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BILINEAR TRANSFORMATION OF INFINITE-DIMENSIONAL STATE-SPACE SYSTEMS AND BALANCED REALIZATIONS OF NONRATIONAL TRANSFER FUNCTIONS*

RAIMUND OBER† AND STEPHEN MONTGOMERY-SMITH‡

Abstract. The bilinear transform maps the open right half plane to the open unit disk and is therefore a suitable tool for carrying over results for continuous-time systems to discrete-time systems and vice versa. Corresponding state-space formulae are widely used and well understood for the case of finite-dimensional systems. In this paper infinite-dimensional generalizations of these formulae are studied for a general class of infinite-dimensional state-space systems. In particular, it is shown that reachability and observability are carried over and that the reachability and observability gramians are preserved under this transformation. Young showed that a wide class of nonrational discrete-time transfer functions admit a balanced state-space representation. It is shown that this result carries over to the continuous-time situation via the bilinear transformation.

Key words. bilinear transformation, infinite-dimensional state-space systems, balanced realizations

AMS(MOS) subject classifications. 93C20, 93B15, 93B20, 93B28

1. Introduction. Balanced realizations for finite-dimensional systems have received a great deal of attention. They were introduced as a means of performing model reduction in an easy fashion [10] and have subsequently been used in H^∞ control theory, for example, to evaluate the Hankel norm of a linear system [5], [3]. Recently, they have been used to study parametrization problems of the set of stable linear systems [11], [13].

The elegant results obtained for finite-dimensional balanced systems brought about some interest in the problem of the extension of the notion of a balanced realization to infinite-dimensional systems. Curtain and Glover [2], as well as Glover, Curtain, and Partington [6] derived continuous-time, balanced realizations for a class of systems with nuclear Hankel operator. Young [20] developed a very general realization theory for infinite-dimensional discrete-time systems.

The motivation for this paper was to show that a large class of systems that includes most H^∞ transfer functions have balanced realizations. Transfer functions in H^∞ are of particular interest since they are precisely the transfer functions of linear systems with L^2 bounded input-output operators. In particular, it is shown here that important systems such as a pure time delay, delayed systems with transfer functions of the form $G(s)\,e^{-sT}$, $G(s)$ nonstrictly proper rational, but also certain transfer functions with singularities on the imaginary axis such as $G(s)=\log(1+1/s)$ admit balanced or, more precisely, parbalanced realizations. These are examples of systems whose corresponding Hankel operator is not nuclear and hence they are not in the class of systems considered by Glover, Curtain, and Partington. The work by Glover, Curtain, and Partington [6] and Ober [12] has shown that balanced realizations can be successfully employed to perform model reduction for certain special classes of infinite-dimensional continuous-time systems. It is hoped that the realization theory for balanced systems developed here is not only of theoretical interest but is also a

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† Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, United Kingdom.
‡ Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, CB2 1SB, United Kingdom. Present address, Department of Mathematics, University of Missouri at Columbia, Columbia, Missouri 65211.

step toward the development of model reduction tools for the important class of H^∞ transfer functions.

The realization problem for infinite-dimensional continuous-time systems has been studied by several authors. Shift realizations of infinite-dimensional continuous-time systems have been investigated, for example, by Fuhrmann [4] and Salamon [16]. Other approaches have been taken by Yamamoto [19] and Hegner [12]. The important role the Hankel operator plays in realization theory is well understood (see, e.g., Fuhrmann [4]). From this point of view it is interesting to note that such a connection is also very apparent in the realization of infinite-dimensional systems in terms of balanced realizations. For example, the realizability conditions on a transfer function are in terms of boundedness conditions and compactness conditions on the Hankel operator corresponding to the transfer function. But these can be expressed in terms of analytical properties of the transfer functions.

System theoretic developments often go in parallel for continuous-time and discrete-time systems. In finite-dimensional system theory it is common practice to derive results for one class of systems and then map these over to the other by using a bilinear transformation or the corresponding state space formulae. With this method it is often possible to avoid the repetition of lengthy derivations if results have already been obtained for one class of systems and similar results are needed for the other. The approach taken to the realization problem considered here is based on the same principal. The work by Young [20] contains very general realization results for discrete-time systems in terms of balanced realizations. We will carry these over to the continuous-time case using infinite-dimensional generalizations of the finite-dimensional methods. A major part of this paper is devoted to establishing infinite-dimensional generalizations of these techniques. It is shown that such generalizations are indeed possible and are especially suited to the study of observability and reachability properties, which are of central importance in linear systems theory. In particular, it is shown that these techniques carry over the observability and reachability operators in such a way that the observability (reachability) operator of a continuous-time system and the observability (reachability) operator of its corresponding discrete-time system are unitarily equivalent. It is hoped that such methods will become as useful in an infinite-dimensional setting as they have proved to be for finite-dimensional systems.

In essence, we will prove infinite-dimensional analogues of the following finite-dimensional results. If $C_n^{p,m}$ is the set of minimal asymptotically stable continuous-time systems $(A_c, B_c, C_c, D_c) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}$ and $D_n^{p,m}$ is the set of minimal asymptotically stable discrete-time systems $(A_d, B_d, C_d, D_d)\in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}$, then the map $T_n: D_n^{p,m}\to C_n^{p,m}$ defined by

$$T_n((A_d, B_d, C_d, D_d))$$
$$=((I+A_d)^{-1}(A_d-I), \sqrt{2}(I+A_d)^{-1}B_d, \sqrt{2}\, C_d(I+A_d)^{-1}, D_d - C_d(I+A_d)^{-1}B_d)$$

is a bijection with inverse $T_n^{-1}: C_n^{p,n}\to D_m^{p,n}$ given by

$$T_n^{-1}((A_c, B_c, C_c, D_c))$$
$$=((I-A_c)^{-1}(I+A_c), \sqrt{2}(I-A_c)^{-1}B_c, \sqrt{2}\, C_c(I-A_c)^{-1}, D_c + C_c(I-A_c)^{-1}B_c).$$

If $(A_c, B_c, C_c, D_c) := T_n((A_d, B_d, C_d, D_d))$, then (A_d, B_d, C_d, D_d) is a realization of the transfer function G_d, i.e., $G_d(z) = C_d(zI-A_d)^{-1}B_d + D_d$, if and only if (A_c, B_c, C_c, D_c) is a realization of the transfer function $G_c(s) := G_d((1+s)/(1-s))$, i.e., $G_c(s) = C_c(sI-A_c)^{-1}B_c + D_c$.

