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systèmes à paramètres distribués

26 - 29 Juin 1989 / Perpignan

Edited by/Edité par

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Stability and Structural Properties of Infinite Dimensional Balanced Realizations

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Abstract

It is shown that a large class of nonrational transfer functions admit output normal realizations. These realizations are proved to be asymptotically stable. Asymptotic stability is also shown for several classes of balanced systems. It is established that a nonrational transfer function whose corresponding Hankel operator is compact has a sign symmetric realization.

Keywords: Asymptotic stability, sign symmetry, infinite dimensional continuous time systems, balanced realizations, output normal realizations

1 Introduction

Balanced realizations for finite-dimensional systems have received a great deal of attention. They were introduced as a means of performing model reduction in an easy fashion ([7]) and have subsequently been used in H^∞ control theory for example to evaluate the Hankel norm of a linear system ([4],[2]). Recently, they have been used to study parametrization problems of the set of stable linear systems ([12],[8]).

The elegant results obtained for finite-dimensional balanced systems brought about some interest in the problem of the extension of the notion of a balanced realization to infinite-dimensional systems. In [5] balanced realizations were derived for a class of systems with nuclear Hankel operator. Young ([17]) developed a very general realization theory for infinite-dimensional discrete-time systems. The approximation of infinite-dimensional balanced systems was considered in ([5], [11]).

In [13] the approach by Young ([17]) was extended to continuous time balanced realizations by studying a bilinear transformation between discrete-time and continuous-time systems. This work will be reviewed here since it is fundamental to the present investigation.

After a review of the realizations results for discrete-time balanced realizations (Section 2) and those for continuous-time balanced realizations (Section 3) we set out the main results in [13] concerning the relationship between discrete- and continuous-time systems. These results will then be used in Section 5 to establish the existence of so called output normal realizations. Such realizations are closely linked to balanced realizations and have in particular the same 'canonical' finite dimensional approximants (see e.g. [5]). It follows from the realization theory that the semigroups associated with balanced realizations as well as with output normal realizations are strongly continuous semigroups of contractions. The question of the stability of these semigroups had however not been studied before. Whereas it is possible to show that output normal realizations are asymptotically stable (Section 5) there is no complete answer for the case of balanced realizations (Section 6).

One reason for the current interest in finite dimensional balanced realizations are their interesting structural properties. Of them the fact that a scalar system admits a sign symmetric realization is of particular significance (see e.g. [16], [12]). In Section 6 it is shown that nonrational transfer functions whose corresponding Hankel operator is compact have an infinite dimensional balanced realization which is sign symmetric.

2 Discrete time balanced realizations

In this section we are first going to define a large class of infinite dimensional discrete-time state space systems and review some facts concerning these systems. Next we will quote a result by N. Young ([17]) which shows that a large class of nonrational transfer functions have a (par-) balanced realization.

Definition 2.1 *The quadruple of operators (A_d, B_d, C_d, D_d) is called an* admissible discrete-time system, *with state space X, output space Y and input space U, where X, U, Y are separable Hilbert spaces, if*

(i) $A_d \in \mathcal{L}(X)$ *is a contraction such that* $-1 \notin \sigma_p(A_d)$.

(ii) $B_d \in \mathcal{L}(U, X)$

(iii) $C_d \in \mathcal{L}(X, Y)$

(iv) $D_d \in \mathcal{L}(U, Y)$

(v) A_d, B_d, C_d *are such that* $\lim_{\substack{\lambda\to 1\\ \lambda>1}} C_d(\lambda I + A_d)^{-1}B_d$ *exists in the norm topology.*

We write $D_X^{U,Y}$ for the set of admissible discrete-time systems with input space U, output space Y and state space X. □

Before we are going to define what we mean by observability and reachability of an admissible discrete-time system we need to consider the notion of a dual system ([13]).

Definition 2.2 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, then the* dual system *$(\tilde A_d, \tilde B_d, \tilde C_d, \tilde D_d)$ of (A_d, B_d, C_d, D_d) is given by*

$$\begin{aligned} \tilde A_d &:= A_d^* : X \to X \\ \tilde B_d &:= C_d^* : Y \to X \\ \tilde C_d &:= B_d^* : X \to U \\ \tilde D_d &:= D_d^* : Y \to U \end{aligned} \qquad \square$$

The following Lemma states that a dual system of an admissible system is admissible and how the transfer function of a system is related to the transfer function of its dual system.

Lemma 2.1 *The dual system $(\tilde A_d, \tilde B_d, \tilde C_d, \tilde D_d)$ of an admissible discrete-time system (A_d, B_d, C_d, D_d) in $D_X^{U,Y}$ is an admissible system in $D_X^{Y,U}$.*

If the discrete time transfer function $G(s) : \mathbb{C}\backslash\bar{\mathbb{D}} \to \mathcal{L}(U, Y)$ has an admissible realization (A_d, B_d, C_d, D_d), then the dual system $(\tilde A_d, \tilde B_d, \tilde C_d, \tilde D_d)$ is a realization of the transfer function $\tilde G(s) : \mathbb{C}\backslash\bar{\mathbb{D}} \to \mathcal{L}(Y, U)$, $s \longmapsto \tilde G(s) := (G(\bar s))^*$, i.e. for all $s \in \mathbb{C}\backslash\bar{\mathbb{D}}$,

$$\tilde G(s) \quad (G(\bar s))^* \quad \tilde C_d(sI - \tilde A_d)^{-1}\tilde B_d + \tilde D_d$$

Next, we define the observability and reachability operators for discrete-time systems.

Definition 2.3 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, then the operator*

$$\begin{aligned} \mathcal{O}_d : D(\mathcal{O}_d) &\to \ell_Y^2 \\ x &\longmapsto (C_d A_d^n x)_{n\geq 0} \end{aligned}$$

is called the observability operator *of the system* (A_d, B_d, C_d, D_d), *where*

$$D(\mathcal{O}_d) = \{x \in X \mid (C_d A_d^n x)_{n\geq 0} \in \ell_Y^2\}\ .$$

If $\mathcal{O}_d$ *is bounded and* $\mathrm{Ker}(\mathcal{O}_d) = \{0\}$, *then the system* (A_d, B_d, C_d, D_d) *is called* observable.

