LINEAR CIRCUITS, SYSTEMS AND SIGNAL PROCESSING: THEORY AND APPLICATION

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1988

NORTH-HOLLAND
AMSTERDAM · NEW YORK · OXFORD · TOKYO

ISBN 0 444 70495 7

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the U.S.A and Canada:
ELSEVIER SCIENCE PUBLISHERS COMPANY. INC.
52 Vanderbilt Avenue
New York. N.Y. 10017
U.S.A.

LIBRARY OF CONGRESS
Library of Congress Cataloging-in-Publication Data

Linear circuits, systems, and signal processing : theory and application / edited by Christopher I. Byrnes and Clyde F. Martin and Richard E. Saeks.
p. cm.
Selected papers from the 8th International Symposium on the Mathematical Theory of Networks and Systems, held June 15-19, 1987, in Phoenix, Ariz.
Bibliography: p.
ISBN 0-444-70495-7
1. Linear systems--Congresses. 2. Electric circuits, Linear--Congresses. 3. Signal processing--Congresses. 4. Linear integrated circuits--Congresses. I. Byrnes, Christopher I., 1949- . II. Martin, Clyde. III. Saeks, Richard, 1941-
IV. International Symposium on the Mathematical Theory of Networks and Systems (8th : 1987 : Phoenix, Ariz.)
QA402.L54 1988
620.7'2--dc19 88-22788
CIP

PRINTED IN THE NETHERLANDS

Linear Circuits, Systems and Signal Processing:
Theory and Application
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The Parametrization of Linear Systems using Balanced Realizations: Relaxation Systems

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Parametrization problems of the set of asymptotically stable relaxation systems of a given order are investigated. The main tool is a canonical form for this class of systems in terms of balanced realizations. It is shown that this class of systems is pathwise connected. Finite dimensional relaxation systems are characterized as systems corresponding to finite rank Hankel operators with nonnegative spectrum. A generalization of this result to infinite dimensional systems is given.

1. Introduction

In this paper we are going to study questions in relation to the parametrization of linear systems. In particular we will be concerned with the set of asymptotically stable relaxation systems. The parametrization of the set of linear systems is of interest e.g. in system identification. For a survey on parametrization problems see ([7]). The reason for addressing the question of the parametrization of relaxation systems is that they admit a canonical form in balanced realizations which reflects in an interesting way the exterior properties of a relaxation system. Relaxation systems form a subclass of the set of multivariable symmetric systems, characterized by the absence of oscillatory behaviour as seen by the following definition. For introductory material see e.g. ([13],[14])

Definition 1.1 : A system (A, B, C) is called a relaxation system if its impulse response $H(t) := Ce^{tA}B$, $t \geqslant 0$, satisfies:

$$(-1)^n \frac{d^n}{dt^n} H(t) \geqslant 0 \quad \text{for} \quad t \geqslant 0 \quad \text{and } n = 0, 1, 2 \ldots \qquad \square$$

The following proposition shows that a relaxation system admits a state-space realization with particularly interesting internal symmetry properties. These symmetry properties are a basic tool in our later investigations.

Proposition 1.2 ([14]) **:** Let $G(s)$ be a real transfer function. Then the following are equivalent:
(i) $G(s)$ is the transfer function of a relaxation system.
(ii) $G(s)$ admits a state-space realization (A, B, C) of relaxation type, i.e. $A = A^T \leqslant 0, B = C^T$.
(iii) For any minimal realization (A, B, C) of $G(s)$, there exists a (unique) matrix $T = T^T > 0$ such that $A = T^{-1}A^T T$, $\quad B = TC^T$. $\square$

Let $C_n^{p,m} = \{(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} | (A, B, C)$ is a minimal and asymptotically stable continuous-time system$\}$ and denote by R_n^m the set of minimal and asymptotically stable state-space representations of n-dimensional relaxation systems with m-dimensional input and output spaces.

In Section 2 we recall the definition of a system to be balanced and present a canonical form for the set of asymptotically stable relaxation systems using balanced realizations. A particular feature of this canonical form is that it immediately leads to a parametrization, with a parameter space of simple geometric structure. This allows us to show that the set of relaxation systems is pathwise connected.