To deal with infinite-dimensional continuous-time systems in their full generality, it is however necessary, in contrast to discrete-time systems, to deal with unbounded input and output operators. This produces serious technical problems, and a careful setup is necessary for the definition of an infinite-dimensional continuous-time system and of the generalization of the transformation T_n.

Our approach to the definition of an infinite-dimensional system is based on the notion of a compatible system, as introduced by Helton [9]. There are, however, several differences in technical details that seem necessary to prove our result. Hedberg [8] used a form of state-space formulae to relate discrete-time shift realizations to continuous-time shift realizations. His method was later reported in the book by Fuhrmann [4]. To derive our results we had to adopt a generalization of the transformation T_n that differs from Hedberg's generalization in several respects.

In § 2 we define the objects of interest to our study, that is, infinite-dimensional discrete- and continuous-time systems. To do this it is necessary to introduce the notion of a rigged Hilbert space, as well as prove several properties of generators of semigroups that are important in our context.

We will need several results from the functional calculus for unbounded functions by Sz.-Nagy and Foias [17]. Section 3 contains a brief introduction to this functional calculus and proves propositions that we will need in later sections. The section can be skipped by readers who are not interested in detailed proofs of the main theorems of the paper.

Section 4 contains our first important results. Here we establish the transformation T relating infinite-dimensional discrete-time systems to continuous-time systems and show that it is a bijection.

State-space systems related by a unitary state-space transformation are studied in § 5. It is established that two discrete-time systems are unitarily equivalent if and only if their corresponding continuous-time systems are unitarily equivalent.

Before § 7, we need to generalize the notion of the dual of a system to infinite-dimensional systems. This is done in § 6.

Section 7 contains one of the main results of this paper. It is shown that the observability operator of a discrete-time system is unitarily equivalent to the observability operator of its corresponding continuous-time system.

Having established all the necessary tools for our treatment of infinite-dimensional state-space systems, we bring them together in § 8 where we prove a general realization result for infinite-dimensional continuous-time transfer functions in terms of balanced systems.

Great emphasis has been placed on a presentation that is as self-contained as possible. It is hoped that this paper might serve some readers as an introduction to infinite-dimensional continuous-time state-space systems.

All Hilbert spaces are assumed to be separable and defined over the complex field. The scalar product $\langle \cdot, \cdot \rangle$ is linear in the first component. The norm of a Hilbert space X is denoted by $\|\cdot\|_X$, or simply by $\|\cdot\|$. The sum of two subsets N and M of a Hilbert space X is defined by $M+N=\{x+y \mid x \in M, y \in N\}$. We denote by $(A, D(A))$ the operator A with domain of definition $D(A)$. The adjoint of the operator $(A, D(A))$ is denoted by $(A^*, D(A^*))$. The space of bounded operators from the Hilbert space X to the Hilbert space Y is denoted by $\mathscr{L}(X, Y)$, whereas $\mathscr{K}(X, Y)$ is the set of compact operators from X to Y. The symbol $\sigma_p(A)$ indicates the point spectrum of the operator A. The abbreviation RHP stands for the open right half plane. The boundary of the open unit disc $\mathbf{D}$ is denoted by $\partial\mathbf{D}$. The real part of a complex number z is denoted by $\operatorname{Re}(z)$.

2. Admissible discrete-time and continuous-time systems. In this section we will define the classes of discrete- and continuous-time systems that we will investigate in later parts of the paper. Whereas we can immediately state what we mean by an admissible discrete-time system, we will have to review the notion of a rigged Hilbert-space before we can give the corresponding definition of an admissible continuous-time system.

An admissible discrete-time system is defined as follows.

DEFINITION 2.1. The quadruple of operators (A_d, B_d, C_d, D_d) is called an admissible discrete-time system, with state space X, output space Y and input space U, where X, U, Y are separable Hilbert spaces, if

(i) $A_d \in \mathscr{L}(X)$ is a contraction such that $-1 \notin \sigma_p(A_d)$,
(ii) $B_d \in \mathscr{L}(U, X)$,
(iii) $C_d \in \mathscr{L}(X, Y)$,
(iv) $D_d \in \mathscr{L}(U, Y)$,
(v) A_d, B_d, C_d are such that $\lim_{\lambda \to 1, \lambda > 1} C_d(\lambda I + A_d)^{-1} B_d$ exists in the norm topology.

We write $D_X^{U,Y}$ for the set of admissible discrete-time systems with input space U, output space Y and state space X.

Remark 2.2. The technical condition (v), which is generally not very restrictive, is not necessary to define infinite-dimensional discrete-time systems. It is, however, important to study the connection between continuous-time and discrete-time systems.

We briefly introduce a number of definitions and results on strongly continuous semigroups of contractions. An excellent reference is Pazy [14].

DEFINITION 2.3. Let X be a Hilbert space. A one-parameter family $(T(t))_{t \geqq 0}$ of contractions in $\mathscr{L}(X)$ is a strongly continuous semigroup of contractions if

(i) $T(0) = I$,
(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geqq 0$,
(iii) $\lim_{t \to 0} T(t)x = x$ for all $x \in X$.

The linear operator $(A, D(A))$ given by

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A), \qquad D(A) = \left\{ x \in X \,\middle|\, \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

is called the generator of the semigroup $(T(t))_{t \geqq 0}$.

It can be shown that the generator $(A, D(A))$ uniquely determines the corresponding semigroup $(T(t))_{t \geqq 0}$. Therefore we write $T(t) =: e^{tA}$, $t \geqq 0$. We note that the generator $(A, D(A))$ is a closed linear operator whose domain $D(A)$ is dense in X. A further important property is that it is dissipative, i.e.,

$$\operatorname{Re} \langle Ax, x \rangle \leqq 0 \quad \text{for all } x \in D(A).$$

