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Asymptotically Stable All-Pass Transfer Functions:

Canonical Form, Parametrisation and Realisation

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Abstract

Based on a canonical form for the set of minimal asymptotically stable systems in terms of balanced realizations, a parametrization is derived for the set of asymptotically stable MIMO all-pass transfer functions. A specialization of this result to the SISO case leads to a particularly simple representation of asymptotically stable scalar all-pass transfer functions. A straightforward state-space realization algorithm for asymptotically stable SISO all-pass transfer functions is given.

Keywords: Balanced realizations. All-pass transfer functions. Parametrization. Canonical form.

I. Introduction

All-pass transfer functions are important in linear systems theory and filter design and it is hoped that the results which will be given in this paper will be useful both in theory and in practice.

The main idea in the derivation of a parametrization result for the set of minimal and asymptotically stable all-pass transfer functions is to use a canonical form in terms of balanced realizations as derived in ([3]). In Section III this canonical form is related to all-pass transfer functions by using a characterization of all-pass transfer functions in terms of their state-space realization ([1]).

Section IV is concerned with the derivation of two further results on asymptotically stable SISO all-pass transfer functions. A parametrization of the coefficients of these rational functions is given. This is followed by the presentation of a realization algorithm for the class of all-pass transfer functions. This algorithm is particularly straightforward since it only requires elementary operations on the coefficients of the all-pass transfer function, i.e. no solution of Lyapunov equations etc. is needed.

II. All-pass Transfer Functions and Balanced Realisations

Definition 2.1:

An $m \times m$ transfer function $G(s)$ is called all-pass if

$$G(s)\,G^T(-s) = I \qquad \text{for all} \quad s \in \mathbb{C}.$$

□

Since it does not involve any additional complications we are going to deal with the slightly more general case where $G(s)G^T(-s) = \sigma^2 I$ for some $\sigma > 0$. In order to relate all-pass transfer functions to their state-space realizations we define balanced realizations as introduced by B.C. Moore ([2]).

Definition 2.2([2]):

Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ be a minimal and asymptotically stable continuous-time system. Then (A, B, C) is called balanced if for

$$W_c = \int_0^\infty e^{At} B B^T e^{tA^T} dt$$

$$W_0 = \int_0^\infty e^{A^T t} C^T C e^{tA}\, dt$$

we have $W_c = W_0 = diag(\sigma_1, \ldots, \sigma_n) =: \Sigma.$

The positive numbers $(\sigma_1, \ldots, \sigma_n)$ are called the singular values of the system (A, B, C). □

An equivalent characterization of a system to be balanced can be given in terms of Lyapunov equations. These are a major tool in working with balanced realizations.

Theorem 2.3 ([2]):

Let $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ be a minimal and asymptotically stable continuous-time system. Then (A,B,C) is balanced iff there exists a diagonal matrix $\Sigma > 0$ such that

$$A\Sigma + \Sigma A^T = -BB^T$$
$$\text{and} \quad A^T\Sigma + \Sigma A = -C^TC.$$

In this case $\Sigma = W_0 = W_C$. □

The following theorem relates balanced realizations to all-pass transfer functions.

Theorem 2.4 ([1]):

Given a realization (A,B,C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, then

(1) If (A,B,C) is controllable and observable the following two statements are equivalent:

  (a) There exists D such that $G(s)\tilde{G}(-s) = \sigma^2 I$ for all $s \in \mathbb{C}$, where $G(s) \triangleq D + C(sI-A)^{-1}B$.

  (b) There exists $P, Q \in \mathbb{R}^{n\times n}$ such that
    (i) $P = P^T$, $Q = Q^T$
    (ii) $AP + PA^T + BB^T = 0$
    (iii) $A^TQ + QA + C^TC = 0$
    (iv) $PQ = \sigma^2 I$.

(2) Given that part (1b) is satisfied then there exists D satisfying

$$D^TD = \sigma^2 I$$
$$D^TC + B^TQ = 0$$
$$\text{and} \quad DB^T + CP = 0$$

and any such D will satisfy part (1a). □

III. The Main Result

We are now able to state and prove the main theorem of this paper.

Theorem 3.1 :

The two following statements are equivalent.

(1) The $m \times m$ transfer function $G(s)$ of MacMillan degreee n is asymptotically stable and $G(s)\tilde{G}(-s) = \sigma^2 I_m$.