After a short summary of some concepts relating to integral Hankel operators we show in Section 3, using the canonical form of Section 2, that R_n^m can be characterized as a set of systems corresponding to rank n Hankel operators with nonnegative eigenvalues. This then implies that an element in R_n^m is fully parametrized by the singular values and the starting points of the Schmidt vectors of the associated Hankel operator. The correspondence between relaxation systems and Hankel operators with positive eigenvalues can be used to derive an upper bound on the multiplicity of a singular value of such an operator. As a last point we consider infinite dimensional systems and show that nuclear infinite dimensional Hankel operators have a characterization in terms of balanced systems analogous to the one given in the finite dimensional case.

2. Balanced Realizations

In this section we will relate balanced realizations to relaxation systems and give a canonical form for R_n^m. This canonical form is then used to show that R_n^m has only one connected component. We now recall the definition of a system to be balanced.

Definition 2.1 ([8]) : Let $(A, B, C) \in C_n^{p,m}$. (A, B, C) is called balanced if

$$\int_0^\infty e^{At} BB^T e^{tA^T} dt = \int_0^\infty e^{tA^T} C^T C e^{tA} dt =: \Sigma =: \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

The positive numbers $\sigma_1, \ldots, \sigma_n$ are called the singular values of the system (A, B, C). Denote by $C_n^{p,m,b} \subseteq C_n^{p,m}$ the subset of all balanced systems. □

The following theorem gives a characterization of relaxation systems in terms of a balanced canonical form.

Theorem 2.2 : The following two statements are equivalent:

(I) $G(s)$ is the transfer function of some $(A, B, C) \in R_n^m$.

(II) $G(s)$ has a realization $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{m\times n}$ given by

B-**matrix:**

(1) Partition $B = \begin{bmatrix} B^1 \\ \vdots \\ B^k \end{bmatrix}$ with $B^j \in \mathbf{R}^{n(j)\times m}$ then for $1 \leqslant j \leqslant k$,

$B^j(B^j)^T = \mathrm{diag}(\lambda_1^{(j)} I_{r(j;1)}, \lambda_2^{(j)} I_{r(j;2)}, \ldots, \lambda_{\ell(j)}^{(j)} I_{r(j;\ell(j))})$

such that $\lambda_1^{(j)} > \lambda_2^{(j)} > \ldots > \lambda_{\ell(j)}^{(j)} > 0$ and $1 \leqslant \sum_{i=1}^{\ell(j)} r(j;i) = n(j) \leqslant m$.

(2) B^j , $1 \leqslant j \leqslant k$, has the following structure:

$$B^j = \begin{bmatrix} B(j;1) \\ \vdots \\ B(j;\ell(j)) \end{bmatrix} \quad \text{with } B(j;i) \in \mathbf{R}^{r(j;i)\times m} \text{ for } 1 \leqslant i \leqslant \ell(j).$$

The precise structure of $B(j;i) =: \left(b(j;i)_{st}\right)_{\substack{1\leqslant s\leqslant r(j;i) \\ 1\leqslant t\leqslant m}}$ is given by the indices:

$$1 \leqslant t(j;i,1) < t(j;i,2) < \ldots < t(j;i,r(j;i)) \leqslant m \text{ for } 1 \leqslant i \leqslant \ell(j).$$

We have

$$\begin{aligned} b(j;i)_{st(j;i,s)} > 0 \quad & \text{for all } 1 \leqslant s \leqslant r(j;i) \\ b(j;i)_{st} = 0 \quad & \text{for all } 1 \leqslant t < t(j;i,s) \text{ and } 1 \leqslant s \leqslant r(j;i) \end{aligned}$$

i.e.

$$B(j;i) = \begin{bmatrix} 0 \, . \, 0 & b(j;i)_{1t(j;i,1)} & & & \\ 0 \, . \, 0 & 0 \dots 0 & b(j;i)_{2t(j;i,2)} & b(j;i)_{st} & \\ \cdot & \cdot & 0 \; \dots . & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 \, . \, 0 & 0 \dots 0 & 0 \dots . 0 & b(j;i)_{r(j;i)t(j,r(j;i))} & \dots \end{bmatrix}$$

A**-matrix:**

A admits a partitioning $A = \big(A(i,j)\big)_{1 \leqslant i,j \leqslant k}$ with $A(i,j) \in \mathbf{R}^{n(i) \times n(j)}$, $1 \leqslant i,j \leqslant k$, with the following properties.