Moreover, $D(A)$ is a Hilbert space with inner product induced by the graph norm

$$\|x\|_A^2 := \|x\|_X^2 + \|Ax\|_X^2, \qquad x \in D(A).$$

Since $\|x\|_A \geqq \|x\|_X$ for $x \in D(A)$, we can embed X in $D(A)^{(\prime)}$, the set of antilinear continuous functionals on $(D(A), \|\cdot\|_A)$, by

$$E: X \to D(A)^{(\prime)}, \qquad x \mapsto (y \mapsto \langle x, y \rangle).$$

Note that $D(A)^{(\prime)}$ is a Hilbert space with norm $\|f\|' := \sup_{\|x\|_A \leqq 1} |f(x)|$. Since $\langle \cdot, \cdot \rangle$ is linear in the first component, the embedding E is linear. By the above, we have the rigged structure

$$D(A) \subseteq X \subseteq D(A)^{(\prime)}.$$

It is well known that if $(A, D(A))$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t\geq 0}$ on a Hilbert space, then the adjoint $(A^*, D(A^*))$ of $(A, D(A))$ is the generator of the adjoint semigroup $(e^{tA})^*_{t\geq 0}$. Hence, we have similarly that

$$D(A^*) \subseteq X \subseteq D(A^*)^{(\prime)}.$$

If M is an operator on X such that $D(A^*) \subseteq X$ is invariant under M^*, then M can be extended to an operator $\tilde{M}$ on $D(A^*)^{(\prime)}$ by

$$\tilde{M}: D(A^*)^{(\prime)} \to D(A^*)^{(\prime)}, \qquad f(\cdot) \mapsto f(M^*(\cdot)).$$

Usually we will not distinguish between M and $\tilde{M}$ and we will write M for $\tilde{M}$.

Also, if we have a map $M: Z \to D(A^*)^{(\prime)}$, Z a Hilbert space, such that $M(Z) \subseteq X^{(\prime)} \subseteq D(A^*)^{(\prime)}$, we can consider $M: Z \to X$ using the Riesz Representation Theorem.

We are now in a position to define admissible continuous-time systems.

DEFINITION 2.4. A quadruple of operators (A_c, B_c, C_c, D_c) is called an admissible continuous-time system with state space X, input space U, and output space Y, where X, U, Y are separable Hilbert spaces, if

(i) $(A_c, D(A_c))$ is the generator of a strongly continuous semigroup of contractions on X.

(ii) $B_c: U \to (D(A_c^*)^{(\prime)}, \|\cdot\|')$ is a bounded linear operator.

(iii) $C_c: D(C_c) \to Y$ is linear with $D(C_c) = D(A_c) + (I - A_c)^{-1} B_c U$ and $C_{c|D(A_c)}: (D(A_c), \|\cdot\|_{A_c}) \to Y$ is bounded.

(iv) $C_c(I - A_c)^{-1} B_c \in \mathscr{L}(U, Y)$.

(v) A_c, B_c, C_c are such that $\lim_{s\in\mathbf{R}, s\to\infty} C_c(sI - A_c)^{-1} B_c = 0$ in the norm topology.

(vi) $D_c \in \mathscr{L}(U, Y)$.

We write $C_X^{U,Y}$ for the set of admissible continuous-time systems with input space U, output space Y, and state space X.

Before we continue to prove two lemmas that show admissible continuous-time systems are well defined, let us remark that the state space X of a system in $C_X^{U,Y}$ has the rigged structure $D(A_c) \subseteq X \subseteq D(A_c^*)^{(\prime)}$.

Remark 2.5. In Helton [9] and Fuhrmann [4] a similar definition was given for continuous-time state-space systems. There are, however, several differences between so-called compatible systems and admissible systems as defined here. Our definition of a rigged Hilbert space is slightly different from that used in Helton and Fuhrmann, where X is embedded in the dual spaces $D(A)'$ and $D(A^*)'$, rather than in the spaces of antilinear functionals $D(A)'$ and $D(A^*)'$ as adopted here. The reason for using our definition is that this naturally leads to a definition of the input operator B_c as a linear, rather than an antilinear operator. Most important, however, for the discussion later, is the imposition of (v) in our definition.

To show that the above definition is well defined, we must show that $C_c(sI - A_c)^{-1} B_c$ is well defined for all $s \in \mathbb{R}$ and that $(I - A_c)^{-1} B_c U \subseteq X$. This follows from the following two lemmas, which also contain technical results that are useful in later sections.

LEMMA 2.6. *Let $(A_c, D(A_c))$ be the generator of a strongly continuous semigroup of contractions $(e^{tA_c})_{t\geq 0}$ on the separable Hilbert space X. Then for $s \in \text{RHP}$,*

(i) *$(sI - A_c)^{-1} X \subseteq D(A_c)$ and the map $(sI - A_c)^{-1}: (X, \|\cdot\|_X) \to (D(A_c), \|\cdot\|_{A_c})$ is bounded.*

(ii) *The map $(sI - A_c)^{-1}: (D(A_c), \|\cdot\|_{A_c}) \to (D(A_c), \|\cdot\|_{A_c})$ is bounded.*

(iii) $(sI-A_c)^{-1}D(A_c^*)^{(\prime)}\subseteq X$ *and the map* $(sI-A_c)^{-1}:(D(A_c^*)^{(\prime)},\|\cdot\|')\to(X,\|\cdot\|_X)$ *is bounded.*

(iv) $e^{tA_c}D(A_c)\subseteq D(A_c)$ *for all* $t\in[0,\infty[$.

Proof. (i) For a proof that $(sI-A_c)^{-1}X\subseteq D(A_c)$, $s\in\text{RHP}$, see Pazy [14, p. 8]. To show that $(sI-A_c)^{-1}:(X,\|\cdot\|_X)\to(D(A_c),\|\cdot\|_{A_c})$ is bounded, $s\in\text{RHP}$, let $x\in X$ and consider

$$\begin{aligned}\|(sI-A_c)^{-1}x\|_{A_c}^2&=\|(sI-A_c)^{-1}x\|_X^2+\|A_c(sI-A_c)^{-1}x\|_X^2\\&\leqq\|(sI-A_c)^{-1}\|^2\|x\|_X^2+\|s(sI-A_c)^{-1}x-(sI-A_c)(sI-A_c)^{-1}x\|_X^2\\&\leqq\|(sI-A_c)^{-1}\|^2\|x\|_X^2+(|s|\,\|(sI-A_c)^{-1}\|\,\|x\|_X+\|x\|_X)^2\\&=(\|(sI-A_c)^{-1}\|^2+(|s|\,\|(sI-A_c)^{-1}\|+1)^2)\|x\|_X^2,\end{aligned}$$

which proves the result.

(ii) This follows from (i) since $\|x\|_X\leqq\|x\|_{A_c}$, for $x\in D(A_c)$.

(iii) This follows by duality from (i).

(iv) See Pazy [14, p. 5]. □

By (iii) of the previous lemma and the definition of B_c we have that $(I-A_c)^{-1}B_c\subseteq X$ and so $D(C_c)$ is well defined. The following lemma shows that $C_c(sI-A_c)^{-1}B_c$ is well defined and in $\mathscr{L}(U,Y)$, for all $s\in\text{RHP}$.