(2) $G(s)$ has a unique balanced state-space realization $G(s) = C(sI-A)^{-1}B + D$ such that

I. For B we have:

(i) There exist unique integer-valued indicies $\ell \geqslant 1$ and $r(i) \geqslant 1$, $1 \leqslant i \leqslant \ell$ such that

$$BB^T = diag(\lambda_1 I_{r(1)}, \ldots, \lambda_\ell I_{r(\ell)}, 0, \ldots, 0)$$

with $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0$,

and $\sum_{i=1}^{\ell} r(i) = \text{rank}(BB^T) =: r$.

(ii) There exist unique integer-valued indicies

$$1 \leqslant t(i,1) < t(i,2) < \ldots < t(i,r) \leqslant m,\ 1 \leqslant i \leqslant \ell$$

such that for

$$B = \begin{bmatrix} B(1) \\ \vdots \\ B(\ell) \\ 0 \end{bmatrix}$$

with $B(i) = (b(i)_{st})_{\substack{1\leqslant s\leqslant r(i) \\ 1\leqslant t\leqslant m}} \in \mathbb{R}^{r(i)\times m}$ for $1 \leqslant i \leqslant \ell$

we have

$b(i)_{st(i,s)} > 0$ for all $1 \leqslant s \leqslant r(i)$
$b(i)_{st} = 0$ for all $1 \leqslant t < t(i,s)$ and $1 \leqslant s \leqslant r(i)$,

i.e. $B(i) =$

$$\begin{bmatrix} 0 & . & 0 & b(i)_{1t(i,1)} & & & & & & \\ 0 & . & 0 & 0 & 0 & . & 0 & b(i)_{2t(i,2)} & & b(i)_{st} \\ . & . & . & . & & . & 0 & 0 & . & \\ . & . & . & . & . & . & . & . & . & \\ . & . & . & . & . & . & . & . & . & \\ 0 & . & . & 0 & 0 & . & 0 & 0 & . & b(i)_{r(i)t(i,r(i))} \end{bmatrix}$$

II. For C we have:

There exists a unique $U \in \mathbb{R}^{m\times r}$ with $U^TU = I_r$, such that

$$C = (U\ O)\, diag(\lambda_1^{1/2} I_{r(1)}, \ldots, \lambda_\ell^{1/2} I_{r(\ell)}, 0 \ldots, 0).$$

III. For $A =: \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11} \in \mathbb{R}^{r\times r}$, we have:

(i) There exists a unique skew-symmetric matrix $\tilde{A}_{11} \in \mathbb{R}^{r\times r}$ and a unique $\sigma > 0$ such that

$$A_{11} = \frac{-1}{2\sigma} diag(\lambda_1 I_{r(1)}, \ldots, \lambda_\ell I_{r(\ell)}) + \tilde{A}_{11}.$$

(ii) There exists unique $q \in \mathbb{N}, q \geqslant 1$ and a set of unique indicies

$$(g(1), h(1)), \ldots, (g(q), h(q)) \in \mathbb{N} \times \mathbb{N}$$

with

$$1 = h(1) < \ldots < h(i) < h(i+1) < \ldots \leqslant n - r$$
$$1 \leqslant g(q) < \ldots < g(i+1) < g(i) < \ldots \leqslant r$$

such that for $A_{12} =: (a_{st})_{\substack{1\leqslant s\leqslant r \\ 1\leqslant t\leqslant n-r}}$ we have

$a_{g(i)h(i)} > 0$ for $1 \leqslant i \leqslant q$,
$a_{g(i)t} = 0$ for $t > h(i)$ where $1 \leqslant i \leqslant q$,

$$a_{st} = 0 \quad \text{for} \quad t \geqslant h(i) \quad \text{and} \quad s > g(i) \quad \text{where} \quad 1 \leqslant i \leqslant q$$

i.e.

$$A_{12} = \begin{bmatrix} z & z & . & z & z & z & . & . \\ . & . & . & . & z & 0 & . & . \\ z & z & . & z & a_{g(2)h(2)} & 0 & & \\ a_{g(1)h(2)} & 0 & . & 0 & 0 & 0 & & \\ 0 & . & . & . & 0 & & & \\ . & . & . & . & & & & \end{bmatrix}$$

