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A Parametrization Approach to Infinite-Dimensional Balanced Systems and their Approximation

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The problem of generalizing the concept of balanced realizations to infinite-dimensional continuous-time systems is considered. The approach taken is based on the generalization of a parametrization of finite dimensional balanced realizations. Using semigroup theory an existence theorem and approximation results are derived.

0. Introduction

BALANCED realizations for finite-dimensional linear systems as introduced by B. C. Moore (1981) have attracted a great deal of attention, mainly due to their interesting properties with respect to model reduction. These properties suggest that balanced realizations could be useful in considering the important problem of the analysis and approximation of infinite-dimensional systems.

While Young (1985) and Curtain (1985) developed a realization theory for discrete-time infinite-dimensional systems in terms of balanced realizations, Curtain & Glover (1985) as well as Glover *et al.* (1986) considered the continuous-time problem for classes of systems with compact Hankel operators. In this paper a 'parametrization' approach is taken to the generalization of the concept of balanced realizations to infinite-dimensional continuous-time systems. The way an infinite-dimensional system will be parametrized is a direct generalization of a parametrization of finite-dimensional balanced realizations. A justification of this approach is given by a result of Glover & Curtain (1986) which shows that balanced realizations of a certain class of systems admit such a parametrization. The motivation behind this approach is that it facilitates the investigations of structural properties of the approximation of infinite-dimensional balanced realizations.

The approximation results in Glover *et al.* (1986) are based on a decomposition result for nuclear Hankel operators, whereas, in this paper, it is the Trotter–Kato theorem for the approximation of strongly continuous semigroups of operators which is the basis of the approximation results. An advantage of this approach is that it allows us to consider the approximation of systems whose Hankel operator does not necessarily satisfy a compactness condition. It further allows us to construct infinite-dimensional systems with a prescribed set of singular values (Ober, 1986b).

In Section 1, definitions and results for finite-dimensional balanced realizations are reviewed. The generalization of a Lyapunov equation to the infinite-dimensional setting is investigated in Section 2. Section 3 contains the proof of the central result of this paper, which states that an infinite-dimensional matrix parametrized in a 'balanced' way gives rise to a generator of a strongly continuous semigroup of contractions. The problem of the approximation of infinite-dimensional balanced realizations is considered in Section 4. A characterization result for selfadjoint infinite-dimensional systems is contained in Section 5.

ℓ^2 denotes the complex Hilbert space of square-summable sequences, where the scalar product is defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \bar{x}_i y_i$$

for $x = (x_i)_{i\geq 1}$ and $y = (y_i)_{i\geq 1} \in \ell^2$. We denote by $L^2[0, \infty)$ the space of square-integrable functions on $[0, \infty)$. A strongly continuous semigroup of operators with generator $(A, \mathfrak{D}(A))$ will be denoted by $(e^{tA})_{t\geq 0}$. The restriction of an operator to a set $\mathscr{X}$ is denoted by $A \restriction \mathscr{X}$. For standard techniques in semigroup theory, refer to Pazy (1983); for Hilbert-space arguments, see Weidmann (1980).

1. Finite-dimensional balanced realizations

B. C. Moore (1981) gave the following definition.

DEFINITION 1.1 Let $(A, \boldsymbol{b}, \boldsymbol{c}^T)$ be a continuous-time minimal asymptotically stable single-input single-output system of order n, i.e.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}u, \qquad A \in \mathbb{R}^{n\times n}, \quad \boldsymbol{b} \in \mathbb{R}^n$$
$$y = \boldsymbol{c}^T\boldsymbol{x}, \qquad \boldsymbol{c} \in \mathbb{R}^n.$$

Then $(A, \boldsymbol{b}, \boldsymbol{c}^T)$ is called *balanced* if

$$\int_0^{\infty} e^{tA}\boldsymbol{b}\boldsymbol{b}^T e^{tA^T}\, dt = \int_0^{\infty} e^{tA^T}\boldsymbol{c}\boldsymbol{c}^T e^{tA}\, dt =: \Sigma,$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) > 0$. The positive numbers $\sigma_1, \ldots, \sigma_n$ are called the *singular values of* $(A, \boldsymbol{b}, \boldsymbol{c}^T)$. □

To talk about balanced realizations is justified by the following theorem.

THEOREM 1.2 (Moore, 1981) *Let $(A, \boldsymbol{b}, \boldsymbol{c}^T)$ be a continuous-time minimal asymptotically stable system of order n, then there exists a nonsingular $T \in \mathbb{R}^{n\times n}$ such that $(TAT^{-1}, T\boldsymbol{b}, \boldsymbol{c}^T T^{-1})$ is balanced.* □

An equivalent definition of balance can be given in terms of Lyapunov equations. These are a major tool in working with balanced realizations.

THEOREM 1.3 (Moore, 1981) *Let $(A, \boldsymbol{b}, \boldsymbol{c}^T)$ be given as in Theorem 1.2. Then $(A, \boldsymbol{b}, \boldsymbol{c}^T)$ is balanced if and only if there exists a diagonal matrix $\Sigma > 0$ such that*

$$A\Sigma + \Sigma A^T = -\boldsymbol{b}\boldsymbol{b}^T, \qquad A^T\Sigma + \Sigma A = -\boldsymbol{c}\boldsymbol{c}^T.$$

In this case,

$$\Sigma = \int_0^\infty e^{tA} \boldsymbol{b}\boldsymbol{b}^\mathsf{T} e^{tA^\mathsf{T}}\, dt = \int_0^\infty e^{tA^\mathsf{T}} \boldsymbol{c}\boldsymbol{c}^\mathsf{T} e^{tA}\, dt. \quad \square$$

For the case of distinct singular values, the following characterization can be given. It is a special case of a canonical form in terms of balanced realizations (Ober, 1986a).

THEOREM 1.4 *Let $(A, \boldsymbol{b}, \boldsymbol{c}^\mathsf{T}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$. Then $(A, \boldsymbol{b}, \boldsymbol{c}^\mathsf{T})$ is a minimal balanced realization of an asymptotically stable system with distinct singular values $\sigma_1, \ldots, \sigma_n$ if and only if*

$$\boldsymbol{b} = [b_1, \ldots, b_n]^\mathsf{T}, \qquad b_i \neq 0 \quad \text{for } 1 \leqslant i \leqslant n,$$

$$\boldsymbol{c} = [s_1 b_1, \ldots, s_n b_n]^\mathsf{T}, \qquad s_i = \pm 1 \quad \text{for } 1 \leqslant i \leqslant n,$$

$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1\leqslant i,j\leqslant n}. \quad \square$$

The concept of a dissipative operator is important in semigroup theory (Pazy, 1983).

DEFINITION 1.5 Let $(A, \mathfrak{D}(A))$ be an operator on a complex Hilbert space. A is called *dissipative* if

$$\operatorname{Re}\langle Ax, x\rangle \leqslant 0 \quad \text{for all } x \in \mathfrak{D}(A). \quad \square$$

The following theorem, which will be needed later, shows the dissipativeness of an A-matrix as parametrized above.

THEOREM 1.6 *Let*

$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1\leqslant i,j\leqslant n},$$

where $b_i \in \mathbb{R}$, $s_i = \pm 1$, and $\sigma_i > 0$, for $1 \leqslant i \leqslant n$, and $\sigma_i \neq \sigma_j$ for $i \neq j$. Then A is a dissipative operator on $\mathbb{C}^n$.