(i) **Block diagonal entries** $A(j,j)$**:**

$A(j,j) = \frac{-1}{2\sigma_j} \operatorname{diag}(\lambda_1^{(j)} I_{r(j;1)}, \dots, \lambda_{\ell(j)}^{(j)} I_{r(j;\ell(j))})$ with $\sigma_1 > \dots > \sigma_j > \dots > \sigma_k > 0$

(ii) **Off-diagonal blocks** $A(i,j) =: \big(a(i,j)_{st}\big)_{\substack{1 \leqslant s \leqslant n(i) \\ 1 \leqslant t \leqslant n(j)}} \in \mathbf{R}^{n(i) \times n(j)}, (i \neq j)$:

$$a(i,j)_{st} = \frac{-1}{\sigma_i + \sigma_j} b(i)_s b(j)_t^T, \quad \text{where } b(i)_s \text{ is the } s^{\text{th}} \text{ row of } B^i.$$

C**-matrix:**

$C^T = B$

Moreover, (A, B, C) as defined in (II) is balanced and has singular values $\sigma_1 > \sigma_2 > \dots > \sigma_k$ of multiplicities $n(1), n(2), \dots, n(k)$. The function which assigns to each asymptotically stable relaxation system of given order the realization given in (II) is a canonical form on the set of state-space realizations for relaxation systems. Each (A, B, C) in canonical form is of relaxation type, i.e $A^T = A$ and $C^T = B$.

Proof: See ([9]) □

As a corollary we state the previous theorem for SISO systems. In this case we obtain a particularly simple structure of the canonical form.

Corollary 2.3 : The following two statements are equivalent:

(I) $g(s)$ is the transfer function of some $(A, b, c) \in R_n^1$.

(II) $g(s)$ has a realization $(A, b, c) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times 1} \times \mathbf{R}^{1 \times n}$ given by

$$\begin{aligned} b &= (b_1, b_2, \dots, b_n)^T, & & b_i > 0 \quad \text{for } 1 \leqslant i \leqslant n \\ c &= b^T, & & \\ A &= \left[\frac{-b_i b_j}{\sigma_i + \sigma_j}\right]_{1 \leqslant i,j \leqslant n}, & & \sigma_1 > \sigma_2 > \dots > \sigma_n > 0 \end{aligned}$$

Moreover, (A, b, c) as defined in (II) is balanced and has singular values $\sigma_1 > \sigma_2 > \dots > \sigma_n$ of multiplicities 1. The function which assigns to each asymptotically stable relaxation system of given order the realization given in (II) is a canonical form on the set of state-space realizations for SISO relaxation systems. Each (A, b, c) in canonical form is of relaxation type, i.e. $A^T = A$, $c^T = b$. □

Remark 2.4 : The above canonical form is a special case of a balanced canonical form for multivariable symmetric and asymptotically stable systems ([9]). A particular feature of this canonical form for symmetric systems is that a system (A, B, C) in canonical form is internally symmetric, i.e. there is a sign matrix $S = \operatorname{diag}(s_1, s_2, \dots, s_n), s_i = \pm 1$, such that $A^T = SAS$ and $C^T = SB$. Since by ([2]) the (matrix –) Cauchy index of an internally symmetric system is given by the trace of S the set R_n^m can be characterized as the subset of $C_n^{m,m}$ with Cauchy index n. □

Remark 2.5 : Note that by the above theorem a relaxation system given in canonical form is completely determined by its set of singular values and its B–matrix. Thus a SISO relaxation system (A,b,c) given in canonical form is parametrized by the vector $(\sigma_1,\sigma_2,\ldots,\sigma_n)\times(b_1,b_2,\ldots,b_n)$, $\sigma_1 > \sigma_2 > \ldots > \sigma_n$ or equivalently by $(\sigma_1-\sigma_2,\sigma_2-\sigma_3,\ldots,\sigma_{n-1}-\sigma_n,\sigma_n)\times(b_1,b_2,\ldots,b_n)\in\mathbf{R}_+^{2n}$, where $\mathbf{R}_+^{2n} := \{(x_i)_{1\leqslant i\leqslant 2n} \mid x_i > 0,\ 1\leqslant i\leqslant 2n\}$. Since each such vector in turn parametrizes a system in R_n^1 we have that R_n^1 can be identified with $\mathbf{R}_+^{2n}$. □