(iii) $A_{21} = -A_{12}^T$

(iv) There exist unique real numbers α_i, $i = 2, 3, \ldots, n-r$, with

$$\alpha_{j_i} = \begin{cases} 0 & \text{if} \quad i = h(j) \quad \text{for some} \quad 1 \leqslant j \leqslant q \\ > 0 & \text{otherwise} \end{cases}$$

such that

$$A_{22} = \begin{bmatrix} 0 & \alpha_2 & & & & \\ -\alpha_2 & 0 & \alpha_3 & & 0 & \\ 0 & -\alpha_3 & 0 & & & \\ & & & . & . & \\ & 0 & & & 0 & \alpha_{n-r} \\ & & & & -\alpha_{n-r} & 0 \end{bmatrix}$$

IV. For D we have:

(i) D is unique.

(ii) $DD^T = \sigma^2 I$.

(iii) $DB^T + \sigma C = 0$.

Proof:

(1) $\Rightarrow$ (2):

By Theorem 2.4 $G(s)$ has a state-space realization with identical singular values $\sigma > 0$. Thus by Theorem 5.7 in ([3]) there exist unique A, B, C as characterized above such that $G(s) = C(sI - A)^{-1}B + G(\infty)$.

Let $D = G(\infty)$

$$\Rightarrow DD^T = \sigma^2 I_m.$$

Following the proof of Theorem 2.4 in ([1], p.32) we obtain that

$$B^T = D^T C T^{-1}$$

for an invertable T such that

$$TA + A^T T - C^T C = 0.$$

But $A^T X + XA = -C^T C$ has a unique positive definitive solution $\Sigma = \sigma I_n$

$$\Rightarrow T = -\sigma I_n$$
$$\Rightarrow DB^T + \sigma C = 0.$$

(ii) $\Rightarrow$ (i)

Follows immediately from Theorem 2.4 by noting that any (A, B, C) as parametrized above is balanced, minimal, asymptotically stable and has identical singular value σ (see Theorem 7.1 in ([3])). □

In the case of SISO all-pass transfer functions we obtain a particularly straightforward characterization.

Corollary 3.2

The following two statements are equivalent:

(1) The scalar transfer function $g(s)$ of McMillan degree n is asymptotically stable and $g(s)g(-s) = \sigma^2$.

(2) $g(s)$ has a unique balanced state-space realization $g(s) = c(sI - A)^{-1}b + d$ such that
for b we have:

There exists unique $b_1 > 0$ such that

$$b^T = (b_1, 0, 0, \ldots, 0)^T \in \mathbb{R}^n$$

for c we have:

There exists unique $s_1 = \pm 1$ such that

$$c^T = s_1 b$$

for A we have:

There exist unique strictly positive real numbers $\sigma, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that

$$A = \begin{bmatrix} a_{11} & \alpha_1 & & & . & . \\ -\alpha_1 & 0 & \alpha_2 & & . & 0 \\ & -\alpha_2 & 0 & \alpha_3 & . & . \\ & & . & . & . & . \\ & 0 & . & . & 0 & \alpha_{n-1} \\ & & . & . & -\alpha_{n-1} & 0 \end{bmatrix}$$

with $a_{11} = \dfrac{-b_1^2}{2\sigma}$

for d we have:

$$d = -s_1 \sigma.$$

Proof:

From $db = \sigma c = 0$ we have $db_1 + \sigma s_1 b_1 = 0$. □

IV. Parametrisation and Realisation of Asymptotically Stable SISO All-Pass Transfer Functions

Before we can state the next lemma which is important for what follows, we have to introduce some notation.

$$\text{Let} \quad \tilde{A} = \begin{bmatrix} s & -\alpha_1 & & & . & . \\ \alpha_1 & s & -\alpha_2 & & . & 0 \\ & \alpha_2 & s & -\alpha_3 & . & . \\ & & . & . & . & . \\ & 0 & . & . & s & -\alpha_{n-1} \\ & & . & . & \alpha_{n-1} & s \end{bmatrix}$$

where $\alpha_i > 0$ for $1 \leqslant i \leqslant n-1$ and $s \in \mathbf{C}$.

Then we denote

$\Delta_n := det(\tilde{A})$

Δ_{n-k}, $1 \leqslant k \leqslant n-1$, is the determinant of the matrix resulting from $\tilde{A}$ by deleting the first k rows and columns.

