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A note on a system theoretic approach to a conjecture by Peller–Khrushchev

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Abstract: Based on the construction of infinite dimensional balanced realizations an alternative solution to the following inverse spectral problem is presented: Given a monotonically decreasing sequence of positive numbers $(\sigma_n)_{n\geq1}$, does there exist a Hankel operator whose sequence of singular values is $(\sigma_n)_{n\geq1}$?

Keywords: Hankel operator, Balanced realizations, Infinite dimensional systems, Moduli of operators, Singular values.

1. Introduction

Motivated by important problems in the prediction theory for Gaussian processes and the approximation theory for rational functions [4,6] Peller and Khrushchev have pointed out the significance of inverse spectral problems for Hankel operators. In this context they conjectured that given any non-increasing sequence $(\sigma_n)_{n\geq1}$ of positive numbers with $\lim_{n\to\infty}\sigma_n = 0$, there exists a Hankel operator Γ whose singular values $(\sigma_n(\Gamma))_{n\geq1}$ satisfy $\sigma_n(\Gamma) = \sigma_n$ for all $n \in \mathbb{N}$.

Here we are considering an integral Hankel operator with kernel $h(t)$ given by

$$\Gamma: L^2([0, \infty[) \to L^2([0, \infty[),$$
$$u(t) \mapsto (\Gamma(u))(s) = \int_0^\infty h(t+s)u(t)\,\mathrm{d}t.$$

Recall [1] that the singular values $\sigma_n(A)$, $n \geq 1$, of a bounded Hilbert space operator A are defined by: Let σ_∞ be the supremum of the limit points of the spectrum and of the eigenvalues of infinite multiplicity of the modulus $(A^*A)^{1/2}$. If those eigenvalues of $(A^*A)^{1/2}$ which are of greater magnitude than σ_∞ are ordered – taking into account their multiplicities – as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_i \geq \cdots \geq \sigma_\infty$, then σ_i is called the i-th singular value of A. In case there is only a finite number $m = 0, 1, 2\ldots$ of those eigenvalues, we set $\sigma_{m+1} = \sigma_{m+2} = \cdots = \sigma_\infty$.

Using delicate function theoretic and functional analytic arguments, Treil and Vasyunin proved the conjecture also for the case when $\lim_{n\to\infty}\sigma_n \neq 0$ [7].

The reason for readdressing this problem in this paper is that for the case of a sequence of distinct positive $(\sigma_n)_{n\geq1}$ an almost elementary solution can be given. This solution is believed to be particularly interesting for system theorists since it is based on the construction of an infinite dimensional balanced system, whose associated Hankel operator has the required property. This reemphasizes the strong connection between Hankel operators and balanced realizations.

The following, slightly more general theorem will be proven.

Theorem 1. *Let $(\sigma_n)_{n\geq1}$ be a bounded sequence of distinct positive numbers. Then (A, b, b^{T}) with*

$$A := \left(\frac{-b_i b_j}{\sigma_i + \sigma_j}\right)_{1\leq i,j<\infty},$$

$$b := (b_1, b_2, b_3, \ldots)^{\mathrm{T}} := \left(\frac{\sigma_1}{1}, \frac{\sigma_2}{2}, \frac{\sigma_3}{3}, \ldots\right)^{\mathrm{T}} \in \ell^2$$

is a system with state space ℓ^2 and bounded and selfadjoint A such that the integral Hankel operator

$$H: L^2([0, \infty[) \to L^2([0, \infty[),$$
$$u(t) \mapsto (H(u))(s) = \int_0^\infty h(t+s)u(t)\,\mathrm{d}t,$$

where $h(t) := \langle b, \mathrm{e}^{tA}b\rangle \in L^2([0, \infty[)$, is positive, bounded and selfadjoint such that we have for the spectrum $\sigma(H)$ (point spectrum $\sigma_p(H)$),

$$\sigma(H)\setminus\{0\} = \overline{\sigma_p(H)}\setminus\{0\} = \overline{\{\sigma_i \mid 1 \leq i < \infty\}}\setminus\{0\}.$$

The eigenspaces corresponding to non-zero eigenvalues are one dimensional.