Let $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ *be the dual system of* (A_d, B_d, C_d, D_d). *If the observability operator* $\tilde{\mathcal{O}}_d$ *of* $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ *is bounded (and hence* $D(\tilde{\mathcal{O}}_d) = X$*), then the adjoint of* $\tilde{\mathcal{O}}_d$ *is called the* reachability operator $\mathcal{R}_d$ *of* (A_d, B_d, C_d, D_d), *i.e.*

$$\mathcal{R}_d := \tilde{\mathcal{O}}_d^*.$$

If $\mathcal{R}_d$ *exists and Range(*$\mathcal{R}_d$*) is dense in* X, *the system* (A_d, B_d, C_d, D_d) *is called* reachable. □

The notion of reachability and observability gramians as defined below is central in the discussion of balanced realizations.

Definition 2.4 *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ *with bounded reachability operator* $\mathcal{R}_d$ *and bounded observability operator* $\mathcal{O}_d$. *Then*

$$\mathcal{W}_d := \mathcal{R}_d\mathcal{R}_d^* : X \to X$$
$$\mathcal{M}_d := \mathcal{O}_d^*\mathcal{O}_d : X \to X$$

are called the reachability *and the* observability gramian *of the system* (A_d, B_d, C_d, D_d). □

The following definition recalls the notion of a balanced system as defined by Moore ([7]) and the notion of a parbalanced as introduced by Young ([17]).

Definition 2.5 *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ *be such that the observability gramian* $\mathcal{M}_d$ *and reachability gramian* $\mathcal{W}_d$ *exist. Then the system is*

(i) parbalanced, *if* $\mathcal{M}_d = \mathcal{W}_d$.

(ii) balanced, *if it is parbalanced and moreover the gramians are diagonal.* □

Before we state any results, we introduce some notation. Let $H : \mathbf{D} \to \mathcal{L}(U,Y)$ be analytic. We say that $H \in P_+L^\infty(\mathbf{D}, \mathcal{L}(U,Y))$ if there exists an analytic function $F : \mathbf{D} \to \mathcal{L}(U,Y)$ such that $H + \bar{F}$ is essentially bounded, where $\bar{F}(z) = F(z^{-1})$. Further, if F can be chosen so that $H + \bar{F} \in C(\mathbf{D}, \mathcal{K}(U,Y))$, where $C(\mathbf{D}, \mathcal{K}(U,Y))$ is the set of norm continuous functions on $\partial\mathbf{D}$ with values in the set of compact operators from U to Y, then H is said to be in $P_+C(\mathbf{D}, \mathcal{K}(U,Y))$. Two systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are called *unitarily equivalent* if there exists a unitary operator V mapping the state space X_1 to the state space X_2, i.e. $V : X_1 \to X_2$ such that

$$(A_1, B_1, C_1, D_1) = (V^*A_2V, V^*B_2, C_2V, D_2).$$

The following theorem by Young ([17]), gives criteria for a (par-) balanced realization to exist of a discrete time transfer function.

Theorem 2.1 *Let* $G_d(z) : \mathbf{C}\backslash\bar{\mathbf{D}} \to \mathcal{L}(U,Y)$ *be analytic with* $G_d(\infty) = D_d \in \mathcal{L}(U,Y)$, *and write*

$$g(z) := \frac{1}{z}(G_d(\frac{1}{z}) - D_d)\ \ ,\ \ z \in \mathbf{D}\,.$$

(i) If $g \in P_+L^\infty(\mathbf{D}, \mathcal{L}(U,Y))$, *then there exists a separable Hilbert space* X *and a discrete-time state-space realization* (A_d, B_d, C_d, D_d) *of* $G_d(z)$ *with state space* X, *such that*

$$A_d \in \mathcal{L}(X) \quad \text{is a contraction}$$
$$B_d \in \mathcal{L}(U,X)$$
$$C_d \in \mathcal{L}(X,Y),$$

and (A_d, B_d, C_d, D_d) *is reachable and observable with bounded reachability and observability operators, such that* (A_d, B_d, C_d, D_d) *is parbalanced, i.e.* $\mathcal{M}_d = \mathcal{W}_d$.
The gramians $\mathcal{M}_d$, $\mathcal{W}_d$ *satisfy the Lyapunov equations*

$$A_d\mathcal{W}_dA_d^* - \mathcal{W}_d = -B_d^*B_d$$
$$A_d^*\mathcal{M}_dA_d - \mathcal{M}_d = -C_dC_d^*.$$

If $(\bar{A}_d, \bar{B}_d, \bar{C}_d, \bar{D}_d)$ *is another parbalanced realization of* $G_d(z)$ *with state space* $\bar{X}$, *then* (A_d, B_d, C_d, D_d) *and* $(\bar{A}_d, \bar{B}_d, \bar{C}_d, \bar{D}_d)$ *are unitarily equivalent.*

(ii) If moreover, $g \in P_+C(\mathbf{D}, \mathcal{K}(U,Y))$, *there is a basis in* X *with respect to which* (A_d, B_d, C_d, D_d) *is balanced.* □

3 Continuous time balanced realizations

In this section we are going to review the results in [13] concerning infinite dimensional continuous time state space systems and balanced realizations of nonrational transfer functions.

It is well known that if A is the generator of a strongly continuous semigroup of operators $(e^{tA})_{t\geq 0}$ with domain of definition $D(A)$, then $D(A)$ is a Hilbert space with inner product induced by the graph norm

$$\|x\|_A^2 := \|x\|_X^2 + \|Ax\|_X^2, \quad x \in D(A).$$

Since $\|x\|_A \geq \|x\|$ for $x \in D(A)$, we can embed X in $D(A)^{(\prime)}$, the set of antilinear continuous functionals on $(D(A), \|\cdot\|_A)$, by

$$E :\ X \to D(A)^{(\prime)}$$
$$x \longmapsto (y \longmapsto < x, y >).$$

Note that $D(A)^{(\prime)}$ is a Hilbert space with norm $\|f\|' := \sup_{\|x\|\leq 1} |f(x)|$. Since $< \cdot, \cdot >$ is linear in the first component, the embedding E is linear. By the above, we have the rigged structure

$$D(A) \subseteq X \subseteq D(A)^{(\prime)}.$$

It is well known that if $(A, D(A))$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t\geq 0}$ on a Hilbert space, then the adjoint $(A^*, D(A^*))$ of $(A, D(A))$ is the generator of the adjoint semigroup $(e^{tA})^*_{t\geq 0}$. Hence, we have similarly that

$$D(A^*) \subseteq X \subseteq D(A^*)^{(\prime)}\ \ .$$

If M is an operator on X such that $D(A^*) \subseteq X$ is invariant under M^*, then M can be extended to an operator $\tilde{M}$ on $D(A^*)^{(\prime)}$ by:

$$\tilde{M} :\ D(A^*)^{(\prime)} \to D(A^*)^{(\prime)}$$
$$f(\cdot) \longmapsto f(M^*(\cdot))\ \ .$$

We will normally not distinguish between M and $\tilde{M}$ and write M for $\tilde{M}$.