LEMMA 2.7. *Let* $A_c:D(A_c)\to X$ *be the generator of a strongly continuous semigroup of contractions. Let* $B_c:U\to(D(A_c^*)^{(\prime)},\|\cdot\|')$ *be bounded and let* $C_c:D(C_c)\to Y$ *be such that* $C_{c|D(A_c)}:(D(A_c),\|\cdot\|_{A_c})\to Y$ *is bounded, where* $D(C_c)=D(A_c)+(I-A_c)^{-1}B_cU$. *Then*

(i) $(sI-A_c)^{-1}B_cU\subseteq D(C_c)$ *for all* $s\in\text{RHP}$.

(ii) *If* $C_c(I-A_c)^{-1}B_c\in\mathscr{L}(U,Y)$, *then* $C_c(sI-A_c)^{-1}B_c\in\mathscr{L}(U,Y)$ *for all* $s\in\text{RHP}$.

Proof. Let $s\in\text{RHP}$; then by the resolvent identity we have

$$(sI-A_c)^{-1}=(I-A_c)^{-1}+(1-s)(I-A_c)^{-1}(sI-A_c)^{-1}.$$

Since $(sI-A_c^*)^{-1}D(A_c^*)\subseteq D(A_c^*)$ we can apply B_c and obtain

$$(sI-A_c)^{-1}B_c=(I-A_c)^{-1}B_c+(1-s)(I-A_c)^{-1}(sI-A_c)^{-1}B_c.$$

Since $B_c:U\to(D(A_c^*)^{(\prime)},\|\cdot\|)$ is bounded, it follows by Lemma 2.6(i), (iii) that

$$(I-A_c)^{-1}(sI-A_c)^{-1}B_c:U\to(D(A_c),\|\cdot\|_{A_c})$$

is continuous. This implies in particular (i), since

$$(sI-A_c)^{-1}B_cU=(I-A_c)^{-1}B_cU+(1-s)(I-A_c)^{-1}(sI-A_c)^{-1}B_cU\subseteq D(C_c).$$

Since $C_{c|D(A_c)}:(D(A_c),\|\cdot\|_{A_c})\to Y$ is bounded and hence

$$C_c(I-A_c)^{-1}(sI-A_c)^{-1}B_c\in\mathscr{L}(U,Y),$$

we have that

$$C_c(sI-A_c)^{-1}B_c=C_c(I-A_c)^{-1}B_c+(1-s)C_c(I-A_c)^{-1}(sI-A_c)^{-1}B_c\in\mathscr{L}(U,Y).\quad\square$$

Remark 2.8. It is useful to note that using the identification of $X^{(\prime)}$ and X via the Riesz Representation Theorem we have that for $u\in U$ the functional $(sI-A_c)^{-1}B_c(u):D(A_c^*)\to\mathbb{C}$ is given by

$$x\mapsto(sI-A_c)^{-1}B_c(u)[x]=B_c(u)[(\bar{s}I-A_c^*)^{-1}x]=\langle(sI-A_c)^{-1}B_cu,x\rangle.$$

3. The functional calculus by Sz.-Nagy–Foias. In this section we will review some results on the functional calculus by Sz.-Nagy-Foias and prove two technical results that are fundamental to the main results of this paper. This section is, however, only necessary for an understanding of the proofs of some of the theorems presented in later sections. Those theorems themselves are largely formulated without reference to the functional calculus discussed here.

Since we do not assume that the reader is fully familiar with the functional calculus as developed in Sz.-Nagy and Foias [17] we give a brief summary of those results necessary for our applications.

We first consider a standard result of functional calculus. Let $\mathcal{A}$ be the set of functions given by

$$a(z)=\sum_{k=0}^{\infty} c_k z^k \quad \text{such that} \quad \sum_{k=0}^{\infty} |c_k|<\infty.$$

Then $\mathcal{A}$ is an algebra with the involution $a \mapsto a^*$ given by $a^*(z) := \overline{a(\bar{z})}$. Note that a function in $\mathcal{A}$ is analytic on $\mathbf{D}$ and continuous on $\bar{\mathbf{D}}$.

For a contraction T on a Hilbert space X, we define $a(T)=\sum_{k=0}^{\infty} c_k T^k$. The sum converges in the operator norm and hence the operator $a(T)$ is well defined. The following theorem states the fundamental result concerning the functional calculus for functions in $\mathcal{A}$.

THEOREM 3.1. *For a contraction T on a Hilbert space X, the map*

$$\mathcal{A} \to \mathcal{L}(X), \qquad a(z) \mapsto a(T)$$

is an algebra homomorphism. In particular, $a(T)b(T)=b(T)a(T)$, for $a, b \in \mathcal{A}$.

The functions that are important in our context are:

(a) $\phi : z \to (z-1)/(z+1)$,

(b) $\varphi_t : z \mapsto e^{t((z-1)/(z+1))}$, $t \geqq 0$,

(c) $\mu : z \mapsto 1/(1+z)$,

(d) $\delta_t : z \mapsto 1/(1+z)\, e^{t((z-1)/(z+1))}$, $t \geqq 0$.

None of these functions are in $\mathcal{A}$ and hence we must consider extensions of the functional calculus of Theorem 3.1. Note, however, that the functions $z \mapsto \phi(rz)$, $z \mapsto \varphi_t(rz)$, $z \mapsto \mu(rz)$, and $z \mapsto \delta_t(rz)$, $0<r<1$ are in $\mathcal{A}$.

Next we exploit the observation that for each function $u \in H^\infty$ the function $z \mapsto u(rz)$, $0<r<1$, is in $\mathcal{A}$ and discuss functions for which the limit $\lim_{r\to 1-0} u(rT)$ is well defined in the following sense.

DEFINITION 3.2. Let T be a contraction on X. H_T^∞ is the set of those functions $u \in H^\infty$ such that

$$u(T) := \lim_{r \to 1-0} u(rT)$$

exists in the strong operator topology.

Before we can describe a subset of H_T^∞, we must consider contractions in some detail. A subspace Y of a Hilbert space X is called reducing for $T \in \mathcal{L}(X)$ if T maps Y onto itself. A contraction T in $\mathcal{L}(X)$ is called completely nonunitary if there is no nonzero reducing subspace Y of X such that $T_{|Y}$ is unitary. To every contraction T on the space X there corresponds a decomposition $X=X_1 \oplus X_2$ into an orthogonal sum of two subspaces X_1 and X_2 reducing T such that $T_1 := T_{|X_1}$ is unitary and $T_2 := T_{|X_2}$ is completely nonunitary. The canonical decomposition of T is denoted by $T=T_1 \oplus T_2$. Recall that each unitary operator U has a spectral decomposition $U=\int_0^{2\pi} e^{it}\, dE_t$ for some spectral family $\{E_t\}_{0 \leqq t \leqq 2\pi}$ and spectral measure E_U on the unit circle.