The following corollary can now be formulated:

Corollary 2. *If $(\sigma_n)_{n\geqslant 1}$ is strictly monotonically decreasing then for H as defined above we have*

$$\sigma_n(H)=\sigma_n \quad \text{for all } n\in\mathbb{N}.$$

2. System theoretic motivation

The approach to the proof of Theorem 1 is based on the following well known facts about balanced realizations (see, e.g. [2]).

Recall the definition of B.C. Moore [5] according to which a minimal, asymptotically stable system $(A, b, c)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{1\times n}$ is called balanced if

$$\int_0^\infty \mathrm{e}^{tA}bb^\mathrm{T}\mathrm{e}^{tA^\mathrm{T}}\,\mathrm{d}t=\int_0^\infty \mathrm{e}^{tA^\mathrm{T}}c^\mathrm{T}c\,\mathrm{e}^{tA}\,\mathrm{d}t=:\Sigma>0,$$

where

$$\Sigma=:\operatorname{diag}(\sigma_1,\sigma_2,\dots,\sigma_n)$$

with $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_n>0$.

$h(t):=c\mathrm{e}^{tA}b$, $t\in[0,\infty[$, is the kernel of a bounded Hankel operator on $L^2([0,\infty[)$ with singular values $\sigma_1,\sigma_2,\dots,\sigma_n$.

If (A, b, c) is balanced this implies that Σ satisfies the Lyapunov equations

$$A\Sigma+\Sigma A^\mathrm{T}=-bb^\mathrm{T},$$
$$A^\mathrm{T}\Sigma+\Sigma A=-c^\mathrm{T}c.$$

For the case of distinct singular values $(\sigma_i)_{1\leqslant i\leqslant n}$, solution of these linear equations shows that (A, b, c) can be parametrized as

$$b=(b_1, b_2,\dots,b_n)^\mathrm{T},$$
$$c=(s_1b_1, s_2b_2,\dots,s_nb_n),$$
$$s_i=\pm1 \quad \text{for } 1\leqslant i\leqslant n,$$
$$A=\left(\frac{-b_ib_j}{s_is_j\sigma_i+\sigma_j}\right)_{1\leqslant i,j\leqslant n}$$

3. An infinite dimensional balanced system

Let $(\sigma_n)_{n\geqslant 1}$ be a bounded sequence of distinct positive numbers. Motivated by the parametrization of finite dimensional balanced realizations we will construct an infinite dimensional balanced system (A, b, b^T) with selfadjoint A, such that

$$\int_0^\infty \mathrm{e}^{tA}bb^\mathrm{T}\,\mathrm{e}^{tA}\,\mathrm{d}t=:\Sigma=\operatorname{diag}(\sigma_1,\sigma_2,\sigma_3,\dots).$$

Proposition 3. *Let $(\sigma_n)_{n\geqslant 1}$ be a bounded sequence of distinct positive numbers. Consider the system (A, b, b^T) with*

$$A=\left(\frac{-b_ib_j}{\sigma_i+\sigma_j}\right)_{1\leqslant i,j<\infty},$$

$$b=(b_1, b_2, b_3,\dots)^\mathrm{T}:=\left(\frac{\sigma_1}{1},\frac{\sigma_2}{2},\frac{\sigma_3}{3},\dots\right)^\mathrm{T}.$$

Then:

(i) *A is a selfadjoint and bounded operator on ℓ^2.*

(ii) *$\Sigma=:\operatorname{diag}(\sigma_1,\sigma_2,\sigma_3,\dots)$ satisfies the Lyapunov equation $A\Sigma+\Sigma A=-bb^\mathrm{T}$.*

(iii) *$\int_0^\infty \mathrm{e}^{tA}bb^\mathrm{T}\,\mathrm{e}^{tA}\,\mathrm{d}t=\Sigma$.*

(iv) *$\langle b, \mathrm{e}^{tA}b\rangle\geqslant 0$ for all $t\in[0,\infty[$.*

Proof. (i) For $A=:(\mathrm{a}_{ij})_{1\leqslant i,j<\infty}$ we have

$$\sum_{j=1}^\infty\sum_{i=1}^\infty a_{ij}^2<\infty,$$

which implies that A is bounded.