Proof. Let $\boldsymbol{x} \in \mathbb{C}^n$. Then $\operatorname{Re}\langle A\boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{2}\langle (A + A^\mathsf{T})\boldsymbol{x}, \boldsymbol{x}\rangle$. We have

$$A\Sigma + \Sigma A^\mathsf{T} = -\boldsymbol{b}\boldsymbol{b}^\mathsf{T}, \qquad A^\mathsf{T}\Sigma + \Sigma A = -\boldsymbol{c}\boldsymbol{c}^\mathsf{T},$$

where $\boldsymbol{c} =: S\boldsymbol{b}$. Thus

$$(A + A^\mathsf{T})\Sigma + \Sigma(A + A^\mathsf{T}) = -(\boldsymbol{b}\boldsymbol{b}^\mathsf{T} + \boldsymbol{c}\boldsymbol{c}^\mathsf{T}).$$

Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of the symmetric matrix $A + A^\mathsf{T}$, with eigenvector $\boldsymbol{x}_0 \neq \boldsymbol{0}$; then

$$\begin{aligned}
&\langle \boldsymbol{x}_0, (A + A^\mathsf{T})\Sigma\boldsymbol{x}_0\rangle + \langle \boldsymbol{x}_0, \Sigma(A + A^\mathsf{T})\boldsymbol{x}_0\rangle \\
&= \langle (A + A^\mathsf{T})\boldsymbol{x}_0, \Sigma\boldsymbol{x}_0\rangle + \langle \boldsymbol{x}_0, \Sigma(A + A^\mathsf{T})\boldsymbol{x}_0\rangle \\
&= 2\lambda_0\langle \boldsymbol{x}_0, \Sigma\boldsymbol{x}_0\rangle \\
&= -\langle \boldsymbol{x}_0, (\boldsymbol{b}\boldsymbol{b}^\mathsf{T} + \boldsymbol{c}\boldsymbol{c}^\mathsf{T})\boldsymbol{x}_0\rangle \leqslant 0 \\
&\qquad\qquad \Rightarrow \lambda_0 \leqslant 0.
\end{aligned}$$

Thus $A + A^\mathsf{T}$ is negative semidefinite, which implies the result. $\square$

2. A Lyapunov equation for infinite-dimensional systems

The characterization of finite-dimensional balanced realizations, as given in Theorem 1.4, is taken as the starting point for the generalization of the concept of balanced realizations to infinite-dimensional systems. This approach is further motivated by a result (Glover & Curtain, 1986) in the connection of the derivation of a balanced realization for a certain class of systems. It follows easily from their result that a balanced realization $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$ of such a system can be parametrized as

$$\boldsymbol{b} = [b_1, b_2, \dots]^{\mathrm{T}}, \qquad b_i \neq 0 \quad \text{for } 1 \leq i < \infty,$$
$$\boldsymbol{c} = [s_1 b_1, s_2 b_2, \dots]^{\mathrm{T}}, \qquad s_i = \pm 1 \quad \text{for } 1 \leq i < \infty,$$

and the generator A has a matrix representation given by

$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1 \leq i,j < \infty}.$$

In what follows, we will thus be concerned with the examination of infinite-dimensional systems that can be associated with the matrix-triple $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$, where

$$\boldsymbol{b} = [b_1, b_2, \dots]^{\mathrm{T}} \in \ell^2, \qquad b_i \in \mathbb{R} \quad \text{for } 1 \leq i < \infty,$$
$$\boldsymbol{c} = [s_1 b_1, s_2 b_2, \dots]^{\mathrm{T}}, \qquad s_i = \pm 1 \quad \text{for } 1 \leq i < \infty,$$
$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1 \leq i,j < \infty},$$

such that $(\sigma_i)_{i \geq 1}$ is a bounded sequence of distinct positive numbers.

Note that we have the formal identity $A^{\mathrm{T}} = SAS$ for $S := \operatorname{diag}(s_1, s_2, \dots)$.

The next two sections will be concerned with showing that the matrix A can be associated with a generator of a strongly continuous semigroup of operators.

We will need the following assumptions on the choice of parameters: the sequences, $(b_i)_{i \geq 1}$, $(s_i)_{i \geq 1}$, and $(\sigma_i)_{i \geq 1}$ are such that

(A) the rows and columns of A are in ℓ^2.
(B) $A\Sigma$ defines a bounded operator on ℓ^2.

In what follows, we will assume (A) and (B).

EXAMPLE 2.1 Conditions (A) and (B) are fulfilled if $(s_i)_{i \geq 1}$ is constant for all $i \in \mathbb{N}$, i.e. $s = \pm 1$.

Proof. Let

$$\boldsymbol{a}_i = -b_i \left(\frac{b_j}{s_i s_j \sigma_i + \sigma_j}\right)_{1 \leq j < \infty}$$

be the ith column of A. Then

$$\|\boldsymbol{a}_i\|_2^2 = b_i^2 \sum_{j=1}^{\infty} \frac{b_j^2}{(\sigma_i + \sigma_j)^2} \leq b_i^2 \sum_{j=1}^{\infty} \frac{b_j^2}{\sigma_i^2} \leq \frac{b_i^2}{\sigma_i^2} \|\boldsymbol{b}\|_2^2 < \infty.$$

Thus the columns of A are in ℓ^2. Similarly, it follows that the rows of A are in ℓ^2.

Let $A\Sigma =: [\tilde{a}_{ij}]_{1\leq i,j<\infty}$. Then $|\tilde{a}_{ij}| = |b_i b_j|/(1+\sigma_i/\sigma_j) \leq |b_i b_j|$. Thus

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \tilde{a}_{ij}^2 \leq \sum_{i=1}^{\infty} b_i^2 \sum_{j=1}^{\infty} b_j^2 < \infty,$$

which implies that $A\Sigma$ is Hilbert–Schmidt (Weidmann, 1980: Thm 6.22) and hence bounded. □

EXAMPLE 2.2 Conditions (A) and (B) are fulfilled if $\sigma_1 > \sigma_2 > \cdots > 0$ and $(s_i)_{i\geq 1}$ is constant for all but finitely many $i \in \mathbb{N}$.

Proof. With the notation as in the previous example,

$$\begin{aligned} \|\boldsymbol{a}_i\|_2^2 &= b_i^2 \sum_{j=1}^{\infty} \frac{b_j^2}{(s_i s_j \sigma_i + \sigma_j)^2} \qquad (i \geq 1) \\ &\leq b_i^2 M \|\boldsymbol{b}\|_2^2 < \infty \qquad \text{(for some } M > 0) \end{aligned}$$

since the assumptions imply that $\liminf_{j\to\infty} (s_i s_j \sigma_i + \sigma_j)^2$ exists. Thus the columns of A are in ℓ^2. Similarly, it follows that the rows of A are in ℓ^2.

Write $A\Sigma =: [\tilde{a}_{ij}]_{1\leq i,j<\infty}$, with

$$\tilde{a}_{ij} = \frac{-b_i b_j \sigma_j}{s_i s_j \sigma_i + \sigma_j} = \frac{-b_i b_j}{s_i s_j \sigma_i/\sigma_j + 1} \qquad (1 \leq i,j < \infty).$$

Then

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \tilde{a}_{ij}^2 < \infty$$

since $\inf\{(s_i s_j(\sigma_i/\sigma_j) + 1)^2 : 1 \leq i,j < \infty\} > 0$ and $\boldsymbol{b} = [b_1, b_2, \ldots]^{\mathrm{T}} \in \ell^2$. Applying Theorem 6.22 of Weidmann (1980), we obtain the result. □

We are now going to define several operators in connection with the matrix $A := [a_{ij}]_{1\leq i,j<\infty}$.

(i) Let

$$A_{\min} : \mathfrak{D}_{\min} \to \ell^2 : \boldsymbol{x} \mapsto \left(\sum_{j=1}^{\infty} a_{ij} x_j\right)_{1\leq i<\infty},$$

with $\mathfrak{D}_{\min} = \mathfrak{D}(A_{\min}) = \operatorname{span}\{\boldsymbol{e}_i : 1 \leq i < \infty\}$, where $\boldsymbol{e}_i$ is the ith element of the standard basis of ℓ^2.