We are now in a position to study the connectivity properties of the set of asymptotically stable relaxation systems, i.e. the quotient space of state-space representations of relaxation systems with respect to system equivalence. The following theorem asserts that there is only one pathwise connected component irrespective of the dimensions of the input and output spaces. To define a topology on the set of asymptotically stable relaxation systems we proceed in the standard way. We embed R_n^m in $\mathbf{R}^{n^2+2nm}$ with the natural topology and consider the quotient space with respect to system equivalence $(\sim)$.

Theorem 2.6 : $R_n^m/\!\sim$ is pathwise connected for all $m\geqslant 1$.
Proof: We will show that to each system (A,B,C) in R_n^m there is a continuous path in R_n^m connecting it to (A_0,B_0,C_0) in R_n^m, where

$$B_0=\begin{pmatrix}1 & 0 & \ldots & 0\\ 1 & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 1 & 0 & \ldots & 0\end{pmatrix}\in\mathbf{R}^{n\times m},\qquad C_o^T=B_o$$

$$A_0=\left(\frac{-1}{\sigma_i+\sigma_j}\right)_{1\leqslant i,j\leqslant n}\qquad \sigma_1=n,\ \sigma_2=n-1,\ldots,\sigma_n=1$$

The result then follows by the continuity of the natural projection $\pi: R_n^m\to R_n^m/\!\sim$.

Assume (A,B,C) is given in the canonical form of Theorem 2.2 with singular values $\sigma_1>\sigma_2>\ldots>\sigma_k$ of multiplicities $n(1),\ldots,n(k)$.
Case 1: $n(j)=1$ for all $1\leqslant j\leqslant k=n$. Since continuous changes of the singular values imply continuous changes of (A,B,C) in R_n^m as long as the multiplicity of any singular value is not increased, we can assume that $\sigma_1=n,\ \sigma_2=n-1,\ldots,\sigma_n=1$. Also, continuous changes in B result in a continuous path in R_n^m provided the first nonzero entry of each row of B is positive at any point of the path. Hence we can assume that $B=B_0$ which implies $(A,B,C)=(A_0,B_0,C_0)$.
Case 2: $n(j)>1$ for some $1\leqslant j\leqslant k$.

In this case repeated application of the following Lemma will give the result.

Lemma 2.7 : Let $(A,B,C)\in R_n^m$ be given in canonical form and parametrized by

$$\begin{aligned}\Sigma &= \operatorname{diag}(\sigma_1 I_{n(1)},\sigma_2 I_{n(2)},\sigma_3 I_{n(3)},\ldots,\sigma_k I_{n(k)})\\ &= \operatorname{diag}(nI_{n(1)},(n-n(1))I_{n(2)},(n-n(1)-n(2))I_{n(3)},\ldots,n(k)I_{n(k)})\\ \text{and}\quad B &= (B^1,B^2,\ldots,B^k)^T,\qquad B^j\in\mathbf{R}^{n(j)\times m}\end{aligned}$$

If $n(j_0)>1$ for some $1\leqslant j_0\leqslant k$, then there is a continuous path in R_n^m connecting (A,B,C) with $(\tilde A,\tilde B,\tilde C)$ where $(\tilde A,\tilde B,\tilde C)$ is parametrized by

$$\tilde\Sigma=\operatorname{diag}(\sigma_1 I_{n(1)},\sigma_2 I_{n(2)},\ldots,\sigma_{j_0} I_{n(j_0)-1},\sigma_{j_0-1}+1,\ldots,\sigma_k I_{n(k)}),\ \text{where } \sigma_0=0$$

and $\tilde B=(B^1,B^2,\ldots,\tilde B^{j_0},\ldots,B^k)^T$, where the first $n(j_0)-1$ rows of B^{j_0} and $\tilde B^{j_0}$ are identical and the last row of B^{j_0} is replaced by $(1\ 0\ 0\ldots0)$ in $\tilde B^{j_0}$.