(ii) Follows from the construction of A.

(iii) We will show first that $\lim_{t\to\infty}\mathrm{e}^{tA}x=0$ for all $x\in\ell^2$.

By considering the spectral representation of A or by applying the more general Theorem 22.3.2 in Hille and Phillips [3], it follows that this is the case if $\sigma_p(A)\subseteq\,]-\infty,0[$. To show that $\sigma_p(A)\subseteq\,]-\infty,0[$ first note that it follows from (ii) that for all $s\geqslant 0$ and $x, y\in\ell^2$,

$$\int_0^s\langle b, \mathrm{e}^{tA}x\rangle\langle b, \mathrm{e}^{tA}y\rangle\,\mathrm{d}t=-\left[\langle \mathrm{e}^{tA}x, \mathrm{e}^{tA}y\rangle\right]_0^s.$$

Let $\lambda\neq 0$ be a nonzero eigenvalue of A with eigenvector $x\in\ell^2$. Then we have

$$\int_0^s\langle b, \mathrm{e}^{tA}x\rangle\langle b, \mathrm{e}^{tA}x\rangle\,\mathrm{d}t$$
$$=\langle b, x\rangle^2\int_0^s \mathrm{e}^{2\lambda t}\,\mathrm{d}t=\langle b, x\rangle^2\frac{1}{2\lambda}\left[\mathrm{e}^{2\lambda t}\right]_0^s$$
$$=-\left[\langle\Sigma\mathrm{e}^{t\lambda}x, \mathrm{e}^{t\lambda}x\rangle\right]_0^s=-\langle\Sigma x, x\rangle\left[\mathrm{e}^{2t\lambda}\right]_0^s$$

and hence

$$2\lambda\langle \Sigma x, x\rangle = -\langle b, x\rangle^2$$

which implies $\lambda \leqslant 0$.

Assume $\lambda = 0$ with eigenvector $x \in \ell^2$; then

$$\int_0^s \langle b, x\rangle^2 \, \mathrm{d}t = -[\langle \Sigma x, x\rangle]_0^s = 0$$

so that $\langle b, x\rangle = 0$. Thus

$$(A\Sigma + \Sigma A)x = A\Sigma x = -bb^{\mathrm{T}}x = 0.$$

So the nullspace of A is invariant under Σ. But an invariant subspace of Σ has the form

$$\overline{\operatorname{span}}\{ e_{n_i} \mid i \in I \}$$

for some index set I. Thus $Ae_{n_i} = 0$ for $i \in I$ and $\langle b, e_{n_i}\rangle = 0$ which is a contradiction to $b_{n_i} = \sigma_{n_i}/n_i$.

We have that $\int_0^\infty \mathrm{e}^{tA}bb^{\mathrm{T}}\mathrm{e}^{tA}\,\mathrm{d}t$ solves $AX + XA = -bb^{\mathrm{T}}$. To show that

$$\int_0^\infty \mathrm{e}^{tA}bb^{\mathrm{T}}\,\mathrm{e}^{tA}\,\mathrm{d}t = \Sigma$$

we show that the solution of $AX + XA = -bb^{\mathrm{T}}$ is unique.

Assume that X_1 and X_2 are solutions of this operator equation. Then $A\Delta X + \Delta XA = 0$, where $\Delta X = X_1 - X_2$.