(ii) $A_{\max}$ is defined to be such that $\mathfrak{D}(A_{\max})$ is the maximal possible domain of definition for the A-matrix, i.e.

$$A_{\max} : \mathfrak{D}(A_{\max}) \to \ell^2 : \boldsymbol{x} \mapsto \left(\sum_{j=1}^{\infty} a_{ij} x_j\right)_{1\leq i<\infty},$$

with

$$\mathfrak{D}(A_{\max}) = \left\{\boldsymbol{x} = (x_i)_{1\leq i<\infty} \in \ell^2 : \left(\sum_{j=1}^{\infty} a_{ij} x_j\right)_{1\leq i<\infty} \in \ell^2\right\}.$$

Note that, if $x = (x_j)_{1 \le j < \infty} \in \ell^2$, then $(\sum_{j=1}^{\infty} a_{ij} x_j)_{1 \le i < \infty}$ exists, since the rows of A are in ℓ^2.

(iii) $A_{\min}^{\mathsf{T}}$ and $A_{\max}^{\mathsf{T}}$ are defined analogously for A^{T}, the transpose of A.

We shall denote the adjoint and closure of the operator $A_{\min}$ (resp. $A_{\min}^{\mathsf{T}}$) by $A_{\min}^*$ and $A_{\min}^{\mathsf{C}}$ (resp. $A_{\min}^{\mathsf{T}*}$ and $A_{\min}^{\mathsf{TC}}$), and similarly for $A_{\max}$ and $A_{\max}^{\mathsf{T}}$.

In what follows, it will be important to know whether $A_{\min}$ and $A_{\max}$ are closable and in what way they relate to $A_{\min}^{\mathsf{T}}$ and $A_{\max}^{\mathsf{T}}$ and to each other. Applying standard techniques of matrix operators (Weidmann, 1980) to our case, we obtain the following proposition.

Proposition 2.3 $A_{\min}^* = A_{\max}^{\mathsf{T}}$ *and* $A_{\max} = A_{\min}^{\mathsf{T}*}$.

Proof. $A_{\min}$ and $A_{\max}^{\mathsf{T}}$ are formal adjoints; i.e. we have

$$\langle A_{\min} x, y \rangle = \langle x, A_{max}^{\mathsf{T}} y \rangle \quad \text{for} \quad x \in \mathfrak{D}(A_{\min}) \quad \text{and} \quad y \in \mathfrak{D}(A_{\max}^{\mathsf{T}}).$$

Thus $A_{\max}^{\mathsf{T}} \subseteq A_{\min}^*$.

To show $\mathfrak{D}(A_{\min}^*) \subseteq \mathfrak{D}(A_{\max}^{\mathsf{T}})$, let $x \in \mathfrak{D}(A_{\min}^*)$. Since $e_i \in \mathfrak{D}(A_{\min})$ for $1 \le i < \infty$, we have

$$\begin{aligned} \langle e_i, A_{\min}^* x \rangle &= \langle A_{\min} e_i, x \rangle \\ &= \left\langle \sum_{j=1}^{\infty} a_{ji} e_j, x \right\rangle \\ &= \sum_{j=1}^{\infty} a_{ji} \langle e_j, x \rangle \quad \text{for all } i \in \mathbb{N}. \end{aligned}$$

Thus

$$\|A_{\max}^{\mathsf{T}} x\|^2 = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ji} \langle e_j, x \rangle \right)^2 = \|A_{\min}^* x\|^2 < \infty.$$

Hence $x \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$. So $A_{\max}^{\mathsf{T}} = A_{\min}^*$. The statement $A_{\max} = A_{\min}^{\mathsf{T}*}$ follows similarly. □

Thus $A_{\max}$ and $A_{\max}^{\mathsf{T}}$ are closed, which implies that $A_{\min}$ and $A_{\min}^{\mathsf{T}}$ are closable with closures $A_{\min}^{\mathsf{C}} \subseteq A_{\max}$ and $A_{\min}^{\mathsf{TC}} \subseteq A_{\max}^{\mathsf{T}}$.

Next, two possible generalizations of the Lyapunov equation for finite-dimensional balanced realizations

$$A\Sigma + \Sigma A^{\mathsf{T}} = -bb^{\mathsf{T}}$$

are considered:

(i) $\langle x, A_{\max} \Sigma y \rangle + \langle x, \Sigma A_{\max}^{\mathsf{T}} y \rangle = -\langle x, bb^{\mathsf{T}} y \rangle$ for all $x, y \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$,

(ii) $\langle \Sigma A_{\max}^{\mathsf{T}} x, y \rangle + \langle x, \Sigma A_{\max}^{\mathsf{T}} y \rangle = -\langle x, bb^{\mathsf{T}} y \rangle$ for all $x, y \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$.

Note that these two equations are not automatically equivalent. The next proposition shows that the identity (i) holds.

Proposition 2.4

(1) If $x \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$, then $\Sigma x \in \mathfrak{D}(A_{\max})$.

(2) For all $x \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$, we have $(A_{\max}\Sigma + \Sigma A_{\max}^{\mathsf{T}})x = -bb^{\mathsf{T}} x$, where Σ and bb^{T} are both interpreted as matrix operators on ℓ^2.

Proof. Since the rows and columns of the matrix A, as defined above, are in ℓ^2, the matrices A and A^{T} can be considered as operators

$$A:\ell^2\to\mathbb{C}^{\mathbb{N}}:\boldsymbol{x}\mapsto\left(\sum_{j=1}^{\infty}a_{ij}x_j\right)_{1\leq i<\infty},\qquad A^{\mathsf{T}}:\ell^2\to\mathbb{C}^{\mathbb{N}}:\boldsymbol{x}\mapsto\left(\sum_{j=1}^{\infty}a_{ji}x_j\right)_{1\leq i<\infty}.$$

Consider

$$\Sigma:\mathbb{C}^{\mathbb{N}}\to\mathbb{C}^{\mathbb{N}}:\boldsymbol{x}\mapsto(\sigma_1x_1,\sigma_2x_2,\ldots).$$

Since $\sup\{\sigma_i:1\leq i<\infty\}<\infty$, we have that $\Sigma\restriction\ell^2$ is a bounded operator on ℓ^2. Thus the map

$$A\Sigma+\Sigma A^{\mathsf{T}}:\ell^2\to\mathbb{C}^{\mathbb{N}}:\boldsymbol{x}\mapsto(A\Sigma+\Sigma A^{\mathsf{T}})\boldsymbol{x}=A\Sigma\boldsymbol{x}+\Sigma A^{\mathsf{T}}\boldsymbol{x}$$

is well defined.

For $\boldsymbol{x}\in\ell^2$, the ith component of $(A\Sigma+\Sigma A^{\mathsf{T}})\boldsymbol{x}$ is given by

$$\begin{aligned}\sum_{j=1}^{\infty}a_{ij}\sigma_jx_j+\sigma_i\sum_{j=1}^{\infty}a_{ji}x_j&=\sum_{j=1}^{\infty}a_{ij}(s_is_j\sigma_i+\sigma_j)x_j\qquad(\text{since } a_{ij}=s_is_ja_{ji})\\&=-\sum_{j=1}^{\infty}b_ib_jx_j=-\langle\boldsymbol{e}_i,\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}\boldsymbol{x}\rangle.\end{aligned}$$

Thus we have that $A\Sigma+\Sigma A^{\mathsf{T}}=-\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}$, as an operator, mapping ℓ^2 into $\mathbb{C}^{\mathbb{N}}$.

Let $\boldsymbol{x}\in\mathfrak{D}(A^{\mathsf{T}}_{\max})$. Since $\Sigma:\ell^2\to\ell^2$, we have $\Sigma A^{\mathsf{T}}_{\max}\boldsymbol{x}\in\ell^2$. From $\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}\boldsymbol{x}\in\ell^2$ it follows that

$$A\Sigma\boldsymbol{x}=-\Sigma A^{\mathsf{T}}\boldsymbol{x}-\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}\boldsymbol{x}\in\ell^2,$$

i.e. $\Sigma\boldsymbol{x}\in\mathfrak{D}(A_{\max})$, which proves the proposition. □

Whereas the previous proposition holds even without imposing condition (B), we need this condition to prove the next result.