$\Delta_0 := 1$.

Lemma 4.1

With the notation as above we have

(1) $\Delta_{n-k} = s\Delta_{n-k-1} + \alpha_{k+1}^2 \Delta_{n-k-2}$ for $0 \leqslant k \leqslant n-2$.

(2) $\Delta_n = s^n + a_{n-2}s^{n-2} + a_{n-4}s^{n-4} + \ldots$

where $a_{n-2\ell} = \sum\limits_{\substack{1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_\ell \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant \ell-1}} \alpha_{i_1}^2 \ldots \alpha_{i_\ell}^2$, $\ell = 1, 2, \ldots$

Proof:

(1) follows by expansion of the determinant.

(2) is shown by induction over n.

for $n = 2$ we have $\Delta_n = s^2 + \alpha_1^2$,

for $n \geqslant 3$ we have

(i) Case: n even

$$\Delta_n = s\Delta_{n-1} + \alpha_1^2 \Delta_{n-2}$$

where by assumption

$$\Delta_{n-1} = s^{n-1} + b_{n-3}s^{n-3} + b_{n-5}s^{n-5} + \ldots + b_1 s$$

$$\Delta_{n-2} = s^{n-2} + c_{n-4}s^{n-4} + c_{n-6}s^{n-6} + \ldots + c_0$$

$$\text{with } b_{n-2r-1} = \sum_{\substack{2 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant r-1}} \alpha_{i_1}^2 \ldots \alpha_{i_r}^2, \quad r = 1, 2, \ldots, \tfrac{n}{2} - 1$$

$$c_{n-2m-2} = \sum_{\substack{3 \leqslant i_1 \leqslant \ldots \leqslant i_m \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant m-1}} \alpha_{i_1}^2 \ldots \alpha_{i_m}^2, \quad m = 2, 3, \ldots, \tfrac{n}{2} - 1$$

So the coefficients of $\Delta_n = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0$ satisfy:

$$a_n = 1$$

$$a_{n-2r+1} = 0 \qquad r = 1, 2, \ldots, \frac{n}{2}$$

$$a_{n-2r} = b_{n-2r-1} + \alpha_1^2 c_{n-2r-2} = \sum_{\substack{1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant r-1}} \alpha_{i_1}^2 \ldots \alpha_{i_r}^2 \qquad r = 1, 2, \ldots, \frac{n}{2} - 1$$

with $b_0 := 1$

$$a_0 = \alpha_1^2 c_0 = \alpha_1^2 \alpha_3^2 \ldots \alpha_{n-1}^2.$$

(ii) Case : n odd

analogous to (i). □

We can now prove a parametrization result for the coefficient of all-pass transfer functions.

Theorem 4.2 :

The following two statements are equivalent:

(1) The scalar transfer function $g(s)$ of MacMillan degree n is asymptotically stable with $g(s)g(-s) = \sigma^2$, $\sigma > 0$.

(2) $g(s)$ is a rational transfer function such that for unique $s_1 = \pm 1$, $\sigma > 0$, $b_1 > 0$ and $\alpha_i > 0$ for $1 \leqslant i \leqslant n-1$ we have

$$g(s) = -s_1 \sigma \frac{(\Delta_n + a_{11}\Delta_{n-1})}{\Delta_n - a_{11}\Delta_{n-1}}$$

where

(i) $a_{11} = \dfrac{-b_1^2}{2\sigma}$

(ii) $\Delta_n = s^n + a_{n-2}s^{n-2} + a_{n-4}s^{n-4} + \ldots$

with $a_{n-2\ell} = \sum\limits_{\substack{2 \leqslant i_1 \leqslant \ldots \leqslant i_\ell \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant \ell-1}} \alpha_{i_1}^2 \ldots \alpha_{i_\ell}^2$, $\ell = 1, 2, \ldots$

(iii) $\Delta_{n-1} = s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \ldots$

with $a_{n-2\ell-1} = \sum\limits_{\substack{2 \leqslant i_1 \leqslant \ldots \leqslant i_\ell \leqslant n-1 \\ |i_j - i_{j+1}| \geqslant 2,\ 1 \leqslant j \leqslant \ell-1}} \alpha_{i_1}^2 \ldots \alpha_{i_\ell}^2$, $\ell = 1, 2, \ldots$