Also, if we have a map $M : Z \to D(A^*)^{(\prime)}, Z$ a Hilbert space, such that $M(Z) \subseteq X^{(\prime)} \subseteq D(A^*)^{(\prime)}$, we can consider $M : Z \to X$ using the Riesz representation theorem.

We are now in a position to define admissible continuous-time systems.

Definition 3.1 *A quadruple of operators* (A_c, B_c, C_c, D_c) *is called an* admissible continuous-time system *with state space* X, *input space* U *and output space* Y, *where* X, U, Y *are separable Hilbert spaces, if*

(i) $(A_c, D(A_c))$ *is the generator of a strongly continuous semigroup of contractions on* X.

(ii) $B_c : U \to (D(A_c^*)^{(\prime)}, \|\cdot\|')$ *is a bounded linear operator.*

(iii) $C_c : D(C_c) \to Y$ *is linear with*
$D(C_c) = D(A_c) + (I - A_c)^{-1}B_cU$ *and*
$C_{c|D(A_c)} : (D(A_c), \|\cdot\|_{A_c}) \to Y$ *is bounded.*

(iv) $C_c(I - A_c)^{-1}B_c \in \mathcal{L}(U,Y)$

(v) A_c, B_c, C_c *are such that* $\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}} C_c(sI - A_c)^{-1}B_c = 0$ *in the norm topology.*

(vi) $D_c \in \mathcal{L}(U,Y)$.

We write $C_X^{U,Y}$ *for the set of admissible continuous-time systems with input space* U, *output space* Y *and state space* X. □

Remark 3.1 *Helton ([6]) and Fuhrmann ([3]) gave a similar definition for continuous-time state-space systems. There are, however, several differences between so-called compatible systems and admissible systems as defined here. Our definition of a rigged Hilbert space is slightly different from that used in Helton and Fuhrmann, where* X *is embedded in the dual spaces* $D(A)'$ *and* $D(A^*)'$, *rather than in the spaces of antilinear functionals* $D(A)^{(\prime)}$ *and* $D(A^*)^{(\prime)}$ *as adopted here. The reason for using our definition is that this naturally leads to a definition of the input operator* B_c *as a linear, rather than an antilinear operator. Most importantly however for the discussion later is the imposition of (v) in our definition.* □

Before defining observability and reachability for continuous-time systems we need to introduce the notion of the dual system of an admissible continuous-time system.

Definition 3.2 *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$. Then the* dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ *of* (A_c, B_c, C_c, D_c) *is given by*

- $(\tilde{A}_c, D(\tilde{A}_c)) = (A_c^*, D(A_c^*))$.
- $\tilde{B}_c : Y \to D(A_c)^{(\prime)}; \quad y \longmapsto \tilde{B}_c(y)[\cdot] := < y, C_c(\cdot) >$
- $\tilde{C}_c : D(\tilde{C}_c) \to U, \quad D(\tilde{C}_c) = D(\tilde{A}_c) + (I - \tilde{A}_c)^{-1}\tilde{B}_c Y$, *where* $\tilde{C}_c x_0$ is defined by

$$\begin{cases} < u, \tilde{C}_c x_0 > & = B_c(u)[x_0], \\ & x_0 \in D(A_c^*), u \in U, \\ < \tilde{C}_c x_0, u > & =< y_0, C_c(I - A_c)^{-1} B_c u >, \\ & x_0 = (I - \tilde{A}_c)^{-1}\tilde{B}_c y_0,\ y_0 \in Y, u \in U, \end{cases}$$

- $\tilde{D}_c := D_c^* : Y \to U.$ □

The following Lemma is the continuous-time equivalent of Lemma 2.1.

Lemma 3.1 *The dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ of an admissible continuous-time system (A_c, B_c, C_c, D_c) is admissible. If the continuous-time transfer function $G(s) : RHP \to L(U,Y)$, has an admissible realization (A_c, B_c, C_c, D_c), then the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a realization of the transfer function $\tilde{G}(s) := (G(\bar{s}))^*$, i.e. for all $s \in RHP$,*

$$\tilde{G}(s) = (G(\bar{s}))^* = \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c + \tilde{D}_c. \qquad \square$$

The definitions of observability and reachability of admissible continuous time systems is now given.

Definition 3.3 *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, then the operator*

$$\begin{array}{rcl} \mathcal{O}_c : D(\mathcal{O}_c) & \to & L_Y^2([0,\infty[) \\ x & \longmapsto & C_c e^{tA_c} x \end{array}$$

is called the observability operator *of the system (A_c, B_c, C_c, D_c), where $D(\mathcal{O}_c) = \{x \in X \mid C_c e^{tA_c} x$ exists for almost all $t \in [0,\infty[,\ C_c e^{tA_c} x \in L_Y^2([0,\infty[)\}$. We say that (A_c, B_c, C_c, D_c) has a bounded observability operator if $D(A_c) \subseteq D(\mathcal{O}_c)$ and $\mathcal{O}_c$ extends to a bounded operator on X. This extension will also be denoted by $\mathcal{O}_c$.*

If (A_c, B_c, C_c, D_c) has bounded observability operator $\mathcal{O}_c$ such that $\mathrm{Ker}(\mathcal{O}_c) = \{0\}$, *then the system (A_c, B_c, C_c, D_c) is called* observable.

Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ be the dual system of (A_c, B_c, C_c, D_c). If the observability operator $\tilde{\mathcal{O}}_c$ of $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a bounded operator on X, the adjoint of $\tilde{\mathcal{O}}_c$ is called the reachability operator *$\mathcal{R}_c$ of (A_c, B_c, C_c, D_c), i.e.*

$$\mathcal{R}_c := \tilde{\mathcal{O}}_c^{\ *}.$$

If $\mathcal{R}_c$ exists and Range($\mathcal{R}_c$) is dense in X, the system (A_c, B_c, C_c, D_c) is called reachable. □

The observability and controllability gramians are defined as follows.

Definition 3.4 *The* reachability gramian $\mathcal{W}_c$ *and the* observability gramian $\mathcal{M}_c$ *of a continuous-time system with bounded reachability operator $\mathcal{R}_c$ and observability operator $\mathcal{O}_c$ are defined to be*

$$\begin{array}{l} \mathcal{W}_c := \mathcal{R}_c\mathcal{R}_c^* : X \to X \\ \mathcal{M}_c := \mathcal{O}_c^*\mathcal{O}_c : X \to X \,. \end{array} \qquad \square$$

The concept of a unitary state space transformation of an admissible continuous-time system is slightly more complicated than in the discrete time case.

Proposition 3.1 *Let $((A_c, D(A_c)), B_c, C_c, D_c) \in C_{X_1}^{U,Y}$. If X_2 is another Hilbert space and $V : X_1 \to X_2$ is a unitary operator, then*

(1) $((VA_cV^*, VD(A_c)), VB_c, (C_cV^*, VD(C_c)), D_c) \in C_{X_2}^{U,Y}$, where

$$(VB_c) : U \to \left((VD(A_c^*))^{(\prime)}, \|\cdot\|'\right)$$

is given by

$$(VB_c)(u)[x] := B_c(u)[V^*x]$$

$u \in U,\ x \in VD(A_c^*)$.

(2) If (A_c, B_c, C_c, D_c) is a state space realization of the transfer function

$$G_c(s) : RHP \to L(U,Y)\,,$$

then $(VA_cV^*, VB_c, C_cV^*, D_c)$ realizes the same transfer function. □

The following definition introduces the standard notation of unitary equivalence of state-space systems. Note that by the previous proposition, unitarily equivalent systems have the same transfer functions.

Definition 3.5 *Two systems $(A_c^i, B_c^i, C_c^i, D_c^i) \in C_{X_i}^{U,Y}, i = 1,2$, are called* unitarily equivalent, *if there exists a unitary operator $V : X_1 \to X_2$ such that*

$$(A_c^2, B_c^2, C_c^2, D_c^2) = (VA_c^1V^*, VB_c^1, C_c^1V^*, D_c^1)$$

□

A (par-) balanced realization for admissible continuous-time systems is defined as follows.

Definition 3.6 *Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ be such that the observability gramian $\mathcal{M}_c$ and reachability gramian $\mathcal{W}_c$ exist. Then the system is*

(i) parbalanced, *if $\mathcal{M}_c = \mathcal{W}_c$.*

(ii) balanced, *if it is parbalanced and moreover the gramians are diagonal.* □

If $H : RHP \to \mathcal{L}(U,Y)$ is analytic, we say that $H \in P_+L^\infty(RHP, \mathcal{L}(U,Y))$ $(P_+C(RHP, \mathcal{K}(U,Y))$ if there is an analytic function $F : RHP \to \mathcal{L}(U,Y)$ $(\mathcal{K}(U,Y))$ such that $H + \tilde{F}$ is essentially bounded (extends to a norm continuous function on the imaginary axis such that $\lim_{\substack{w\in\mathbb{R} \\ w\to\infty}} (H + \tilde{F})(iw) = \lim_{\substack{w\in\mathbb{R} \\ w\to\infty}} (H + \tilde{F})(-iw))$, where $\tilde{F}(s) = F(-s)$.

In [13] the following realization theorem was proved for non-rational continuous-time transfer functions.

Theorem 3.1 *Let $G_c : RHP \to \mathcal{L}(U,Y)$ be a continuous time transfer function, which is analytic and such that $\lim_{\substack{s\in\mathbb{R} \\ s\to\infty}} G_c(s) \in \mathcal{L}(U,Y)$ exists.*

(i) If $G_c \in P_+L^\infty(RHP, \mathcal{L}(U,Y))$, then there exists a separable Hilbert space X and a parbalanced admissible continuous time state space realization (A_c, B_c, C_c, D_c) of G_c with state space X. This system is reachable, observable and has bounded reachability and observability operators. If $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ is another par-balanced realization of $G_c(s)$, then (A_c, B_c, C_c, D_c) and $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ are unitarily equivalent.

(ii) If moreover $G_c \in P_+C(RHP, \mathcal{K}(U,Y))$, then there is a basis in X with respect to which (A_c, B_c, C_c, D_c) is balanced. □

As a corollary we give further sufficient conditions for the existence of (par-) balanced realizations.

Corollary 3.1 *Let $G_c(s) : RHP \to \mathcal{L}(U,Y)$ be a continuous time transfer function, such that $\lim_{\substack{s\in\mathbb{R} \\ s\to\infty}} G_c(s) \in \mathcal{L}(U,Y)$ exists and $G_c(s)$ is analytic in RHP.*

(i) If $G_c(s)$ is bounded in the RHP, i.e. $\sup_{s\in RHP} \|G_c(s)\| < \infty$, then $G_c(s)$ has a par-balanced realization.

(ii) If in particular $G_c(s) : RHP \mapsto \mathcal{K}(U,Y)$, such that G_c is bounded in the RHP and $G_c(s)$ is norm continuous on the imaginary axis including at the points $+\infty$ and $-\infty$, i.e. $w \longmapsto G_c(iw)$, $w \in \mathbb{R}$, is norm continuous and $\lim_{w\to-\infty} G_c(iw) = \lim_{w\to+\infty} G_c(iw)$, then $G_c(s)$ has a balanced realization. □

4 Connection between continuous and discrete time systems

The realization result quoted in Section 3 was proved in [13] by relating discrete time systems to continuous time systems using an infinite dimensional generalization of a well known bilinear transformation for finite dimensional systems. Thereby it was possible to carry Young's results for discrete time systems over to continuous time systems.