Proof: Recall that relaxation systems are completely determined by their singular values and their B-matrices. To connect $(\tilde{A}, \tilde{B}, \tilde{C})$ with (A, B, C) we first continuously change the last row of $\tilde{B}^{j_0}$, i.e. $(1\ 0\ 0 \ldots 0)$, to the last row of $\tilde{B}^{j_0}$ such that the first nonzero entry of this row remains positive throughout. Hence the path lies in R_n^m by Theorem 2.2. Also note that continuous changes in B imply continuous changes in $(\tilde{A}, \tilde{B}, \tilde{C})$ which implies the continuity of the path. We obtain a system $(\tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{C}})$ such that $\tilde{\tilde{B}} = B$,

$$\tilde{\tilde{A}} = \begin{pmatrix} A_{11} & \tilde{\tilde{A}}_{12} & A_{13} \\ \tilde{\tilde{A}}_{21} & \tilde{\tilde{A}}_{22} & \tilde{\tilde{A}}_{23} \\ A_{31} & \tilde{\tilde{A}}_{32} & A_{33} \end{pmatrix} \quad \text{with} \quad \begin{array}{l} A_{11} \in \mathbf{R}^{(n(1)+\ldots+n(j_0-1))\times(n(1)+\ldots+n(j_0-1))} \\ \tilde{\tilde{A}}_{22} \in \mathbf{R}^{n(j_0)\times n(j_0)} \end{array}$$

Then A_{11}, A_{13}, A_{31} and A_{33} are identical to the corresponding blocks of A. The orthogonality of the rows of B^{j_0} implies that the off-diagonal terms of $\tilde{\tilde{A}}_{22}$ are zero and $\tilde{\tilde{A}}_{22}$ is given by

$$\tilde{\tilde{A}}_{22} = \mathrm{diag}\left(\frac{-b^1(b^1)^T}{2\sigma_{j_0}}, \ldots, \frac{-b^{n(j_0)-1}(b^{n(j_0)-1})^T}{2\sigma_{j_0}}, \frac{-b^{n(j_0)}(b^{n(j_0)})^T}{2(\sigma_{j_0-1}+1)}\right),$$

where b^i is the i^{th} row of B^{j_0}.

Next, we continuously change the singular value $\sigma_{j_0-1}+1$ to σ_{j_0}, which implies a continuous path connecting $(\tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{C}})$ with (A, B, C). □ □

3. Hankel Operators

An important feature of a balanced realization is the close relationship to its corresponding Hankel operator. Here we consider an integral Hankel operator with kernel $H(t) \in \mathbf{R}^{p\times m}$, $t \geqslant 0$, given by

$$\Gamma: L^2_{\mathbf{R}^m}([0,\infty[) \to L^2_{\mathbf{R}^p}([0,\infty[)$$
$$u(t) \mapsto (\Gamma(u))(s) = \int_0^\infty H(t+s)u(t)dt$$

Recall (see e.g. [12]) that the singular values $\sigma_n(A), n \geqslant 1$, of a compact Hilbert space operator A are defined to be the ordered eigenvalues of the modulus $(A^*A)^{\frac{1}{2}}$, such that $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_i \geqslant \ldots$, taking into account their multiplicities.

The significance of a Hankel operator in a system theoretic context is that for a system $(A, B, C) \in C_n^{p,m}$ the Hankel operator with kernel $H(t) := Ce^{tA}B$ can be interpreted as an operator mapping past inputs to future outputs. Conversely, standard realization theory shows that for any finite rank Hankel operator with kernel $H(t) \in \mathbf{R}^{p\times m}$, $t \geqslant 0$, there is a $(A, B, C) \in C_n^{p,m}$ such that $H(t) = Ce^{tA}B$, $t \geqslant 0$, almost everywhere. It can be easily verified that for a balanced realization $(A, B, C) \in C_n^{p,m,b}$ we have that the singular values of (A, B, C) equal the nonzero singular values of the corresponding Hankel operator (see e.g. [4]).

Under certain conditions on Γ, e.g. compactness, there exist pairs of normalized vectors $(v_i, w_i)_{i\geqslant 1}$, the so-called Schmidt-pairs, such that for $i \geqslant 1$, $\Gamma v_i = \sigma_i w_i, \Gamma^* w_i = \sigma_i v_i$.