Proof:

The proof follows from Corollary 3.2 by noting that for (A, b, c, d) as parametrized there we have

$$\begin{aligned} g(s) &= c(sI - A)^{-1}b + d \\ &= s_1 b_1^2 \frac{\Delta_{n-1}}{\det(sI - A)} + d \\ &= \frac{s_1 b_1^2 \Delta_{n-1}}{s\Delta_{n-1} - a_{11}\Delta_{n-1} + \alpha_1^2 \Delta_{n-2}} + d \\ &= \frac{s_1 b_1^2 \Delta_{n-1} + (-s_1\sigma)[\Delta_n - a_{11}\Delta_{n-1}]}{\Delta_n - a_{11}\Delta_{n-1}} \\ &= -s_1 \sigma \frac{\Delta_n + a_{11}\Delta_{n-1}}{\Delta_n - a_{11}\Delta_{n-1}} \end{aligned}$$ □

Next we are going to consider the realization problem for asymptotically stable all-pass transfer functions:

Given an asymptotically stable all-pass transfer function $g(s)$, s.t. $g(s)g(-s) = \sigma^2$, $\sigma > 0$, find a minimal state-space realization (A, b, c, d) of $g(s)$ in balanced canonical form.

It is clear from Corollary 3.2 that this amounts to the problem of finding an algorithm yielding the parameters $s_1, b_1, \alpha_1, \ldots, \alpha_{n-1}$ in terms of the coefficients of $g(s)$. Such an algorithm is now presented.

Theorem 4.3 :

Let $g(s)$ be an asymptotically stable all-pass transfer of MacMillan degree n, such that $g(s)g(-s) = \sigma^2$, for some $\sigma > 0$

(i) Write $g(s) = \frac{n(s)}{d(s)}$
such that $n(s)$ and $d(s)$ are relatively prime and the leading coefficient of $d(s)$ is 1.

(ii) Let $d := -s_1\sigma$ be the leading coefficient of $n(s)$

where $s_1 = \pm 1$, $\sigma > 0$.

(iii) Write $d(s) =: s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0$.

Let
$\Delta_n(s) := s^n + a_{n-2}s^{n-2} + a_{n-4}s^{n-4} + \ldots$
$-a_{11}\Delta_{n-1}(s) := a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \ldots$
such that the leading coefficient of $\Delta_{n-1}(s)$ is 1
and let
$b_1^2 := 2\sigma a_{n-1}$ for $b_1 > 0$.

(iv) For $k = 1, 2, \ldots, n-1$ define recursively

$$\alpha_k^2\Delta_{n-k-1}(s) := \Delta_{n-k+1}(s) - s\Delta_{n-k}(s)$$

where Δ_{n-k-1} has leading coefficient 1
and $\alpha_k > 0$.

Then a minimal state-space realization for $g(s)$ in balanced canonical form is given by (A, b, c, d), where

$$A = \begin{bmatrix} a_{11} & \alpha_1 & & & \cdot & \cdot \\ -\alpha_1 & 0 & \alpha_2 & & \cdot & 0 \\ & -\alpha_2 & 0 & \alpha_3 & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & 0 & \cdot & \cdot & 0 & \alpha_{n-1} \\ & & \cdot & \cdot & -\alpha_{n-1} & 0 \end{bmatrix}$$

$b = (b_1, 0, \ldots, 0)^T$
$c = (s_1 b_1, 0, \ldots, 0)$
$d = -s_1\sigma.$

Proof:

By Corollary 3.2 $g(s)$ has a unique balanced state-space realization (A, b, c, d) which is parametrized by $s_1 = \pm 1$, $b_1 > 0$, $\sigma > 0$, $\alpha_1 > 0, \ldots, \alpha_{n-1} > 0$.

By Theorem 4.2 $g(s)$ can be written as

$$g(s) = -s_1\sigma\frac{(\Delta_n + a_{11}\Delta_{n-1})}{\Delta_n - a_{11}\Delta_{n-1}} =: \frac{n(s)}{d(s)}$$

where a_{11}, Δ_{n-1}, Δ_n are defined as in Theorem 4.2.

Thus $-s_1\sigma$ is the leading coefficient of $n(s)$. The same representation justifies (iii).

(iv) follows from Lemma 4.1. □

V. Acknowledgements

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VI. References

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