PROPOSITION 2.5

$$\langle\Sigma A^{\mathsf{T}}_{\max}\boldsymbol{x},\boldsymbol{y}\rangle+\langle\Sigma\boldsymbol{x},A^{\mathsf{T}}_{\max}\boldsymbol{y}\rangle=-\langle\boldsymbol{x},\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}\rangle$$

for all $\boldsymbol{x},\boldsymbol{y}\in\mathfrak{D}(A^{\mathsf{T}}_{\max})$.

Proof. By Proposition 2.4, we know that

$$\langle\boldsymbol{x},A_{\max}\Sigma\boldsymbol{y}\rangle+\langle\boldsymbol{x},\Sigma A^{\mathsf{T}}_{\max}\boldsymbol{y}\rangle=-\langle\boldsymbol{x},\boldsymbol{b}\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y}\rangle\quad\text{for all}\quad\boldsymbol{x},\boldsymbol{y}\in\mathfrak{D}(A^{\mathsf{T}}_{\max}).$$

It remains to be shown that

$$\langle\boldsymbol{x},A_{\max}\Sigma\boldsymbol{y}\rangle=\langle\Sigma A^{\mathsf{T}}_{\max}\boldsymbol{x},\boldsymbol{y}\rangle\quad\text{for all}\quad\boldsymbol{x},\boldsymbol{y}\in\mathfrak{D}(A^{\mathsf{T}}_{\max}).$$

Let P_n be the projection

$$P_n:[x_1,x_2,\ldots,x_n,x_{n+1},\ldots]^{\mathsf{T}}\mapsto[x_1,\ldots,x_n,0,\ldots]^{\mathsf{T}};$$

then

$$\langle\boldsymbol{x},A_{\max}\Sigma\boldsymbol{y}\rangle=\lim_{n\to\infty}\langle\boldsymbol{x},A_{\max}\Sigma P_n\boldsymbol{y}\rangle$$

since we have, by assumption (B), that $(A\Sigma \restriction \mathfrak{D}(A_{\max}^{\mathsf{T}}) = (A_{\max}\Sigma) \restriction (\mathfrak{D}(A_{\max}^{\mathsf{T}})$ is bounded. Thus

$$\langle \boldsymbol{x}, A_{\max}\Sigma \boldsymbol{y}\rangle = \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij}\sigma_j y_j \bar{x}_i = \lim_{n\to\infty} \langle \Sigma A_{\max}^{\mathsf{T}} \boldsymbol{x}, P_n \boldsymbol{y}\rangle$$
$$= \langle \Sigma A_{\max}^{\mathsf{T}} \boldsymbol{x}, \boldsymbol{y}\rangle. \quad \square$$

3. The matrix A as the generator of a semigroup

In this section, the question will be considered whether the infinite-dimensional A-matrix defined in Section 2 gives rise to a strongly continuous semigroup of operators.

The following two propositions are the cornerstones of the main theorem.

PROPOSITION 3.1 *$A_{\min}^{\mathsf{C}}$ is dissipative.*

Proof. We first show that $A_{\min}$ is dissipative. Let

$$\tilde{A}_n = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1\le i,j\le n} \quad \text{for } n \in \mathbb{N}.$$

By Theorem 1.6, we know that $\tilde{A}_n$ is dissipative. For $\boldsymbol{x} = (x_i)_{1\le i<\infty} \in \mathfrak{D}_{\min}$, we have

$$\operatorname{Re}\langle A_{\min}\boldsymbol{x}, \boldsymbol{x}\rangle = \operatorname{Re}\langle \tilde{A}_n \tilde{\boldsymbol{x}}_n, \tilde{\boldsymbol{x}}_n\rangle \le 0$$

for n sufficiently large, where $\tilde{\boldsymbol{x}}_n = [x_1, \ldots, x_n]^{\mathsf{T}}$.

The fact that $A_{\min}^{\mathsf{C}}$ is dissipative now follows from Thm 4.5, p. 15, of Pazy (1983) $\square$

The next proposition strongly relies on the results of the previous section.

PROPOSITION 3.2

$$(I - A_{\min}^{\mathsf{C}})\mathfrak{D}(A_{\min}^{\mathsf{C}}) = \ell^2.$$

Proof. We show that $(I - A_{\min}^{\mathsf{C}})\mathfrak{D}_{\min} = (I - A_{\min})\mathfrak{D}_{\min}$ is dense in ℓ^2. The statement then follows from Lemma 2.10, p. 52, of Nagel (1986).

Assume $[(I - A_{\min})\mathfrak{D}_{\min}]^{\mathsf{C}} \ne \ell^2$; then there exists $\boldsymbol{x}_0 = (x_i)_{1\le i<\infty} \in \ell^2$ such that $\boldsymbol{x}_0 \ne \boldsymbol{0}$ and

$$\langle \boldsymbol{y}, \boldsymbol{x}_0\rangle = 0 \quad \text{for all } \boldsymbol{y} \in (I - A_{\min})\mathfrak{D}_{\min}.$$

So, especially, $\langle (I - A_{\min})\boldsymbol{e}_n, \boldsymbol{x}_0\rangle = 0$ for all $n \in \mathbb{N}$. But

$$\langle (I - A_{\min})\boldsymbol{e}_n, \boldsymbol{x}_0\rangle = \langle \boldsymbol{e}_n, \boldsymbol{x}_0\rangle - \langle A_{\min}\boldsymbol{e}_n, \boldsymbol{x}_0\rangle$$
$$= x_n - \sum_{j=1}^{\infty} a_{jn} x_j = 0$$
$$\Rightarrow \quad x_n = \sum_{j=1}^{\infty} a_{jn} x_j \quad \text{for all } n \in \mathbb{N}.$$

Thus $\boldsymbol{x}_0 \in \mathfrak{D}(A_{\max}^{\mathsf{T}})$ and $A_{\max}^{\mathsf{T}} \boldsymbol{x}_0 = \boldsymbol{x}_0$.

From Proposition 2.5, it follows that

$$\langle \Sigma A_{\max}^{\mathsf{T}} \boldsymbol{x}_0, \boldsymbol{x}_0 \rangle + \langle \Sigma \boldsymbol{x}_0, A_{\max}^{\mathsf{T}} \boldsymbol{x}_0 \rangle = 2\langle \Sigma \boldsymbol{x}_0, \boldsymbol{x}_0 \rangle = -\langle \boldsymbol{x}_0, \boldsymbol{b}\boldsymbol{b}^{\mathsf{T}} \boldsymbol{x}_0 \rangle,$$

which is a contradiction to $\boldsymbol{x}_0 \neq \mathbf{0}$. □

We are now in a position to state the main theorem.

THEOREM 3.3 *$A_{\min}^{\mathsf{C}}$ generates a semigroup of contractions.*

Proof. The result follows from Proposition 3.1 and Proposition 3.2 by applying the Lumer–Phillips criterion; see Theorem 4.3, p. 14, of Pazy (1983). □

Remark. Theorem 3.3 implies that $A_{\min}^{\mathsf{C}}$ is the generator of the semigroup of the balanced realization derived in Curtain & Glover (1985), provided that $\boldsymbol{b} \in \ell^2$ and the parameters $(b_i)_{i\geq 1}$, $(\sigma_i)_{i\geq 1}$, and $(s_i)_{i\geq 1}$ satisfy the above conditions (A) and (B). □

The previous result allows us to clarify the relationship between $A_{\min}^{\mathsf{C}}$, $A_{\max}$, $A_{\min}^{\mathsf{TC}}$, and $A_{\max}^{\mathsf{T}}$.

