diag}\,(\lambda_1^{(j)}, \ldots, \lambda_{r_0(j)}^{(j)}, 0, \ldots, 0)$$

with $\lambda_i^{(j)} > 0$ for all $1 \leqslant i \leqslant r_0(j)$. So $x = (0 \quad \tilde{x})^{\mathrm{T}}$ with $\tilde{x} \in \mathbb{C}^{n(j) - r_0(j)}$.

Now write

$$A(j,j) = \begin{bmatrix} A(j,j)_{11} & A(j,j)_{12} \\ A(j,j)_{21} & A(j,j)_{22} \end{bmatrix}, \quad A(j,j)_{11} \in \mathbb{R}^{r_0(j) \times r_0(j)}$$

Since

$$A(j,j)\begin{bmatrix} 0 \\ \tilde{x} \end{bmatrix} = iw\begin{bmatrix} 0 \\ \tilde{x} \end{bmatrix}$$

we have

$$A(j,j)_{12}\tilde{x} = 0$$

and

$$A(j,j)_{22}\tilde{x} = iw\tilde{x}$$

This allows us to show component-wise that $\tilde{x} = 0$.

Write $\tilde{x}^{\mathrm{T}} = (x_1, x_2, \ldots, x_{n(j) - r_0(j)})$.

Owing to the specific structure of $A(j,j)_{12}$ we obtain from $A(j,j)_{12}\tilde{x}=0$ that

$$a_{g(j;1)h(j;1)}x_1 = 0$$
$$\Rightarrow x_1 = 0 \quad \text{since } a_{g(j;1)h(j;1)} \neq 0$$

From

$$A(j,j)_{22}\begin{bmatrix} 0 \\ x_2 \\ \vdots \\ x_{n(j)-r_0(j)} \end{bmatrix} = iw\begin{bmatrix} 0 \\ x_2 \\ \vdots \\ x_{n(j)-r_0(j)} \end{bmatrix}$$

we obtain by considering the first component of $A(j,j)_{22}\tilde{x}$ that

$$\alpha(j)_2 x_2 = iw0 = 0$$

Case 1: $\alpha(j)_2 \neq 0$. This implies $x_2 = 0$.

Case 2: $\alpha(j)_2 = 0$. Then $(g(j;2), h(j;2)) = (g(j;2), 2)$ and by assumption $a_{g(j;2)h(j;2)} \neq 0$. Thus

$$A(j,j)_{12}\tilde{x} = \begin{bmatrix} \vdots \\ a_{g(j;2)h(j;2)}x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow x_2 = 0$$

Now proceed inductively to show that $x_i = 0$ for $3 \leqslant i \leqslant n(j) - r_0(j)$ by considering

$$A(j,j)_{22}\tilde{x} = iw\tilde{x} \quad \text{if } \alpha(j)_i \neq 0$$

and

$$A(j,j)_{12}\tilde{x} = 0 \quad \text{if } \alpha(j)_i = 0$$

This shows that $\tilde{x} = 0$, and hence $x = 0$.

Thus we have the asymptotic stability of $(A(j,j), B^j, C^j)$. Controllability of $(A(j,j), B^j)$ follows from Lemma 2.6. The same argument shows controllability of $(A^T, (C^j)^T)$ and hence observability of $(C^j, A(j,j))$. Thus we have minimality of $(A(j,j), B^j, C^j)$.

The result is now a consequence of Proposition 4 in Kabamba (1985). □

The above parametrization result is particularly interesting from the point of view that the parameters can be continuously varied within a region, which is given in a straightforward way, without hitting a non-minimal system. This is in contrast to other canonical forms, like the controller canonical form, where the regions of non-minimality are defined by complicated algebraic equations.

8. Model reduction

The next theorem is a generalization of the truncation result given in Theorem 2.5 to the case when a system is given in balanced canonical form.

Theorem 8.1

Let (A, B, C) be parametrized as in Theorem 7.1 with observability gramian $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathbb{R}^{k\times k}$ for $1 \leqslant k \leqslant n$. Let (A, B, C) be partitioned accordingly, i.e.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{k\times k} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad C = (C_1 \quad C_2)$$

Then

(i) $(A_{11}, B_1, C_1) \in C_k^{p,m,b}$

(ii) If Σ_1 and Σ_2 have no eigenvalues in common then also $(A_{22}, B_2, C_2) \in C_{n-k}^{p,m,b}$.

(iii) For $(A, B, C) \in \Gamma_n(C_n^{p,m})$, we have $(A_{11}, B_1, C_1) \in \Gamma_k(C_k^{p,m})$.

Proof

The truncations fulfil the assumptions of Theorem 7.1. □

Via the transformation T_n, two alternative model reduction procedures are introduced.

Proposition 8.2

Let $\tilde{\Gamma}_n = T_n \Gamma_n T_n^{-1} : D_n^{p,m} \to D_n^{p,m,b}$ be the canonical form induced by $\Gamma_n : C_n^{p,m} \to C_n^{p,m,b}$. Let $(A, B, C) \in \tilde{\Gamma}_n(D_n^{p,m})$ have observability gramian $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathbb{R}^{k\times k}$, for $1 \leqslant k \leqslant n$.

Partition $T_n^{-1}((A, B, C)) =: (\tilde{A}, \tilde{B}, \tilde{C})$ according to

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in \mathbb{R}^{k\times k}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{B}_1 \in \mathbb{R}^{k\times m}$$

$$\tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2), \quad \tilde{C}_1 \in \mathbb{R}^{p\times k}$$

Then $(A_{11}, B_1, C_1) := T_k((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)) \in D_k^{p,m,b}$.

Proof

Note that $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma_n(C_n^{p,m})$. By Proposition 8.1 $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1) \in \Gamma_k(C_k^{p,m})$. The result follows from Proposition 4.1. □

Similarly, the truncation property for discrete-time balanced systems leads to another model reduction procedure for continuous-time systems.

Proposition 8.3

Let $(A, B, C) \in C_n^{p,m,b}$ with observability gramian $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, $\Sigma_1 \in \mathbb{R}^{k\times k}$, for $1 \leqslant k < n$, such that $\sigma_{\min}(\Sigma_1) > \sigma_{\max}(\Sigma_2)$. As in Proposition 8.2, partition $T_n((A, B, C)) =: (\tilde{A}, \tilde{B}, \tilde{C})$ according to Σ. Then

$$(A_{11}, B_1, C_1) := T_k^{-1}((\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)) \in C_k^{p,m}$$

Proof

Apply Theorem 2.11 and Propositions 8.1 and 4.1. □

9. Conclusions

Using balanced realizations, a canonical form has been derived for the set of minimal and asymptotically stable MIMO systems of given order n.

A parametrization result showed that systems of a particular form are necessarily asymptotically stable and minimal. This permitted us to show that the canonical form has a certain truncation property.

Two alternative approaches to model reduction have been suggested.

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