If $(A, B, C) \in C_n^{p,m,b}$, then $v_i(t) = \frac{1}{\sqrt{\sigma_i}} B^T e^{tA^T} e_i$ and $w_i(t) = \frac{1}{\sqrt{\sigma_i}} Ce^{tA} e_i$, $1 \leqslant i \leqslant n$. Since $v_i(0) = \frac{1}{\sqrt{\sigma_i}} B^T e_i$ and $w_i(0) = \frac{1}{\sqrt{\sigma_i}} Ce_i$, $1 \leqslant i \leqslant n$, the starting points of the Schmidt vectors are fully determined by the B and C matrices and the singular values.

Notice that if we consider the Hankel operator Γ corresponding to a symmetric system (A, B, C), the eigenvectors and Schmidt-vectors of Γ coincide, since Γ is selfadjoint. If moreover (A, B, C) is internally symmetric then the eigenvectors respectively Schmidt-vectors are given by $v_i(t) = \frac{1}{\sqrt{\sigma_i}} B^T e^{tA^T} e_i = \frac{s_i}{\sqrt{\sigma_i}} Ce^{tA} e_i$, $1 \leqslant i \leqslant n$, and the eigenvalue corresponding to v_i is given by

$$(\Gamma(v_i))(t) = \int_0^\infty H(t+s)v_i(s)ds = \frac{1}{\sqrt{\sigma_i}} \int_0^\infty Ce^{(t+s)A} BB^T e^{tA^T} Se_i dt = s_i \sigma_i v_i(t),$$

where $S =: \operatorname{diag}(s_1, s_2, \ldots, s_n)$ is the sign-matrix corresponding to (A, B, C).

The following theorem characterizes relaxation systems in terms of Hankel operators with nonnegative spectrum.

Theorem 3.1 : Let $H(t) = H^T(t) \in \mathbf{R}^{m\times m}$, $t \geqslant 0$, be the kernel of a Hankel operator Γ on $L^2_{\mathbf{R}^m}([0,\infty[)$ of rank n having only nonnegative eigenvalues. Then $H(t) = Ce^{tA}B$, $t \geqslant 0$, almost everywhere, where (A, B, C) is a minimal realization of an n-dimensional asymptotically stable relaxation system.

Conversely, let (A, B, C) be a minimal realization of an n-dimensional asymptotically stable relaxation. Then $H(t) := Ce^{tA}B$, $t \geqslant 0$, is the kernel of a Hankel operator Γ of rank n having only nonnegative eigenvalues.

Proof: By standard realization theory there exists an asymptotically stable and minimal n-dimensional state-space realization (A, B, C) of $H(t)$, i.e. $H(t) = Ce^{tA}B$, $t \geqslant 0$, a.e.. Since $H(t)$ is symmetric we can assume by that (A, B, C) has a balanced and internally symmetric realization (see Remark 2.4). Since Γ has positive nonzero eigenvalues, the above remarks imply that for the sign-symmetry matrix we have $S = I$. Hence (A, B, C) is a relaxation system. The convers follows immediately by the above remarks. □

The following corollary shows that there is a bound on the multiplicity of a singular value of a Hankel operator with positive spectrum.

Corollary 3.2 : Let $H(t) \in \mathbf{R}^{m\times m}$, $t \geqslant 0$, be the kernel of a Hankel operator Γ acting on $L^2_{\mathbf{R}^m}([0,\infty[)$ such that $H(t) = H^T(t)$, $t \geqslant 0$. If Γ is of finite rank and has only nonnegative eigenvalues, then the multiplicity of each of the eigenvalues and singular values of Γ is at most m.

Proof: Since Γ is selfadjoint and has nonnegative spectrum we have that the notions of eigenvalue and singular value are equivalent. The result follows by inspection of the canonical form of Theorem 2.2. □

Remark 3.3 : The above theorem, together with Remark 2.5, implies that a finite rank Hankel operator with nonnegative spectrum is uniquely determined by its singular values and the starting points of its Schmidt-vectors. □

We are going to consider a generalization of the notion of relaxation systems to an infinite dimensional setting. The basis for this generalization is Theorem 3.1, in which relaxation systems are characterized by Hankel operators with nonnegative spectrum. The next theorem states that infinite dimensional nuclear Hankel operators with nonnegative spectrum can be characterized by infinite dimensional balanced systems with selfadjoint generator. For background material concerning Hilbert spaces see ([12]). A reference for semigroup theory is ([11]).