PROPOSITION 3.4
(i) $A_{\min}^{\mathsf{C}} = A_{\max}$ *and* $A_{\min}^{\mathsf{TC}} = A_{\max}^{\mathsf{T}}$.
(ii) $A_{\max} = A_{\max}^{\mathsf{T}*}$.

In the special case where $S = \pm I$, we have $A_{\max}^{*} = A_{\max}$.

Proof. (i) We know that $A_{\min}^{\mathsf{C}} \subseteq A_{\max}$. To show that $A_{\max} \subseteq A_{\min}^{\mathsf{C}}$, let

$$\boldsymbol{y} = A_{\max}\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathfrak{D}(A_{\max}).$$

Since $(I - A_{\min}^{\mathsf{C}})\mathfrak{D}_{\min}$ is dense in ℓ^2, there exists a sequence $(\boldsymbol{x}_n)_{1\leq n<\infty}$ in $\mathfrak{D}_{\min}$ such that

$$\boldsymbol{z}_n = (I - A_{\min}^{\mathsf{C}})\boldsymbol{x}_n \to \boldsymbol{x} - \boldsymbol{y} = (I - A_{\min}^{\mathsf{C}})\boldsymbol{x} \quad (n \to \infty).$$

Since $A_{\min}^{\mathsf{C}}$ generates a semigroup of contractions, 1 is not a spectral value of A, i.e. $(I - A_{\min}^{\mathsf{C}})^{-1}$ exists and is bounded. Thus,

$$\boldsymbol{x}_n = (I - A_{\min}^{\mathsf{C}})^{-1}\boldsymbol{z}_n \to (I - A_{\min}^{\mathsf{C}})^{-1}(I - A_{\min}^{\mathsf{C}})\boldsymbol{x} = \boldsymbol{x} \quad (n \to \infty),$$

$$A_{\min}^{\mathsf{C}}\boldsymbol{x}_n = \boldsymbol{x}_n - \boldsymbol{z}_n \to \boldsymbol{x} - (\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{y} \quad (n \to \infty).$$

So $\boldsymbol{y} = A_{\min}^{\mathsf{C}}\boldsymbol{x}$; hence $A_{\max} \subseteq A_{\min}^{\mathsf{C}}$. The identity $A_{\min}^{\mathsf{TC}} = A_{\max}^{\mathsf{T}}$ follows similarly.

(ii) From (i) we have that $A_{\min}^{\mathsf{TC}} = A_{\max}^{\mathsf{T}}$. But

$$A_{\max} = A_{\min}^{\mathsf{T}*} = A_{\min}^{\mathsf{TC}*} = A_{\max}^{\mathsf{T}*}. \quad \square$$

Remark. In view of this result, we shall write $A := A_{\max} = A_{\min}^{\mathsf{C}}$ and, similarly, $A^{\mathsf{T}} := A_{\max}^{\mathsf{T}} = A_{\min}^{\mathsf{TC}}$.

For the case $S = \pm I$, i.e. when the A-matrix is symmetric, we have the following stability property.

THEOREM 3.5 *Assume $(s_i)_{1\leq i<\infty}$ is constant and $b_i \neq 0$ for $1 \leq i < \infty$. Then*

$$e^{tA}x \to 0 \text{ as } t \to \infty \quad \text{for } x \in \ell^2.$$

Before we can prove this theorem, we need the following two lemmas.

LEMMA 3.6 *For all $s \geq 0$ and $x, y \in \mathfrak{D}(A^{\mathsf{T}})$, we have*

$$\int_0^s \langle e^{tA^{\mathsf{T}}}x, b\rangle\langle b, e^{tA^{\mathsf{T}}}y\rangle \, dt = -\left[\langle \Sigma e^{tA^{\mathsf{T}}}x, e^{tA^{\mathsf{T}}}y\rangle\right]_0^s,$$

where $(e^{tA^{\mathsf{T}}})_{t\geq 0}$ is the adjoint semigroup of $(e^{tA})_{t\geq 0}$, which is generated by A^{T}.

Proof. We have

$$\begin{aligned}\frac{d}{dt}\langle \Sigma e^{tA^{\mathsf{T}}}x, e^{tA^{\mathsf{T}}}y\rangle &= \langle \Sigma A^{\mathsf{T}} e^{tA^{\mathsf{T}}}x, e^{tA^{\mathsf{T}}}y\rangle + \langle \Sigma e^{tA^{\mathsf{T}}}x, A^{\mathsf{T}} e^{tA^{\mathsf{T}}}y\rangle \\ &= -\langle \Sigma e^{tA^{\mathsf{T}}}x, bb^{\mathsf{T}} e^{tA^{\mathsf{T}}}y\rangle = -\langle \Sigma e^{tA^{\mathsf{T}}}x, b\rangle\langle b, e^{tA^{\mathsf{T}}}y\rangle\end{aligned}$$

for $x, y \in \mathfrak{D}(A^{\mathsf{T}})$. The last equality holds by Proposition 2.5. Note that $x \in \mathfrak{D}(A^{\mathsf{T}})$ implies that $e^{tA^{\mathsf{T}}}x \in \mathfrak{D}(A^{\mathsf{T}})$ for $0 \leq t < \infty$. The result follows by integration. □

LEMMA 3.7 *Assume $S = \pm I$. Then 0 is not an eigenvalue of A if $b_i \neq 0$ for all $i \in \mathbb{N}$.*

Proof. Assume there exists $x_0 \in \mathfrak{D}(A)$, with $x_0 \neq 0$, such that $Ax_0 = 0$. By Theorem 2.4(d), p. 5, of Pazy (1983), we have

$$\int_0^t e^{sA}Ax_0 \, ds = e^{tA}x_0 - x_0,$$

which implies that $e^{tA}x_0 = x_0$ for all $t \in [0, \infty)$. By Lemma 3.6, we obtain

$$\begin{aligned}\int_0^s |\langle b, e^{tA}x_0\rangle|^2 \, dt = \int_0^s |\langle b, x_0\rangle|^2 \, dt &= -\left[\langle \Sigma e^{tA}x_0, e^{tA}x_0\rangle\right]_0^s \\ &= -\left[\langle \Sigma x_0, x_0\rangle\right]_0^s \quad (\text{for all } s \geq 0) \\ &= 0 \\ &\Rightarrow \langle b, x_0\rangle = 0.\end{aligned}$$

Applying the Lyapunov equation, it follows that

$$A\Sigma x_0 + \Sigma A x_0 = A\Sigma x_0 = -bb^{\mathsf{T}}x_0 = 0.$$

Thus the closed nullspace of A is invariant under Σ. But the closed invariant subspaces of Σ are of the form

$$\text{span}^{\mathsf{C}}\{e_i : i \in \mathcal{I}\} \quad \text{for some index set } \mathcal{I} \subseteq \mathbb{N}.$$

Hence $Ae_i = 0$ for some $i \in \mathcal{I}$. So, $\langle b, e_i\rangle = b_i = 0$ for some $i \in \mathcal{I}$, which is a contradiction to the assumptions on b. □

Proof of Theorem 3.5. Since A is selfadjoint, the result follows from Thm 22.3.2 of Hille & Phillips (1957). □

4. Finite-dimensional approximations

In this section, finite-dimensional approximations of an infinite-dimensional balanced system are considered.

Let $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$ be the balanced system constructed in the previous section, i.e.

$$A = \left[\frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j}\right]_{1 \leq i,j < \infty}, \qquad \boldsymbol{b} = [b_1, b_2, \ldots]^{\mathrm{T}} \in \ell^2, \quad b_i \in \mathbb{R},$$

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots), \qquad S = \operatorname{diag}(s_1, s_2, \ldots), \quad s_i = \pm 1,$$

$$\boldsymbol{c}^{\mathrm{T}} = \boldsymbol{b}^{\mathrm{T}} S,$$

such that conditions (A) and (B) are satisfied.