THEOREM 3.3. *H_T^∞ contains the functions $u \in H^\infty$ for which the set*

$$C_u = \{z \in \partial\mathbf{D} \mid u(z) \text{ has no nontangential limit at } z\}$$

has measure zero with respect to the spectral measure E_{T_1} corresponding to the unitary part T_1 of T.

Remark 3.4. Now we consider the functions φ_t, $t \geqq 0$, as defined in (b). We clearly have that $\varphi_t \in H^\infty$ and $C_{\varphi_t} \subseteq \{-1\}$. For a contraction T such that -1 is not an eigenvalue of T, -1 is also not an eigenvalue of the unitary part T_1 of T and hence $E_{T_1}(\{-1\}) = 0$, which shows that $\varphi_t \in H_T^\infty$, $t \geqq 0$.

We will not explore the properties of H_T^∞ in general, but consider the special case of the functions φ_t, $t \geqq 0$. These are of importance in connection with semigroup theory. Before we can state the next theorem establishing this role, we need to introduce some additional notation. Let A_c be the generator of a strongly continuous semigroup of contractions $(e^{tA_c})_{t\geqq 0}$; then

$$A_d = (I + A_c)(I - A_c)^{-1}$$

is called the cogenerator of the semigroup $(e^{tA_c})_{t\geqq 0}$ that can be shown to be a contraction such that -1 is not an eigenvalue of A_d. The generator A_c can be expressed by A_d as

$$A_c = (I + A_d)^{-1}(A_d - I).$$

The following theorem states that if given a contraction T such that -1 is not an eigenvalue of T, then $(\varphi_t(T))_{t\geqq 0}$ is a semigroup of contractions with generator $(I+T)^{-1}(T-I)$ and cogenerator T.

THEOREM 3.5. *Let T be a contraction on X. In order that there exists a strongly continuous semigroup of contractions $(T(t))_{t\geqq 0}$ whose cogenerator equals T, it is necessary and sufficient that -1 is not an eigenvalue of T. If this is the case, then $(T(t))_{t\geqq 0}$ is determined by*

$$T(t) = \varphi_t(T), \qquad t \geqq 0$$

with generator $A_c = (I+T)^{-1}(T-I)$.

Proof. The proof follows from Sz.-Nagy and Foias [17, p. 142], replacing T by $-T$. □

We will now consider unbounded functions in order to deal with ϕ, μ, and δ_t. If $T \in \mathscr{L}(X)$ is a contraction such that $-1 \notin \sigma_p(T)$, then it is easily checked that ϕ, μ, and δ_t, $t \geqq 0$ are in the set of functions N_T defined as follows.

DEFINITION 3.6. For a contraction T in $\mathscr{L}(X)$, denote by K_T^∞ the class of functions $v \in H_T^\infty$ for which $v(T)^{-1}$ exists and has dense domain. Let N_T be the class of functions w that admit a representation

$$w = \frac{u}{v}, \quad u \in H_T^\infty, \quad v \in K_T^\infty.$$

For $w \in N_T$, we define $w(T) = v(T)^{-1}u(T)$.

The following proposition states that for a certain subset of N_T we, in fact, have the commutativity property $w(T) = v(T)^{-1}u(T) = u(T)v(T)^{-1}$.

PROPOSITION 3.7. *Let u, v be continuous on $\bar{\mathbf{D}}$, analytic on $\mathbf{D}$ and have no common zeros in $\bar{\mathbf{D}}$. If $v \in K_T^\infty$ for a contraction T, then*

$$v(T)^{-1}u(T) = u(T)v(T)^{-1}.$$

Remark 3.8. Let T be a contraction T such that $-1 \notin \sigma_p(T)$; then, applying the previous proposition to ϕ, we obtain $(I-T)(I+T)^{-1} = (I+T)^{-1}(I-T)$.

The following theorem provides us with techniques to deal with functions in N_T.

THEOREM 3.9. (i) *Let T be a contraction in $\mathcal{L}(X)$ and let $w \in N_T$ be analytic on* **D**. *If for $x \in X$ we have that*

$$\sup_{0<r<1} \|w(rT)x\| < \infty,$$

it follows that $x \in D(w(T))$ and

$$w(rT)x \to w(T)x$$

weakly as $r \to 1-0$.

(ii) *Suppose the functions u, v are continuous on $\bar{\mathbf{D}}$, analytic on $\mathbf{D}$, and have no common zeros in $\bar{\mathbf{D}}$. We assume that v has no zeros in $\mathbf{D}$ and that it does not vanish on $\partial\mathbf{D}$ except at points of measure zero with respect to the spectral measure E_{T_1} of the unitary part T_1 of T. Moreover, we assume that there exists a constant M such that $|v(\lambda)/v(r\lambda)| \leqq M$ for $\lambda \in \mathbf{D}$, $0<r<1$. Then $w = u/v$ belongs to the class N_T and is analytic in $\mathbf{D}$.*

The condition

$$\sup_{0<r<1} \|w(rT)x\| < \infty$$

characterizes the vectors in $D(w(T))$.

For each $x \in D(w(T))$,

$$w(rT)x \to w(T)x$$

strongly as $r \to 1-0$.

Having reviewed the functional calculus by Sz.-Nagy–Foias, we are now in a position to prove two results that will be a key to results in later sections. Whereas the first proposition deals with the function μ, the second proposition establishes properties of the function δ_t.

PROPOSITION 3.10. *Let T be a contraction on X such that -1 is not an eigenvalue of T. Then*

$$\lim_{\substack{\lambda \to 1 \\ \lambda > 1}} (\lambda I + T)^{-1}x = (I+T)^{-1}x,$$

for $x \in D((I+T)^{-1})$.

Moreover, $x \in D((I+T)^{-1})$ if and only if $\sup_{0<r<1} \|(I+rT)^{-1}x\| < \infty$.