In the following theorem the map $T : D_X^{U,Y} \to C_X^{U,Y}$ is introduced.

Theorem 4.1 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, then $T((A_d, B_d, C_d, D_d)) := (A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, where*

(i) $A_c := (I + A_d)^{-1}(A_d - I) = (A_d - I)(I + A_d)^{-1}$, $D(A_c) := D((I + A_d)^{-1})$. A_c generates a strongly continuous semigroup of contractions on X given by $\varphi_t(A_d), t \geq 0$, with $\varphi_t(z) = e^{t\frac{z-1}{z+1}}$.

(ii) $B_c := \sqrt{2}(I + A_d)^{-1}B_d : U \to D(A_c^*)^{(\prime)}$

$$u \longmapsto \sqrt{2}(I + A_d)^{-1}B_d(u)[C_d]$$

(iii) $$:= \; < B_d(u), (I + A_d^*)^{-1}(C_d) >_X .$$

$$C_c : D(C_c) \to Y$$
$$x \longmapsto \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} \sqrt{2}C_d(\lambda I + A_d)^{-1}x,$$

where $D(C_c) = D(A_c) + (I - A_c)^{-1}B_cU$.
On $D(A_c)$ we have,

$$C_{c|D(A_c)} = \sqrt{2}C_d(I + A_d)^{-1}.$$

(iv) $D_c := D_d - \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} C_d(\lambda I + A_d)^{-1}B_d$.

Moreover, let the admissible discrete-time system (A_d, B_d, C_d, D_d) be a realization of the transfer function

$$G_d(z) : \mathbb{C} \setminus \bar{\mathbf{D}} \to \mathcal{L}(U, Y) ,$$

i.e. $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$ for $z \in \mathbb{C} \setminus \bar{D}$.

Then, $(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d))$ is an admissible continuous-time realization of the transfer function

$$G_c(s) := G_d\left(\frac{1+s}{1-s}\right) : RHP \to \mathcal{L}(U, Y) . \qquad \square$$

The inverse map is considered in the next theorem.

Theorem 4.2 *Let $(A_c, B_c, C_c, D_c) \in D_X^{U,Y}$, then $T^{-1}((A_c, B_c, C_c, D_c)) := (A_d, B_d, C_d, D_d) \in C_X^{U,Y}$, where*
(i) $A_d := (I + A_c)(I - A_c)^{-1}$, and for $x \in D(A_c)$ we have that $A_d x = (I - A_c)^{-1}(I + A_c)x$.
(ii) $B_d := \sqrt{2}(I - A_c)^{-1}B_c$
(iii) $C_d := \sqrt{2}C_c(I - A_c)^{-1}$
(iv) $D_d := C_c(I - A_c)^{-1}B_c + D_c$.
Moreover, let the admissible continuous-time system (A_c, B_c, C_c, D_c) be a realization of the transfer function

$$G_c(s) : RHP \to \mathcal{L}(U, Y) ,$$

i.e., $G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$ for $s \in RHP$.
Then, $(A_d, B_d, C_d, D_d) = T^{-1}((A_c, B_c, C_c, D_c))$ is an admissible discrete-time realization of the transfer function

$$G_d(z) := G_c\left(\frac{z-1}{z+1}\right) : \mathbb{C} \setminus \bar{D} \to \mathcal{L}(U, Y) . \qquad \square$$

Next we are going to quote two technical results which will be of importance in the following sections. The first one states that the map T preserves the unitary equivalence of systems.

Proposition 4.1 *Let $(A_d^i, B_d^i, C_d^i, D_d^i) \in D_{X_i}^{U,Y}$, $i = 1, 2$. Let $(A_c^i, B_c^i, C_c^i, D_c^i) := T((A_d^i, B_d^i, C_d^i, D_d^i))$, $i = 1, 2$, be the associated continuous-time systems.*

Then, $(A_c^1, B_c^1, C_c^1, D_c^1)$ and $(A_c^2, B_c^2, C_c^2, D_c^2)$ are unitarily equivalent with unitary state space transformation V if and only if $(A_d^1, B_d^1, C_d^1, D_d^1)$ and $(A_d^2, B_d^2, C_d^2, D_d^2)$ are unitarily equivalent with unitary state space transformation V. $\square$

The next proposition states that similarly the duality of systems is preserved under the map T.

Proposition 4.2 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ and define $(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$. Let $(A_d^1, B_d^1, C_d^1, D_d^1) \in D_X^{U,Y}$ be another discrete time system and let $(A_c^1, B_c^1, C_c^1, D_c^1) := T((A_d^1, B_d^1, C_d^1, D_d^1))$ be its corresponding admissible continuous-time system. Then,*

$(A_d^1, B_d^1, C_d^1, D_d^1)$ is the dual system of (A_d, B_d, C_d, D_d)
if and only if
$(A_c^1, B_c^1, C_c^1, D_c^1)$ is the dual system of (A_c, B_c, C_c, D_c). $\square$

The following theorem shows that observability and reachability as well as the gramians are preserved under T. This implies that the transformation preserves balancing of systems.

Theorem 4.3 *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ and $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ such that*

Then, $$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d))$$
(1) (A_c, B_c, C_c, D_c) is observable (reachable) if and only if (A_d, B_d, C_d, D_d) is observable (reachable).
(2) if the reachability gramians $\mathcal{W}_c, \mathcal{W}_d$ (observability gramians $\mathcal{M}_c, \mathcal{M}_d$) of (A_d, B_d, C_d, D_d) and (A_c, B_c, C_c, D_c) are defined, then

$$\mathcal{W}_c = \mathcal{W}_d \qquad (\mathcal{M}_c = \mathcal{M}_d). \qquad \square$$

5 Output normal realizations

In this section we are going to define what we mean by output normal realizations. Such realizations were first defined by Moore ([7]) and then for infinite dimensional discrete time systems by Young ([17]). They are of particular interest here because of their strong connections to balanced realizations. The main result of this section is a theorem stating that admissible continuous time output normal realizations are asymptotically stable. The approach will be to prove asymptotic stability for discrete time output normal realizations and to carry this result over to the continuous time case using the transformation introduced in Section 4.