For $n \in \mathbb{N}$, let $\tilde{A}_n$ be parametrized in the same way by

$$\tilde{\boldsymbol{b}}_n := [b_1, \ldots, b_n, 0, 0, \ldots]^{\mathrm{T}}, \quad \tilde{\Sigma}_n := \Sigma, \quad \tilde{S} := S.$$

Since $\tilde{A}_n$ $(n \in \mathbb{N})$ has only finitely many nonzero entries, $\tilde{A}_n$ defines a bounded operator on l^2. Because the parameters of $\tilde{A}_n$ satisfy the conditions of Theorem 3.3, $\tilde{A}_n$ $(n \in \mathbb{N})$ generates a semigroup of contractions.

Note that $(\tilde{A}_n, \tilde{\boldsymbol{b}}_n, \tilde{\boldsymbol{c}}_n^{\mathrm{T}}) := (\tilde{A}_n, \tilde{\boldsymbol{b}}_n, \tilde{\boldsymbol{b}}_n^{\mathrm{T}} S)$ can be interpreted as an n-dimensional system. If moreover $b_i \neq 0$, for $1 \leq i \leq n$, then Theorem 1.4 shows that $(\tilde{A}_n, \tilde{\boldsymbol{b}}_n, \tilde{\boldsymbol{c}}_n^{\mathrm{T}})$ can be considered as an n-dimensional asymptotically stable minimal system.

The following theorem of the Trotter–Kato type is the key result in this section.

THEOREM 4.1 *Let the operator* A $(= A^{\mathrm{C}}_{\min})$ *be given as in Theorem* 3.3. *Then, for* $(\tilde{A}_n)_{n \geq 1}$ *defined as above, we have*

$$\mathrm{e}^{t\tilde{A}_n}\boldsymbol{x} \rightarrow \mathrm{e}^{tA}\boldsymbol{x} \quad (n \rightarrow \infty, t \geq 0, \boldsymbol{x} \in \ell^2).$$

The limit is uniform in t for t in bounded intervals.

Proof. Since $\lim_{n\to\infty} \tilde{A}_n \boldsymbol{e}_i = A\boldsymbol{e}_i$ for all $i \in \mathbb{N}$, we have that

$$\lim_{n \to \infty} \tilde{A}_n \boldsymbol{x} = A\boldsymbol{x}$$

for all $\boldsymbol{x} \in \mathfrak{D}_{\min}$. In the proof of Proposition 3.2, we have shown that $(I - A)\mathfrak{D}_{\min} = (I - A_{\min})\mathfrak{D}_{\min}$ is dense in ℓ^2. Since $\tilde{A}_n$ generates a semigroup of contractions for all $n \in \mathbb{N}$, we have the result by applying Thm 4.5, p. 88, of Pazy (1983). □

For a system $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$, the observability operator θ—if it exists—is defined by

$$\theta : \ell^2 \rightarrow \mathrm{L}^2[0, \infty) : \boldsymbol{x} \mapsto (t \mapsto \langle \boldsymbol{c}, \mathrm{e}^{tA}\boldsymbol{x} \rangle).$$

The reachability operator R—if it exists—is given as

$$R : L^2\,[0, \infty) \rightarrow \ell^2 : u \mapsto \int_0^\infty \mathrm{e}^{tA}\boldsymbol{b}u(t)\,\mathrm{d}t.$$

In order to relate the observability operator to the reachability operator, we need the following Lemma.

LEMMA 4.2 *Let A be given as in Theorem* 3.3. *Then*

$$SAS = A^{\mathrm{T}}, \qquad Se^{tA}S = e^{tA\mathrm{T}}.$$

Proof. We have $A \upharpoonright \mathfrak{D}_{\min} = S(A^{\mathrm{T}} \upharpoonright \mathfrak{D}_{\min})S = S(A^{\mathrm{TC}} \upharpoonright \mathfrak{D}_{\min})S$. Thus $A = (A \upharpoonright \mathfrak{D}_{\min})^{\mathrm{C}} = [S(A^{\mathrm{T}} \upharpoonright \mathfrak{D}_{\min})S]^{\mathrm{C}} = S(A^{\mathrm{T}} \upharpoonright \mathfrak{D}_{\min})^{\mathrm{C}} S = SA^{\mathrm{T}}S$. The second statement follows from the remarks on similar semigroups of Nagel (1986), p. 13. □

THEOREM 4.3 *Let $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$ be an infinite-dimensional balanced system, with A as in Theorem* 3.3. *Then*:
(i) *θ and R are bounded.*
(ii) *$R = S\theta^*$.*
(iii) *If $(\theta_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ are the observability and reachability operators of the approximating systems $(\tilde{A}_n, \tilde{\boldsymbol{b}}_n, \tilde{\boldsymbol{c}}_n^{\mathrm{T}})_{n\geq 1}$, then*

$$\theta_n \to \theta \quad \textit{weakly}, \qquad R_n \to R \quad \textit{weakly}.$$

(iv) *If we have, moreover,*

$$\int_0^\infty e^{tA}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}e^{tA^{\mathrm{T}}}\,dt = \int_0^\infty e^{tA^{\mathrm{T}}}\boldsymbol{c}\boldsymbol{c}^{\mathrm{T}}e^{tA}\,dt = \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots)$$

then

$$\theta_n \to \theta \quad \textit{strongly}, \qquad R_n \to R \quad \textit{strongly}.$$

Proof. Let $\boldsymbol{x} \in \ell^2$; then

$$\begin{aligned}\int_0^\infty |\theta_n(\boldsymbol{x})(t)|^2\,dt &= \int_0^\infty |\langle \tilde{\boldsymbol{c}}_n, e^{t\tilde{A}_n}\boldsymbol{x}\rangle|^2\,dt \\ &= \left\langle \boldsymbol{x}, \left(\int_0^\infty e^{t\tilde{A}_n^{\mathrm{T}}}\tilde{\boldsymbol{c}}_n\tilde{\boldsymbol{c}}_n^{\mathrm{T}}e^{t\tilde{A}_n}\,dt\right)x\right\rangle \\ &= \langle \boldsymbol{x}, \Sigma_n\boldsymbol{x}\rangle \qquad (\text{where } \Sigma_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_n, 0, 0, \ldots)) \\ &\leq \langle \boldsymbol{x}, \Sigma\boldsymbol{x}\rangle\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that the θ_n are uniformly bounded.

For $0 \leq a < b < \infty$, Theorem 4.1 gives

$$\begin{aligned}\int_0^\infty \theta_n(x)(t)1_{[a,b]}(t)\,dt &= \int_0^\infty \langle \tilde{\boldsymbol{c}}_n, e^{t\tilde{A}_n}\boldsymbol{x}\rangle 1_{[a,b]}(t)\,dt \\ &\to \int_0^\infty \langle \boldsymbol{c}, e^{tA}\boldsymbol{x}\rangle 1_{[a,b]}(t)\,dt \\ &= \int_0^\infty \theta(\boldsymbol{x})(t)1_{[a,b]}(t)\,dt\end{aligned}$$

where

$$1_{[a,b]}(t) = \begin{cases} 1 & \text{if } t \in [a, b], \\ 0 & \text{otherwise.}\end{cases}$$

Since the sequence $(\theta_n)_{n\geq 1}$ is uniformly bounded, this implies that θ is bounded and that

$$\theta_n \to \theta \quad \text{weakly.}$$

Thus

$$\theta_n^* \to \theta^* \quad \text{weakly.}$$

To show the corresponding results for the reachability operators we prove (ii).
The adjoint of the observability operator is given by

$$\theta^* : \mathrm{L}^2[0,\infty) \to \ell^2 : u \mapsto \int_0^\infty \mathrm{e}^{tA^\mathrm{T}} \boldsymbol{c} u(t)\, \mathrm{d}t$$