Proof. Let $u=1$ and $v=1+z$, so v only vanishes at $z=-1$. Since -1 is not an eigenvalue of T we have that $E_{T_1}(\{-1\})=0$. Using $w := u/v$ and $r := 1/\lambda$, the result now follows from Theorem 3.9(ii). □

PROPOSITION 3.11. *Let T be a contraction on X such that -1 is not an eigenvalue of T. If $x \in D((I+T)^{-1})$, then for all $t \geqq 0$,*

(1) $\sup_{0<r<1} \|(I+rT)^{-1} e^{t(rT+I)^{-1}(rT-I)}x\| \leqq \sup_{0<r<1} \|(I+rT)^{-1}x\| < \infty$,

(2) $(I+rT)^{-1} e^{t(rT+I)^{-1}(rT-I)}x \to (I+T)^{-1} e^{t(T+I)^{-1}(T-I)}x$ *weakly as $r \to 1-0$.*

Proof. Write for $\delta_t(z) = 1/(1+z)\, e^{t((z-1)/(z+1))} = \mu(z)\varphi_t(z)$, with $\mu(z) = 1/(z+1)$ and $\varphi_t(z) = e^{t((z-1)/(z+1))}$, $t \geqq 0$. Then we have that $\delta_t \in N_T$, $t \geqq 0$, since $\mu^{-1} \in K_T^\infty$ and since $\varphi_t \in H_T^\infty$, $t \geqq 0$, by Remark 3.4.

As $\varphi_t(rz), \mu(rz) \in \mathcal{A}$, for $0<r<1$, $t \geqq 0$, we have that $\varphi_t(rT)\mu(rT) = \mu(rT)\varphi_t(rT)$. Also note that $\varphi_t(rT)$ is a contraction by Theorem 3.5. Hence we obtain for $t \geqq 0$ and

$x \in D(\mu(T)) = D((I+T)^{-1})$ that

$$\sup_{0<r<1} \|\delta_t(rT)x\| = \sup_{0<r<1} \|\mu(rT)\varphi_t(rT)x\| = \sup_{0<r<1} \|\varphi_t(rT)\mu(rT)x\|$$
$$\leqq \sup_{0<r<1} \|\varphi_t(rT)\| \, \|\mu(rT)x\| \leqq \sup_{0<r<1} \|(I+rT)^{-1}x\|$$
$$<\infty$$

where the last inequality follows from Proposition 3.10. The chain of inequalities shows (1).

Thus we have by Theorem 3.9(i) that $D((I+T)^{-1}) \subseteq D(\delta_t(T))$ and that for $x \in D((I+T)^{-1})$ we have,

$$\varphi_t(rT)x \to \varphi_t(T)x$$

weakly as $r \to 1-0$, which proves (2). □

4. A transformation between discrete- and continuous-time systems. We will now introduce a transformation T relating systems in $D_X^{U,Y}$ to systems in $C_X^{U,Y}$ and vice versa. This transformation, which is inspired by a bilinear transformation mapping the unit disk to the right-half plane, is often used for finite-dimensional systems to carry over results from discrete-time systems to continuous-time systems (see, e.g., Glover [5], Ober [11]. Hedberg [8] and Fuhrmann [4] used this approach to prove the existence of state-space realizations for continuous-time systems with transfer function in a certain class of H^∞ functions. The same idea is used here, the specific definitions are, however, somewhat different to avoid certain technical problems.

We first consider the map $T: D_X^{U,Y} \to C_X^{U,Y}$.

THEOREM 4.1. *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$; *then* $T((A_d, B_d, C_d, D_d)) := (A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, *where*

(i) $A_c := (I+A_d)^{-1}(A_d - I) = (A_d - I)(I+A_d)^{-1}$, $D(A_c) := D((I+A_d)^{-1})$, *and* A_c *generates a strongly continuous semigroup of contractions on* X *given by* $\varphi_t(A_d)$, $t \geqq 0$, *with* $\varphi_t(z) = e^{t((z-1)/(z+1))}$.

(ii) $B_c := \sqrt{2}(I+A_d)^{-1}B_d : U \to D(A_c^*)^{(\prime)}$,

$u \mapsto \sqrt{2}(I+A_d)^{-1}B_d(u)[\cdot] := \sqrt{2}\langle B_d(u), (I+A_d^*)^{-1}(\cdot)\rangle_X$.

(iii) $C_c: D(C_c) \to Y, \quad x \mapsto \lim_{\substack{\lambda\to 1\\ \lambda>1}} \sqrt{2}\, C_d(\lambda I + A_d)^{-1}x,$

where $D(C_c) = D(A_c) + (I-A_c)^{-1}B_cU$. *On* $D(A_c)$ *we have,*

$$C_{c|D(A_c)} = \sqrt{2}\, C_d(I+A_d)^{-1}.$$

(iv) $D_c := D_d - \lim_{\lambda\to 1, \lambda>1} C_d(\lambda I + A_d)^{-1}B_d$.

Moreover, let the admissible discrete-time system (A_d, B_d, C_d, D_d) *be a realization of the transfer function*

$$G_d(z): \mathbb{C}\backslash\bar{\mathbf{D}} \to \mathscr{L}(U, Y),$$

i.e., $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$ *for* $z \in \mathbb{C}\backslash\bar{\mathbf{D}}$.

Then, $(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d))$ *is an admissible continuous-time realization of the transfer function*

$$G_c(s) := G_d\left(\frac{1+s}{1-s}\right): \text{RHP} \to \mathscr{L}(U, Y).$$

Proof. We must check that conditions (i)–(vi) of Definition 2.4 are satisfied.

(i) This follows from Theorem 3.5. The fact that $A_c=(A_d-I)(I+A_d)^{-1}=(I+A_d)^{-1}(A_d-I)$ was shown in Remark 3.8.

(ii) Let $u\in U$, $x\in D(A^*)$. Then, since $\frac{1}{2}(I-A_c)=(I+A_d)^{-1}$,

$$\begin{aligned}|B_c(u)[x]| &= |\sqrt{2}\langle B_d(u),(I+A_d^*)^{-1}[x]\rangle_X| \\ &\leqq \frac{1}{\sqrt{2}}\|B_d(u)\|_X\|(I-A_c^*)x\|_X \\ &\leqq \|B_d\|_{\mathscr{L}(U,X)}\|u\|_U(\|x\|_X^2+\|A_c^*x\|_X^2)^{1/2}.\end{aligned}$$

This implies that $B_c(u)\in D(A_c^*)^{(\prime)}$ and that $B_c\colon U\to D(A_c^*)^{(\prime)}$ is continuous.

(iii) We first note that, by Proposition 3.10, C_c is defined on $D(A_c)=D((I+A_d)^{-1})$, and that $C_{c|D(A_c)}=\sqrt{2}\,C_d(I+A_d)^{-1}$.