Output normal realizations are defined to be those for which the observability gramian $\mathcal{W}_d$, $(\mathcal{W}_c)$ is the identity.

Before quoting a slightly modified version of a result by Young ([17]) on the existence and properties of output normal realizations we need the following Lemma.

Lemma 5.1 *Let (A_d, B_d, C_d, D_d) be a discrete-time system (not necessarily admissible) system such that A_d is a contraction and B_d, C_d and D_d are bounded. If (A_d, B_d, C_d, D_d) is reachable or observable then $\sigma_p(A_d) \subseteq \mathbf{D}$.*

Proof:
Assume (A_d, B_d, C_d, D_d) is observable with observability gramian $\mathcal{M}_d$ then it is easily verified that

$$A_d^*\mathcal{M}_d A_d - \mathcal{M}_d = -C_d^*C_d .$$

Assume $\lambda \in \sigma_p(A_d)$, $|\lambda| = 1$ with eigenvector $x \neq 0$, then

$$< x, A_d^*\mathcal{M}_d A_d x > - < x, \mathcal{M}_d x > = - < x, C_d^*C_d x >$$

and hence

$$< A_d x, \mathcal{M}_d A_d x > - < x, \mathcal{M}_d x > = 0 = -\|C_d x\|^2$$

which implies that $C_d x = 0$. Hence for all $n \geq 0$, $\mathcal{O}_d x := (C_d A_d^n x)_{n \geq 0} = (\lambda)^n C_d x = 0$ which is a contradiction to the observability of (A_d, B_d, C_d, D_d).

If (A_d, B_d, C_d, D_d) is reachable the result follows by considering the dual system. $\square$

We now state the result by Young ([17]). For the precise definition of a restricted shift realization see e.g. [3], [17].

Theorem 5.1 *With the same notation and the same conditions as in Theorem 2.1 the transfer function G_d has an output normal reachable ande observable realization (A_d, B_d, C_d, D_d) where A_d is a contraction and B_d, C_d and D_d are bounded. This realization is unitarily equivalent to a restricted shift realization.*
$\square$

We call a discrete-time system (A_d, B_d, C_d, D_d) *asymptotically stable* if $A_d^n x \to 0$, for all $x \in X$. Similarly, a continuous-time system (A_c, B_c, C_c, D_c) is called *asymptotically stable* if $e^{tA_c}x \to 0$ for all $x \in X$.

The following theorem shows that output normal realizations of transfer functions considered in Theorem 2.1 and Theorem 3.1 are asymptotically stable.

Theorem 5.2 *Let G_d be a discrete-time transfer function defined as in Theorem 2.1 then an output normal realization of G_d is asymptotically stable.*

If G_c is a continuous-time transfer functions as defined in Theorem 3.1 then G_c admits an observable and reachable output normal realization in $C_X^{U,Y}$. Such realizations are asymptotically stable.

Proof:
First consider the case of discrete-time systems. We know by Theorem 5.1 that G_d admits an output normal realization which is unitarily equivalent to a restricted shift realization (A_s, B_s, C_s, D_s). Hence the output normal realization is asymptotically stable if and only if the restricted shift realization is asymptotically stable. But A_s is a left shift on a Hilbert space and is therefore asymptotically stable.

Now consider the continuous-time case. Let G_c be as in Theorem 3.1 and let $G_d : \mathbb{C} \setminus \bar{\mathbf{D}} \to \mathcal{L}(U,Y)$ be the associated discrete-time transfer function $G_d(z) = G_c\left(\frac{z-1}{z+1}\right)$. It is straightforward to verify that G_d satisfies the conditions of Theorem 2.1. Hence G_d has an asymptotically stable minimal observable and reachable realization (A_d, B_d, C_d, D_d) which is output normal and admissible by Lemma 5.1. Then Theorem 4.1 shows that $(A_c, B_c, C_c, D_c) := T(A_d, B_d, C_d, D_d)$ is a reachable and observable realization of G_c which is output normal by Theorem 4.3. The asymptotic stability of (A_c, B_c, C_c, D_c) follows from Proposition 9.1, p.148, in [14]. There it is shown that the Cayley transformation preserves asymptotic stability, i.e. A_d is asymptotically stable if and only if $(e^{tA_c})_{t\geq 0}$ is asymptotically stable.
□

6 Sign symmetry and stability of balanced realizations

In the previous section we showed that a large class of transfer functions have asymptotically stable output normal realizations. In this section we address the stability problem for balanced realizations. It can be easily seen that in order to balance an output normal system it is necessary to perform a state space transformation with a bounded operator whose inverse is not necessarily bounded. Since the asymptotic stability is not necessarily preserved under such a state space transformation we can not generally deduce the asymptotic stability of a balanced realization from the asymptotic stability of the corresponding output normal realization. In what follows we shall however establish the asymptotic stability of balanced realizations at least for several interesting special cases.

First we are going to discuss the existence of so called *sign symmetric realizations* for *real symmetric* transfer functions. We call a discrete time (continuous-time) transfer function real symmetric if $G(z) = (G(\bar{z}))^*$ for all $z \in \mathbb{C} \setminus \bar{\mathbf{D}}$ ($z \in RHP$). In finite dimensions they have been extensively studied (for a study without the imposition of the balancing constraint see e.g. [15]). The existence of sign symmetric balanced realizations for rational transfer functions has been shown for example in ([16] [8]).

The following Lemma shows an interesting property of parbalanced realizations of real symmetric transfer functions.

Lemma 6.1 *Let G_d satisfy the conditions of Theorem 2.1 (i) and assume that G_d is real symmetric. If (A_d, B_d, C_d, D_d) is a parbalanced realization of G_d with gramian Σ then the dual system is again a parbalanced realization of G_d with gramian Σ.*

Proof:
For $z \in \mathbf{D}$ we have

$$\begin{aligned} G_d(z) &= (G_d(\bar{z}))^* \\ &= (C_d(\bar{z}I - A_d)^{-1}B_d + D_d)^* \\ &= B_d^*(zI - A_d^*)^{-1}C_d^* + D_d^*, \end{aligned}$$

which shows that the dual of (A_d, B_d, C_d, D_d) is a realization of G_d. That it is a parbalanced realization with the same gramian follows from the definition of the observability and controllability gramians. □

The following Lemma will be useful later.