But

$$\theta^*(u) = \int_0^\infty \mathrm{e}^{tA^\mathrm{T}} \boldsymbol{c} u(t)\, \mathrm{d}t = S \int_0^\infty \mathrm{e}^{tA} S S \boldsymbol{b} u(t)\, \mathrm{d}t = S \int_0^\infty \mathrm{e}^{tA} \boldsymbol{b} u(t)\, \mathrm{d}t$$
$$= SR(u).$$

Thus we have the boundedness of R, and that

$$R_n = S\theta_n^* \to R = S\theta^* \quad \text{weakly.}$$

To show that $\theta_n \to \theta$ strongly, it is sufficient to show that, for $\boldsymbol{x} \in \ell^2$, we have $\limsup_{n\to\infty} \|\theta_n(\boldsymbol{x})\| \leq \|\theta(\boldsymbol{x})\|$. Now

$$\|\theta(\boldsymbol{x})\|_2^2 = \int_0^\infty |\langle \boldsymbol{c}, \mathrm{e}^{tA}\boldsymbol{x}\rangle|^2\, \mathrm{d}t = \left\langle \boldsymbol{x}, \left(\int_0^\infty \mathrm{e}^{tA^\mathrm{T}} \boldsymbol{c}\boldsymbol{c}^\mathrm{T} \mathrm{e}^{tA}\, \mathrm{d}t\right)\boldsymbol{x}\right\rangle$$
$$= \langle \boldsymbol{x}, \Sigma \boldsymbol{x}\rangle$$
$$\geq \|\theta_n(\boldsymbol{x})\|_2^2 \quad \text{for all } n \in \mathbb{N}.$$

Since $(\|\theta_n\|)_{n\geq 1}$ is bounded, we have that $(\|R_n\|)_{n\geq 1} = (\|S\theta_n^*\|)_{n\geq 1}$ is bounded. Thus it is sufficient to show that

$$R_n(1_{[a,b]}) \to R(1_{[a,b]}) \quad (0 \leq a < b < \infty).$$

But

$$R_n(1_{[a,b]}) = \int_0^\infty \mathrm{e}^{t\tilde{A}_n} \boldsymbol{b}_n 1_{[a,b]}(t)\, \mathrm{d}t \to \int_0^\infty \mathrm{e}^{tA} \boldsymbol{b} 1_{[a,b]}(t)\, \mathrm{d}t,$$

by Theorem 4.1. □

COROLLARY 4.4 *Let* $(A, \boldsymbol{b}, \boldsymbol{c}^\mathrm{T})$ *be given as in Theorem* 4.3. *Then, for the impulse responses* $h_n(t) = \langle \tilde{\boldsymbol{c}}_n, \mathrm{e}^{t\tilde{A}_n} \tilde{\boldsymbol{b}}_n\rangle$ *of the approximating systems, we have*

$$h_n(t) \to h(t) = \langle \boldsymbol{c}, \mathrm{e}^{tA}\boldsymbol{b}\rangle \quad \textit{weakly in } \mathrm{L}^2[0,\infty).$$

If moreover

$$\int_0^\infty \mathrm{e}^{tA} \boldsymbol{b}\boldsymbol{b}^\mathrm{T} \mathrm{e}^{tA^\mathrm{T}}\, \mathrm{d}t = \int_0^\infty \mathrm{e}^{tA^\mathrm{T}} \boldsymbol{c}\boldsymbol{c}^\mathrm{T} \mathrm{e}^{tA}\, \mathrm{d}t = \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots) > 0,$$

then

$$h_n(t) \to h(t) \quad in\ \mathrm{L}^2[0, \infty).$$

Proof. The result follows by noting that $h(t) = \theta(\boldsymbol{b})(t)$ for $t \in [0, \infty)$. □

The Hankel operator of a system is of great importance in model reduction. It is defined as:

$$H : \mathrm{L}^2[0, \infty) \to \mathrm{L}^2[0, \infty) : u \mapsto H(u) = \left(s \mapsto \int_0^\infty h(t+s)u(t)\,\mathrm{d}t\right),$$

where $h(t) = \langle \boldsymbol{c}, \mathrm{e}^{tA}\boldsymbol{b} \rangle$ is the impulse response of $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$. Note that H can be factored as $H = \theta R$.

Corollary 4.5 *Let $(A, \boldsymbol{b}, \boldsymbol{c}^{\mathrm{T}})$ be given as in Theorem* 4.3. *For the sequence $(H_n)_{n \geq 1}$ of the Hankel operators of the approximating systems, we have*

$$H_n \to H \quad weakly.$$

If we have moreover

$$\int_0^\infty \mathrm{e}^{tA}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{tA^{\mathrm{T}}}\boldsymbol{c}\boldsymbol{c}^{\mathrm{T}}\mathrm{e}^{tA}\,\mathrm{d}t = \Sigma = \mathrm{diag}\,(\sigma_1, \sigma_2, \ldots),$$

then

$$H_n \to H \quad strongly. \quad \square$$

5. A Characterization for infinite-dimensional balanced realizations

For the case of symmetric systems, we are going to prove a result analogous to the characterization of finite-dimensional balanced realizations as given in Theorem 1.4. We need the following statements on the solution of the Lyapunov equation.

Proposition 5.1 (i) *If* $\int_0^\infty \mathrm{e}^{tA}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\,\mathrm{d}t$ *exists and is bounded, then it solves the Lyapunov equation*

$$\langle X\boldsymbol{x}, A^{\mathrm{T}}\boldsymbol{y} \rangle + \langle XA^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{y} \rangle = -\langle \boldsymbol{x}, \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \rangle \quad for\ all \quad \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^{\mathrm{T}}),$$

provided $\langle \boldsymbol{b}, \mathrm{e}^{tA^{\mathrm{T}}}\boldsymbol{x} \rangle \to 0$ for $\boldsymbol{x} \in \mathfrak{D}(A^{\mathrm{T}})$.

(ii) *If* $\mathrm{e}^{tA^{\mathrm{T}}}\boldsymbol{x} \to 0$ *as* $t \to \infty$, *where* $\boldsymbol{x} \in \mathfrak{D}(A^{\mathrm{T}}_{\max})$, *then the Lyapunov equation*

$$\langle X\boldsymbol{x}, A^{\mathrm{T}}\boldsymbol{y} \rangle + \langle XA^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{y} \rangle = -\langle x, \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}y \rangle \quad for\ all \quad \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^{\mathrm{T}})$$

has at most one bounded solution.

Proof.

(i)
$$\left\langle\left(\int_0^\infty \mathrm{e}^{tA}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\,\mathrm{d}t\right)A^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{y}\right\rangle + \left\langle\left(\int_0^\infty \mathrm{e}^{tA}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\,\mathrm{d}t\right)\boldsymbol{x}, A^{\mathrm{T}}\boldsymbol{y}\right\rangle$$
$$= \int_0^\infty (\langle \mathrm{e}^{tA^{\mathrm{T}}}A^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}\boldsymbol{y} \rangle + \langle \mathrm{e}^{tA^{\mathrm{T}}}\boldsymbol{x}, \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\mathrm{e}^{tA^{\mathrm{T}}}A^{\mathrm{T}}y \rangle)\,\mathrm{d}t$$

$$= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \boldsymbol{b}\boldsymbol{b}^\mathsf{T} \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle \, \mathrm{d}t$$

$$= \lim_{s\to\infty} \left[\langle \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \boldsymbol{b}\boldsymbol{b}^\mathsf{T} \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle \right]_0^s = -\langle x, \boldsymbol{b}\boldsymbol{b}^\mathsf{T} y \rangle \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^\mathsf{T}).$$