Theorem 3.4 : Let $h(t) \in L^2_{\mathbf{R}}([0,\infty[) \cap L^1_{\mathbf{R}}([0,\infty[)$ be such that $h(t)$ is the kernel of a Hankel operator Γ on $L^2_{\mathbf{R}}$ $([0,\infty[)$ such that

H(i) Γ is nuclear, i.e. $\sum_{i=1}^{\infty} \sigma_i(\Gamma) < \infty$.

H(ii) all nonzero eigenvalues of Γ are positive.

H(iii) $\sum_{i=1}^{\infty} \sigma_i(\Gamma)|v_i(0)|^2 < \infty$ where $(v_i(t),\, w_i(t))$ is the i^{th} Schmidt pair of Γ.

Then $(\sigma_i(\Gamma))_{i\geqslant 1}$ and $(b_i)_{i\geqslant 1}$ with $b_i = \sqrt{\sigma_i(\Gamma)}|v_i(0)|$ parametrize an infinite dimensional balanced system (A, b, b^T) with state-space ℓ^2 such that $h(t) = < b, e^{tA}b >, t \geqslant 0$, almost everywhere, where A has the matrix representation

$$A = \left(\frac{-b_i b_j}{\sigma_i(\Gamma) + \sigma_j(\Gamma)}\right)_{1\leqslant i,j<\infty}.$$

Conversely, let $b = (b_i)_{i\geqslant 1} \in \ell^2, b_i > 0, \quad (\sigma_i)_{i\geqslant 1}, \quad \sigma_1 > \sigma_2 > \ldots > \sigma_i > \ldots > 0$ be such that $\sum_{i=1}^{\infty} \sigma_i < \infty$ and $A_{min} : D_{min} = D(A_{min}) \to \ell^2; \quad x \mapsto \tilde{A}x = (\sum_{j=1}^{\infty} a_{ij}x_j)_{1\leqslant i<\infty}$ with $D_{min} = \text{span}\{e_i | 1 \leqslant i < \infty\}$, where e_i is the i^{th} element of the standard basis of ℓ^2 and

$$\tilde{A} = \left(\frac{-b_i b_j}{\sigma_i + \sigma_j}\right)_{1\leqslant i,j<\infty}$$

Then A_{min} is well-defined and admits a closure $A := \overline{A}_{min}$ such that A generates a semigroup of contractions $(e^{tA})_{t\geqslant 0}$. Then the impulse response $h(t) =< b, e^{tA}b >$, $t \geqslant 0$, of the system (A, b, c) is the kernel of a Hankel operator Γ with singular values $\sigma_n(\Gamma) = \sigma_n$, for all $n \in \mathbb{N}$, such that properties H(i), H(ii) and H(iii) hold and $h(t) \in L^1_{\mathbb{R}}([0,\infty[) \cap L^2_{\mathbb{R}}([0,\infty[)$.

Proof: Part 1 follows by applying the realization results of ([3],[5]) to our case. The conditions on Γ guarantee the existence of a balanced realization (A, b, c) on the state-space ℓ^2 such that $h(t) = < c, e^{tA}b >, t \geqslant 0$, almost everywhere.

It can be shown ([6]) by D_0-chain arguments as in ([1]) that under the present assumptions Γ has only distinct singular values.

Since all nonzero eigenvalues of Γ are positive, we have that $w_i(t) = v_i(t)$ almost everywhere, $i \geqslant 1$. Hence $c_i = \sqrt{\sigma_i(\Gamma)}w_i(0) = \sqrt{\sigma_i(\Gamma)}v_i(0) = b_i$, $i \geqslant 1$. As $v_i(0) \neq 0$, $i \geqslant 0$, we have $b_i \neq 0$ and hence w.l.o.g. $b_i > 0$. $b \in \ell^2$ follows since $\sum_{i=1}^{\infty} b_i^2 = \sum_{i=1}^{\infty} \sigma_i(\Gamma)|v_i(0)|^2 < \infty$

Moreover, the domain of A contains the finite sequences ([3]) and A has the matrix representation

$$A = \left(\frac{-b_i b_j}{\sigma_i + \sigma_j}\right)_{1\leqslant i,j<\infty}$$