Lemma 6.2 *Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two admissible discrete-time systems which are either controllable or observable. If the two systems are unitarily equivalent with respect to a unitary state space transformation V, then V is unique.*

Proof:
Assume the two systems are observable and that there are two unitary operators V_j, $j = 1, 2$, such that

$$V_j A_1 V_j^* = A_2, \quad V_j B_1 = B_2, \quad C_1 V_j^* = C_2, \quad j = 1, 2.$$

Then we have for the observability operators $\mathcal{O}_i$, $i = 1, 2$, of the two systems that

$$\mathcal{O}_1 = \mathcal{O}_2 V_j^*, \quad j = 1, 2,$$

and hence

$$\mathcal{O}_2(V_1^* - V_2^*) = 0.$$

This implies that $V_1 = V_2$ since $\mathcal{O}_2$ has zero kernel. If the system is reachable the statement follows by duality. □

The following theorem states in part (1) an important uniqueness property of balanced realizations and in part (2) it is shown that real symmetric transfer functions have balanced realizations with the sign symmetry property. A realization (A, B, C, D) is called *sign symmetric* if there exists a basis in the state space X such that

$$A^* = SAS, \quad C^* = SB,$$

where S has a matrix representation with respect to this basis which is given by

$$S = diag(s_1, s_2, s_3, \ldots)$$

with $s_j = \pm 1$.

Theorem 6.1 *(1) Let G_d be a discrete time transfer function satisfying the conditions in part (ii) of Theorem 2.1. Then by Theorem 2.1 G_d has an admissible balanced discrete-time realization (A_d, B_d, C_d, D_d) with observability/reachability gramian Σ which for a given basis $(e_j)_{j\geq 1}$ in the state space X has the matrix representation*

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \sigma_3 I_{n_3}, \ldots)$$

where $\sigma_1 > \sigma_2 > \sigma_3 > \ldots > 0$ such that $\lim_{j\to\infty}\sigma_j = 0$.

All admissible discrete-time balanced systems with state space X and gramian Σ can be written as

$$(QA_dQ^*, QB_d, C_dQ^*, D_d),$$

where Q has a matrix representation with respect to the basis $(e_j)_{j\geq 1}$ given by

$$Q = diag(Q_1, Q_2, Q_3, \ldots)$$

where $Q_j \in \mathbb{C}^{n_j \times n_j}$, $Q_j^* Q_j = I_{n_j}$, $j \geq 1$.

(2) Let G_d be a real symmetric transfer function such that the conditions of part (ii) in Theorem 2.1 are satisfied. Then G_d has a balanced and admissible discrete-time realization which is sign symmetric.

Proof:
(1) Let (A_d, B_d, C_d, D_d) be a balanced realization of G_d. Note that the diagonal entries of Σ are the singular values of the Hankel operator corresponding to G_d ([17]). The assumptions on G_d imply that the Hankel operator is compact. Hence the only accumulation point of the singular values is 0 and the multiplicities of the nonzero singular values is finite. Therefore, possibly after rearranging the basis $(e_j)_{j\geq 1}$, we can assume that Σ has the claimed structure.

Let $(\tilde{A}_d,\tilde{B}_d,\tilde{C}_d,\tilde{D}_d)$ be another balanced realization of G_d with gramian Σ. Then by Theorem 2.1 the two systems are unitarily equivalent with a unitary state space transformation Q. Using the definition of the observability gramian we obtain

$$Q\Sigma Q^* = \Sigma$$

and hence $Q\Sigma = \Sigma Q$. This implies the claimed structure of Q.
(2) By Lemma 6.1 we know that if (A_d, B_d, C_d, D_d) is a balanced realization of G_d with gramian

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \sigma_3 I_{n_3}, \ldots)$$

then the dual system is also a balanced realization of G_d with the same gramian. Hence by part (1) we have that

$$A_d^* = QA_dQ^*, \quad C_d^* = QB_d, \quad B_d^* = C_dQ^*,$$

where $Q = diag(Q_1, Q_2, Q_3, \ldots)$, with $Q_j \in \mathbb{C}^{n_j \times n_j}$, $Q_j^*Q_j = I_{n_j}$, $j \geq 1$. From these identities it follows that

$$A_d^* = Q^*A_dQ, \quad C_d^* = Q^*B_d, \quad B_d^* = C_dQ.$$

Hence by the uniqueness of the state space transformation we have that $Q = Q^*$ and thus $Q_j^* = Q_j$, $j \geq 1$. Since Q_j is unitary and selfadjoint we can therefore find

$$S_j = diag(s_1^j, \ldots, s_{n_j}^j), \; s_i^j = \pm 1, \; j \geq 1, \; 1 \leq i \leq n_j$$

and unitary matrices V_j, $j \geq 1$, such that

$$Q_j = V_j^* S_j V_j, \quad j \geq 1.$$

Setting $V = diag(V_1, V_2, V_3, \ldots)$ and $S = diag(S_1, S_2, S_3, \ldots)$ we can perform a state space transformation of (A_d, B_d, C_d, D_d) with V. Note that the resulting system

$$(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d) = (VA_dV^*, VB_d, C_dV^*, D_d)$$

is a balanced realization of G_d. It is straightforward to verify that $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ is sign symmetric. □

The results in Section 4 imply that the previous theorem carries over to the continuous-time case. In particular we have the following corollary.

Corollary 6.1 *(1) Let G_c be a continuous-time transfer function satisfying the conditions of part (ii) in Theorem 3.1. Then by Theorem 3.1 G_c has an admissible balanced continuous-time realization (A_c, B_c, C_c, D_c) with observability/reachability gramian Σ which for a given basis $(e_j)_{j\geq 1}$ in the state space X has the matrix representation*

$$\Sigma = diag(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \sigma_3 I_{n_3}, \ldots)$$

where $\sigma_1 > \sigma_2 > \sigma_3 > \ldots > 0$ such that $\lim_{j\to\infty}\sigma_j = 0$.