To show that $C_{c|D(A_c)}$ is continuous with respect $\|\cdot\|_{A_c}$, we see that for $x\in D(A_c)$, we have

$$\begin{aligned}\|C_cx\|_Y &= \frac{1}{\sqrt{2}}\|C_d(I-A_c)x\|_Y \\ &\leqq \|C_d\|_{\mathscr{L}(X,Y)}(\|x\|_X^2+\|A_cx\|_X^2)^{1/2}.\end{aligned}$$

It remains to show that $\lim_{\lambda\to1,\lambda>1}C_d(\lambda I+A_d)^{-1}x$ exists for $x\in(I-A_c)^{-1}B_cU$. First note that $(I-A_c)^{-1}B_c=(1/\sqrt{2})B_d$, for if $x\in D(A_c^*)$, $u\in U$, then

$$\begin{aligned}(I-A_c)^{-1}B_c(u)[x] &= B_c(u)[(I-A_c^*)^{-1}x] \\ &= \sqrt{2}\,\langle B_d(u),(I+A_d^*)^{-1}(I-A_c^*)^{-1}x\rangle \\ &= \frac{1}{\sqrt{2}}\langle B_d(u),x\rangle\end{aligned}$$

where we have used the identity $(I-A_c^*)^{-1}=\frac{1}{2}(I+A_d^*)$. Now we see that

$$\lim_{\substack{\lambda\to1\\ \lambda>1}}\sqrt{2}\,C_d(\lambda I+A_d)^{-1}(I-A_c)^{-1}B_cu=\lim_{\substack{\lambda\to1\\ \lambda>1}}C_d(\lambda I+A_d)^{-1}B_du$$

exists by the admissibility of (A_d, B_d, C_d, D_d).

(iv) We must show that $C_c(I-A_c)^{-1}B_c\in\mathscr{L}(U,Y)$. But by the proof of (iii), we know that $(I-A_c)^{-1}B_c=(1/\sqrt{2})B_d$, and hence that

$$C_c(I-A_c)^{-1}B_c=\frac{1}{\sqrt{2}}C_cB_d=\lim_{\substack{\lambda\to1\\ \lambda>1}}C_d(\lambda I+A_d)^{-1}B_d\in\mathscr{L}(U,Y)$$

by the admissibility of (A_d, B_d, C_d, D_d).

(v) This will be shown after the remaining parts of the theorem have been proved.

(vi) The boundedness of D_c follows since $D_d\in\mathscr{L}(U,Y)$ and $\lim_{\lambda\to1,\lambda>1}C_d(\lambda I+A_d)^{-1}B_d\in\mathscr{L}(U,Y)$.

We will now prove the statements on the transformation of transfer functions. We have for $s\in\text{RHP}$ that $(1+s)/(1-s)\in\mathbb{C}\backslash\bar{\mathbf{D}}$ and hence,

$$\begin{aligned}G_c(s)=G_d\left(\frac{1+s}{1-s}\right) &= C_d\left(\left(\frac{1+s}{1-s}\right)I-A_d\right)^{-1}+D_d \\ &= (1-s)C_d((I-A_d)+s(I+A_d))^{-1}B_d+D_d \\ &= (1-s)C_d(I+A_d)^{-1}(sI-(A_d-I)(I+A_d)^{-1})^{-1}B_d+D_d.\end{aligned}$$

The last identity is well defined, since

$$(sI-(A_d-I)(I+A_d)^{-1})^{-1}B_dU=(sI-A_c)^{-1}B_dU\subseteq D(A_c)=D((I+A_d)^{-1}).$$

Hence, $G_c(s)=(1-s)(1/\sqrt{2})C_c(sI-A_c)^{-1}B_d+D_d$.

Now if we extend the range of $(sI-A_c)^{-1}B_d$ to $D(A_c^*)^{(\prime)}$, then we can show that $(sI-A_c)^{-1}B_d=\sqrt{2}/(1-s)(sI-A_c)^{-1}B_c-1/(1-s)B_d$. For if $x\in D(A_c^*)$, then we have, using the resolvent identity, that

$$\begin{aligned}
&\langle (sI-A_c)^{-1}B_d(u),x\rangle_X\\
&=\langle B_d(u),(I-A_c^*)(I-A_c^*)^{-1}(\bar{s}I-A_c^*)^{-1}x\rangle_X\\
&=\left\langle B_d(u),(I-A_c^*)\frac{1}{(1-\bar{s})}[(\bar{s}I-A_c^*)^{-1}-(I-A_c^*)^{-1})]x\right\rangle_X\\
&=\frac{1}{1-s}(\langle B_d(u),(I-A_c^*)(\bar{s}I-A_c^*)^{-1}x\rangle_X-\langle B_d(u),x\rangle_X)\\
&=\frac{\sqrt{2}}{1-s}[(sI-A_c)^{-1}B_c(u)](x)-\frac{1}{1-s}\langle B_d(u),x\rangle_X.
\end{aligned}$$

But we know that $B_dU\subseteq D(C_c)$ and $(sI-A_c)^{-1}B_cU\subseteq D(C_c)$ for $s\in\text{RHP}$. Hence

$$\begin{aligned}
G_c(s)&=(1-s)\frac{1}{\sqrt{2}}C_c(sI-A_c)^{-1}B_d+D_d\\
&=C_c(sI-A_c)^{-1}B_c-\frac{1}{\sqrt{2}}C_cB_d+D_d\\
&=C_c(sI-A_c)^{-1}B_c-\lim_{\substack{\lambda\to 1\\ \lambda>1}}C_d(\lambda I+A_d)^{-1}B_d+D_d\\
&=C_c(sI-A_c)^{-1}B_c+D_c,
\end{aligned}$$

and so (A_c, B_c, C_c, D_c) is a state-space realization of $G_c(s)$.

To finish the proof, it remains to show (v) of Definition 2.4. By the admissibility of (A_d, B_d, C_d, D_d) we obtain

$$\begin{aligned}
\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}}C_c(sI-A_c)^{-1}B_c&=\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}}G_c(s)-D_c\\
&=\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}}C_d\left(\left(\frac{1+s}{1-s}\right)I-A_d\right)^{-1}B_d+D_d-D_c\\
&=-\lim_{\substack{\lambda\to 1\\ \lambda>1}}C_d(\lambda I+A_d)^{-1}B_d+D_d-D_c\\
&=0,
\end{aligned}$$

which completes the proof. □

Before we consider the map $T^{-1}:C_X^{U,Y}\to D_X^{U,Y}$ we need the following lemma, which gives a version of the resolvent identity for not necessarily bounded resolvents.