To show the converse, note that A_{min} is well-defined since the rows of $\tilde{A}$ are in ℓ^2. By Theorem 5.2 in ([10]) we have that the closure $A := \overline{A}_{min}$ of A_{min} exists and that $\int_0^\infty e^{tA}bb^Te^{tA}dt = \text{diag}(\sigma_1, \sigma_2, \ldots)$. By Theorem 4.3 in ([10]) and Corollary VI.5.3 in ([9]) we obtain that $h(t) :=< b, e^{tA}b >\in L^1_{\mathbb{R}}([0,\infty[) \cap L^2_{\mathbb{R}}([0,\infty[)$. This implies that $h(t)$ is the kernel of a compact Hankel operator Γ. It can be easily verified that for $i \geqslant 1$, σ_i is a singular value of Γ with Schmidt vectors $w_i(t) = v_i(t) = \frac{1}{\sqrt{\sigma_i}} < b, e^{tA}e_i >, \quad i \geqslant 1$. To show that indeed all nonzero singular values are given by $(\sigma_n)_{n\geqslant 1}$ consider $E := \overline{span}\{< b, e^{At}e_i > | 1 \leqslant i < \infty\}$. For $u \in E^\perp$, the orthogonal complement of E, we have for all $1 \leqslant i < \infty$,

$$0 =< u(t), < b, e^{At}e_i >>_{L^2} = \int_0^\infty < b, e^{tA}e_i > u(t)dt =< e_i, \int_0^\infty e^{tA}bu(t)dt > .$$

Thus $\int_0^\infty e^{tA}bu(t)dt = 0$ and hence $(\Gamma(u))(s) =< b, e^{sA} \int_0^\infty e^{tA}bu(t)dt >= 0$ for all $s \in [0,\infty[$. So $E^\perp = \text{Ker}(\Gamma)$. This implies H(i). H(ii) is a consequence of the fact that $\Gamma v_i = \sigma_i w_i = \sigma_i v_i$. Since $\sum_{i=1}^{\infty} \sigma_i |v_i(0)|^2 = \sum_{i=1}^{\infty} b_i^2 < \infty$ we have H(iii). □

Remark 3.5 : Analogously to Remark 2.5 we have that infinite dimensional relaxation systems as defined through their associated Hankel operators having nonnegative spectrum are parametrized by the set

$$\{(\sigma_1-\sigma_2, \sigma_2-\sigma_3, \ldots, \sigma_{i-1}-\sigma_i, \ldots)\times(b_1, b_2, \ldots, b_i, \ldots) | \sigma_1 > \sigma_2 > \ldots, \sum_{i=1}^{\infty} \sigma_i < \infty;\ b_i > 0,\ i \geqslant 1\} = \ell^1_+ \times \ell^2_+,$$

where $\ell^{1\,(2)}_+ := \{(x_i)_{i\geqslant 1} \in \ell^{1\,(2)} | x_i > 0,\ i \geqslant 1\}$. □

Remark 3.6 : Let (A, b, b^T) be an infinite dimensional balanced system as defined in Theorem 3.4 and let $(A(n), b(n), b(n)^T)$, $n \geqslant 1$, be the finite dimensional balanced approximants of (A, b, b^T) given by

$$b(n) = (b_1, b_2, \ldots, b_n)^T, \qquad A(n) = \left(\frac{-b_i b_j}{\sigma_i + \sigma_j}\right)_{1 \leqslant i,j \leqslant n}.$$

Then by Corollary 2.3 $(A(n), b(n), b(n)^T)$, $n \geqslant 1$, is in R^1_n. Using the techniques of ([9],[10]) it can be shown that the following approximation results hold for the semigroups and the impulse responses $h_n(t)$ of the approximating systems.

(i) $\|e^{tA(n)} P_n x - P_n e^{tA} x\| \to 0$, uniformly for t in bounded intervals, $x \in \ell^2$, where $P_n((x_i)_{i \geqslant 1}) = (x_i)_{1 \leqslant i \leqslant n}$.

(ii) $\lim_{n \to \infty} h_n(t) = h(t)$ pointwise, in $L^2([0, \infty[)$ and in $L^1([0, \infty[)$ □

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