All admissible continuous-time balanced systems with state space X and gramian Σ can be written as

$$(QA_cQ^*, QB_c, C_cQ^*, D_c),$$

where Q has a matrix representation with respect to the basis $(e_j)_{j\geq 1}$ given by

$$Q = diag(Q_1, Q_2, Q_3, \ldots),$$

where $Q_j \in \mathbb{C}^{n_j \times n_j}$, $Q_j^*Q_j = I_{n_j}$, $j \geq 1$.

(2) Let G_c be a real symmetric transfer function such that the conditions of part (ii) in Theorem 3.1 are satisfied. Then G_c has a balanced and admissible continuous-time realization which is sign symmetric.

Proof:
(1) This follows from the results in Section 4.
(2) The property is a consequence of the results in Section 4 together with the fact that a continuous time transfer function is real symmetric if and only if its associate discrete time transfer function is real symmetric. □

In the next theorem we are going to give an interpretation of the signs in the sign symmetry matrix S.

Theorem 6.2 *Let G_d be a real symmetric transfer function satisfying the conditions in Theorem 2.1. If Σ is the observability/reachability gramian of a balanced sign symmetric realization of G_d with sign symmetry matrix S, then the diagonal entries of ΣS are the nonzero eigenvalues of the Hankel operator*

$$\begin{aligned} \mathcal{H} : \ell_U^2 \; &\to \ell_Y^2 \\ (u_1, u_2, u_3, \ldots)^T \; &\mapsto (y_1, y_2, y_3, \ldots)^T \\ &= \begin{pmatrix} H_1 & H_2 & H_3 & . & \cdots \\ H_2 & H_3 & H_4 & . & \cdots \\ H_3 & H_4 & H_5 & . & \cdots \\ . & . & . & . & \cdots \\ \vdots & \vdots & \vdots & & \cdots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ \vdots \end{pmatrix}, \end{aligned}$$

where $H(i)$, $i \geq 1$, are the Markov parameters of G_d, which are given by $H_i = C_dA_d^{i-1}B_d$, $i \geq 1$, where (A_d, B_d, C_d, D_d) is an admissible realization of G_d.

Proof:

The statement follows by noting that $\mathcal{H}$ has the decomposition $\mathcal{H} = \mathcal{OR}$. For a sign symmetric realization which has sign symmetry matrix S with respect to a basis $(e_j)_{j\geq 1}$ we obtain $\mathcal{H} = \mathcal{O}S\mathcal{O}^*$. Setting $\tilde{u}_j = \mathcal{O}e_j$, $j \geq 1$, we obtain

$$\mathcal{H}\tilde{u}_j = \mathcal{O}S\mathcal{O}^*\mathcal{O}\tilde{u}_j = s_j\sigma_j\mathcal{O}e_j = s_j\sigma_j\tilde{u}_j$$

for $j \geq 1$, which implies the result. It is easily seen that all nonzero eigenvalues of $\mathcal{H}$ are given in this way. □

The following theorem states the asymptotic stability of a balanced realizations for a special class of real symmetric transfer functions.

Theorem 6.3 *Let G_d (G_c) be a real symmetric transfer function satisfying the conditions in Theorem 2.1 (Theorem 3.1). If the sign symmetry matrix S of the sign symmetric realization (A_d, B_d, C_d, D_d) $((A_c, B_c, C_c, D_c))$ is such that $S = \pm I$ then A_d $\left((e^{tA_c})_{t\geq 0}\right)$ is asymptotically stable.*

Proof:

First note that A_d is selfadjoint. Hence the spectrum of A_d is real. By Lemma 5.1 A_d has no eigenvalues on the unit circle. Since the state space X is a seperable Hilbert space the result follows from the stability result in [1].

The asymptotic stability of $(e^{tA_c})_{t\geq 0}$ is a consequence of the fact that $(e^{tA_c})_{t\geq 0}$ is asymptotically if and only if the cogenerator is asymptotically stable ([14]). This is the case since the continuous-time system is sign symmetric with sign symmetry matrix S if and only if the associated discrete-time system is sign symmetric with sign symmetry matrix S (Proposition 4.1). □

In the next theorem we show asymptotic stability for another special class of (par-) balanced realizations.

Theorem 6.4 *If a discrete-time par-balanced system is such that $\mathcal{M}_d = \mathcal{W}_d$ is bounded below, i.e. for some $m > 0$, $\|\mathcal{M}_dx\| \geq m\|x\|$ for all $x \in X$ then the system is asymptotically stable.*

The same result holds for par-balanced continuous time systems.

Proof:

If (A_d, B_d, C_d, D_d) is par-balanced with $\mathcal{M}_d$ bounded below then

$$(\mathcal{M}_d^{1/2}A_d\mathcal{M}_d^{-1/2}, \mathcal{M}_d^{1/2}B_d, C_d\mathcal{M}_d^{-1/2}, D_d)$$

is an output normal realization. But by Theorem 5.2 this system is asymptotically stable. Note that this output normal realization is well defined since both $\mathcal{M}_d^{1/2}$ and $\mathcal{M}_d^{-1/2}$ are bounded operators. The asymptotic stability of (A_d, B_d, C_d, D_d) is now a consequence of the boundedness of these operators. The standard arguments imply the result for the continuous-time case. □

We have therefore shown that in two special cases we have asymptotic stability of (par-) balanced realizations. Whether (par-) balanced systems are in general asymptotically stable is not clear. We know that for a (par-) balanced discrete-time system A_d has no point spectrum on the unit circle and that A_d is a contraction. Similarly for a (par-) balanced continuous-time system we know that A_c has no point spectrum on the imaginary axis and that $(e^{tA_c})_{t\geq 0}$ is a semi group of contractions. It is, however, well known that this is not sufficient to guarantee the asymptotic stability of the systems.

We conclude with a remark concerning exponentially stable systems.

Remark 6.1 *Since Hankel operators corresponding to transfer functions play an important role in realization theory it might be suggested that certain compactness assumptions on the Hankel operator imply the existence of exponentially stable balanced realizations. That this is not the case follows from a construction in [10], [9], where for a given sequence of singular values a Hankel operator was constructed via a balanced continuous-time state space system. This system was shown to be asymptotically stable. Since the dissipative operator A_c was constructed to be a Hilbert-Schmidt operator this implies that its eigenvalues converge to zero. This however excludes the exponential stability of A_c.* □

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