(ii) Assume X_1 and X_2 are bounded and solve the Lyapunov equation. Then

$$\langle X_\Delta A^\mathsf{T} \boldsymbol{x}, \boldsymbol{y} \rangle + \langle X_\Delta \boldsymbol{x}, A^\mathsf{T} \boldsymbol{y} \rangle = 0 \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^\mathsf{T}).$$

where $X_\Delta = X_1 - X_2$. Now consider

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle X_\Delta \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle = \langle X_\Delta A^\mathsf{T} \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle + \langle X_\Delta \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, A^\mathsf{T} \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle$$

$$= 0 \quad \text{for} \quad t \in [0, \infty) \text{ and } \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^\mathsf{T}).$$

Note that $\boldsymbol{x} \in \mathfrak{D}(A^\mathsf{T})$ implies that $\mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x} \in \mathfrak{D}(A^\mathsf{T})$ for $t \in [0, \infty)$. Hence

$$\langle X_\Delta \boldsymbol{x}, \boldsymbol{y} \rangle = \langle X_\Delta \mathrm{e}^{0A^\mathsf{T}} \boldsymbol{x}, \mathrm{e}^{0A^\mathsf{T}} \boldsymbol{y} \rangle$$

$$= \langle X_\Delta \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle \quad \text{for } t \in [0, \infty).$$

By the asymptotic stability of $(\mathrm{e}^{tA^\mathsf{T}})_{t\geq 0}$, however, we have

$$\langle X_\Delta \boldsymbol{x}, \boldsymbol{y} \rangle = \lim_{t\to\infty} \langle X_\Delta \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{x}, \mathrm{e}^{tA^\mathsf{T}} \boldsymbol{y} \rangle = 0 \quad \text{for all } \boldsymbol{x},\boldsymbol{y} \in \mathfrak{D}(A^\mathsf{T}).$$

Since $\mathfrak{D}(A^\mathsf{T})$ is dense in ℓ^2 and X_Δ is bounded, we have $X_\Delta = 0$. □

For the case of a selfadjoint generator, we can now formulate a characterization result similar to the finite-dimensional case.

THEOREM 5.2 *Let $(A, \mathfrak{D}(A))$ be an operator on ℓ^2 such that $\boldsymbol{e}_i \in \mathfrak{D}(A)$ for all $i \in \mathbb{N}$. Assume $\boldsymbol{b} = (b_i)_{i\geq 1} \in \ell^2$. Then the following two statements are equivalent:*

(I) *A has a matrix representation*

$$A = \left[\frac{-b_i b_j}{\sigma_i + \sigma_j} \right]_{1\leq i,j<\infty}, \qquad b_i \neq 0 \quad \text{for } 1 \leq i < \infty,$$

with $0 < \sigma_i < M < \infty$ for all $i \in \mathbb{N}$, and $\sigma_i \neq \sigma_j$ for all $i \neq j$.

(II) *The closure A^C of A exists and has the properties:*

(i) *A^C is selfadjoint and generates a semigroup of contractions $(\mathrm{e}^{tA^\mathrm{C}})_{t\geq 0}$.*

(ii)

$$\int_0^\infty \mathrm{e}^{tA^\mathrm{C}} \boldsymbol{b}\boldsymbol{b}^\mathsf{T} \mathrm{e}^{tA^\mathrm{C}} \, \mathrm{d}t = \operatorname{diag}(\sigma_1, \sigma_2, \ldots),$$

with $0 < \sigma_i < M < \infty$ for all $i \in \mathbb{N}$, and $\sigma_i \neq \sigma_j$ for all $i \neq j$.

Proof. (I)⇒(II). Statement (i) follows from Theorem 3.3. and Proposition 3.4.

The identity $\int_0^\infty \mathrm{e}^{tA^\mathrm{C}} \boldsymbol{b}\boldsymbol{b}^\mathsf{T} \mathrm{e}^{tA^\mathrm{C}} \, \mathrm{d}t = \operatorname{diag}(\sigma_1, \sigma_2, \ldots) =: \Sigma$ follows from Proposition

5.1 and Proposition 3.5, provided we show that $\int_0^\infty e^{tA^C} bb^T e^{tA^C}\, dt$ is a bounded operator on ℓ^2. This is a consequence of the fact that

$$RR^* = \int_0^\infty e^{tA^C} bb^T e^{tA^C}\, dt$$

and the boundedness of R (Theorem 4.3).

(II)$\Rightarrow$(I). We first show that $e^{tA^C}x \to 0$ as $t \to \infty$ $(x \in \ell^2)$. This follows from Theorem 22.3.2 of Hille & Phillips (1957) if 0 is not an eigenvalue of A^C.

Assume that there exists $x_0 \in \mathfrak{D}(A^C)$ such that $A^C x_0 = 0$. Then, as in the proof of Lemma 3.7, we have $e^{tA^C} x_0 = x_0$ for $t \in [0, \infty)$. Hence

$$\begin{aligned}\langle x_0, \Sigma x_0\rangle &= \left\langle x_0, \int_0^\infty e^{tA^C} bb^T e^{tA^C}\, dt\, x_0\right\rangle \\ &= \int_0^\infty |\langle b, e^{tA^C} x_0\rangle|^2\, dt = \int_0^\infty |\langle b, x_0\rangle|^2\, dt \\ &= \begin{cases} 0 & \text{if } \langle b, x_0\rangle = 0, \\ \infty & \text{otherwise,}\end{cases}\end{aligned}$$

which implies that $x_0 = 0$. For $x, y \in \mathfrak{D}(A^C)$ we have

$$\begin{aligned}&\langle A^C x, \Sigma y\rangle + \langle \Sigma x, A^C y\rangle \\ &= \left\langle A^C x, \left(\int_0^\infty e^{tA^C} bb^T e^{tA^C}\, dt\right) y\right\rangle + \left\langle \left(\int_0^\infty e^{tA^C} bb^T e^{tA^C}\, dt\right) x, A^C y\right\rangle \\ &= \int_0^\infty (\langle e^{tA^C} A^C x, bb^T e^{tA^C} y\rangle + \langle e^{tA^C} x, bb^T e^{tA^C} A^C y\rangle)\, dt \\ &= \int_0^\infty \left(\frac{d}{dt}\langle e^{tA^C} x, bb^T e^{tA^C} y\rangle\right) dt \\ &= \lim_{s\to\infty} \left[\langle e^{tA^C} x, b\rangle\langle b, e^{tA^C} y\rangle\right]_0^s = -\langle x, b\rangle\langle b, y\rangle.\end{aligned}$$

Since $e_i \in \mathfrak{D}(A) \subseteq \mathfrak{D}(A^C)$ for all $i \in \mathbb{N}$, the result follows by considering

$$\langle Ae_i, \Sigma e_j\rangle + \langle \Sigma e_i, Ae_j\rangle = -\langle e_i, b\rangle\langle b, e_j\rangle.$$

Then $\sigma_j a_{ij} + \sigma_i a_{ij} = -b_i b_j$, giving $a_{ij} = -b_i b_j/(\sigma_i + \sigma_j)$ where $a_{ij} := \langle Ae_j, e_i\rangle$ for $1 \le i,j < \infty$. To show that $b_i \neq 0$ for all $i \in \mathbb{N}$, assume that $b_{i_0} \neq 0$ for some $i_0 \in \mathbb{N}$. Since A^C has the matrix representation

$$A^C = \left[\frac{-b_i b_j}{\sigma_i + \sigma_j}\right]_{1\le i,j<\infty},$$

it follows that $A^C e_{i_0} = 0$. This is in contraction to A^C having zero kernel □.

6. Concluding remarks

In this paper, results were derived for the existence, characterization, and approximation of infinite-dimensional balanced systems. The definition of an infinite-dimensional realization was based on a generalization of a parametrization for finite-dimensional balanced systems. We restricted ourselves to the case where the input and output vectors are bounded. The more general case of unbounded input and output vectors will be considered in a later publication.

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