Thus for $t \in [0, \infty[$ and $x, y \in \ell^2$,

$$0 = \langle \Delta X\,\mathrm{e}^{tA}x, A\,\mathrm{e}^{tA}y\rangle + \langle \Delta XA\,\mathrm{e}^{tA}x, \mathrm{e}^{tA}y\rangle$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\langle \Delta X\,\mathrm{e}^{tA}x, \mathrm{e}^{tA}y\rangle.$$

So $\langle \Delta X\,\mathrm{e}^{tA}x, \mathrm{e}^{tA}y\rangle$ is constant as $t \to \infty$. But since $\mathrm{e}^{tA}x \to 0$ for all $x \in \ell^2$, we have

$$\langle \Delta X\,\mathrm{e}^{tA}x, \mathrm{e}^{tA}y\rangle \to 0 \quad \text{as } t \to \infty.$$

Hence

$$\langle \Delta X\,\mathrm{e}^{0A}x, \mathrm{e}^{0A}y\rangle = \langle \Delta Xx, y\rangle = 0$$

for all $x, y \in \ell^2$, which implies that $\Delta X = 0$.

(iv) $(\mathrm{e}^{tA})_{t \geqslant 0}$ is a semigroup of selfadjoint operators, thus

$$\langle \mathrm{e}^{tA}b, b\rangle = \langle \mathrm{e}^{tA/2}b, \mathrm{e}^{tA/2}b\rangle \geqslant 0$$

for $t \in [0, \infty[$. □

Proof of Theorem 1. To show that H is bounded we write $H = \theta\theta^*$ where

$$\theta: \ell^2 \to L^2([0, \infty[), \quad x \mapsto \langle b, \mathrm{e}^{tA}x\rangle.$$

Since

$$\int_0^\infty ((\theta(x))(t))^2\,\mathrm{d}t = \int_0^\infty \langle b, \mathrm{e}^{tA}x\rangle^2\,\mathrm{d}t$$
$$= \left\langle x, \left(\int_0^\infty \mathrm{e}^{tA}bb^{\mathrm{T}}\,\mathrm{e}^{tA}\,\mathrm{d}t\right)x\right\rangle = \langle x, \Sigma x\rangle$$
$$\leqslant \|\Sigma\|\,\|x\|^2,$$

we have that θ is bounded, which implies the boundedness of H.

The selfadjointness and positivity of H are clear from Proposition 3. It can be justified in a straightforward way that

$$(\langle b, \mathrm{e}^{At}\Sigma e_i\rangle)_{1 \leqslant i < \infty}$$

is an orthogonal system of eigenvectors of H with $(\sigma_n)_{n \geqslant 1}$ being the corresponding sequence of eigenvalues.

Let

$$E := \overline{\operatorname{span}}\{\langle b, \mathrm{e}^{At}\Sigma e_i\rangle \mid 1 \leqslant i < \infty\}.$$

For $u \in E^{\perp}$, the orthogonal complement of E, we have for all $1 \leqslant i < \infty$,

$$0 = \langle u(t), \langle b, \mathrm{e}^{At}\Sigma e_i\rangle\rangle_{L^2}$$
$$= \int_0^\infty \langle b, \mathrm{e}^{tA}\Sigma e_i\rangle u(t)\,\mathrm{d}t$$
$$= \sigma_i\left\langle e_i, \int_0^\infty \mathrm{e}^{tA}bu(t)\,\mathrm{d}t\right\rangle.$$

Thus $\int_0^\infty \mathrm{e}^{tA}bu(t)\,\mathrm{d}t = 0$, hence

$$(H(u))(s) = \left\langle b, \mathrm{e}^{sA}\int_0^\infty \mathrm{e}^{tA}bu(t)\,\mathrm{d}t\right\rangle$$
$$= 0 \quad \text{for all } s \in [0, \infty[.$$

So $E^{\perp} = \operatorname{Ker}(H)$. Hence $L^2([0, \infty[)$ is spanned by an orthogonal set of eigenvectors which implies that $\sigma(H) = \overline{\sigma_p(H)}$. This gives the result. □

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