LEMMA 4.2. *Let $A_d:X\to X$ be a contraction such that $-1\notin\sigma_p(A_d)$. Then for $z\in\mathbb{C}$, such that $|z|>1$ and for $x\in D((I+A_d^*)^{-1})$, we have*

$$(\bar{z}+1)(I+A_d^*)^{-1}(\bar{z}I-A_d^*)^{-1}x=(I+A_d^*)^{-1}x+(\bar{z}I-A_d^*)^{-1}x.$$

Proof. We first must show that if $x \in D((I+A_d^*)^{-1})$, then $(\bar{z}I-A_d^*)^{-1}x \in D((I+A_d^*)^{-1})$. We know by Theorem 4.1 that $A_c=(I+A_d)^{-1}(A_d-I)$ is the generator of a strongly continuous semigroup of contractions, such that $D(A_c^*)=D((I+A_d^*)^{-1})$.

Since $|z|>1$, we have that $s=(z-1)/(z+1) \in \text{RHP}$. Hence $(\bar{s}I-A_c^*)^{-1}$ is bounded. But

$$(\bar{s}I-A_c^*)^{-1}=(\bar{z}+1)(\bar{z}(I-A_c^*)-(I+A_c^*))^{-1}=(\bar{z}+1)(\bar{z}I-A_d^*)^{-1}(I-A_c^*)^{-1}.$$

Thus $(\bar{s}I-A_c^*)^{-1}(I-A_c^*)=(\bar{z}+1)(\bar{z}I-A_d^*)^{-1}$ and hence, since $(\bar{s}I-A_c^*)^{-1}X \subseteq D(A_c^*)$ by Lemma 2.6, we have that

$$\begin{aligned}(\bar{z}+1)(\bar{z}I-A_d^*)^{-1}D(A_c^*)&=(\bar{s}I-A_c^*)^{-1}(I-A_c^*)D(A_c^*)\\ &\subseteq(\bar{s}I-A_c^*)^{-1}X\subseteq D(A_c^*)=D((I+A_d^*)^{-1}),\end{aligned}$$

which shows the claim.

To prove the statement of the lemma, let $y:=(\bar{z}I-A_d^*)^{-1}x \in D((I+A_d^*)^{-1})$. Then,

$$(\bar{z}+1)y=(\bar{z}I-A_d^*)y+(I+A_d^*)y.$$

Since $y \in D((I+A_d^*)^{-1})$, we can apply $(I+A_d^*)^{-1}$ from the left to obtain

$$(\bar{z}+1)(I+A_d^*)^{-1}y=(I+A_d^*)^{-1}(\bar{z}I-A_d^*)y+y,$$

and hence $(\bar{z}+1)(I+A_d^*)^{-1}(\bar{z}I-A_d^*)^{-1}x=(I+A_d^*)^{-1}x+(\bar{z}I-A_d^*)^{-1}x$. □

THEOREM 4.3. *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$; *then* $T^{-1}((A_c, B_c, C_c, D_c)):=(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$, *where*

(i) $A_d:=(I+A_c)(I-A_c)^{-1}$, *and for* $x \in D(A_c)$ *we have that* $A_d x=(I-A_c)^{-1}(I+A_c)x$.

(ii) $B_d:=\sqrt{2}(I-A_c)^{-1}B_c$.

(iii) $C_d:=\sqrt{2}\,C_c(I-A_c)^{-1}$.

(iv) $D_d:=C_c(I-A_c)^{-1}B_c+D_c$.

Moreover, let the admissible continuous-time system (A_c, B_c, C_c, D_c) *be a realization of the transfer function*

$$G_c(s):\text{RHP}\to\mathscr{L}(U, Y),$$

i.e., $G_c(s)=C_c(sI-A_c)^{-1}B_c+D_c$ *for* $s \in \text{RHP}$.

Then, $(A_d, B_d, C_d, D_d)=T^{-1}((A_c, B_c, C_c, D_c))$ *is an admissible discrete-time realization of the transfer function*

$$G_d(z):=G_c\left(\frac{z-1}{z+1}\right):\mathbb{C}\backslash\bar{\mathbf{D}}\to\mathscr{L}(\mathbf{U},\mathbf{Y}).$$

Proof. We must show that (A_d, B_d, C_d, D_d) satisfies conditions (i)–(v) of Definition 2.1.

(i) Let $x \in X$ and define $y=(I-A_c)^{-1}x \in D(A_c)$; then

$$\begin{aligned}\|A_d x\|^2&=\|(I+A_c)y\|^2=\langle y, y\rangle+\langle A_c y, A_c y\rangle+2\,\text{Re}\,\langle A_c y, y\rangle\\ &=\langle (I-A_c)y, (I-A_c)y\rangle+4\,\text{Re}\,\langle A_c y, y\rangle=\|x\|^2+4\,\text{Re}\,\langle A_c y, y\rangle\\ &\leqq\|x\|^2\end{aligned}$$

since $\text{Re}\,\langle A_c y, y\rangle \leqq 0$ as A_c is dissipative, being the generator of a strongly continuous semigroup of contractions. This shows that A_d is a contraction.

It is easily verified that $(I+A_c)(I-A_c)^{-1}x=(I-A_c)^{-1}(I+A_c)x$, $x\in D(A_c)$, as claimed in the theorem and that $-1\notin\sigma_p(A_d)$.

(ii) This follows in a straightforward way from Lemma 2.6.

(iii) Since

$$C_{c|D(A_c)}:(D(A_c),\|\cdot\|_{A_c})\to Y$$

and

$$(I-A_c)^{-1}:(X,\|\cdot\|_X)\to(D(A_c),\|\cdot\|_{A_c})$$

are continuous, we have that

$$C_d=\sqrt{2}\,C_c(I-A_c)^{-1}:(X,\|\cdot\|_X)\to Y$$

is continuous.

(iv) Since by assumption $C_c(I-A_c)^{-1}B_c\in\mathscr{L}(U,Y)$ and $D_c\in\mathscr{L}(U,Y)$, we have that $D_d\in\mathscr{L}(U,Y)$.

(v) Before we prove (v) we first show the last statement of the theorem.

Let $z\in\mathbb{C}$, such that $|z|>1$; then $s=(z-1)/(z+1)\in\text{RHP}$. By definition

$$G_d(z)=C_c\left(\frac{z-1}{z+1}I-A_c\right)^{-1}B_c+D_c.$$

Consider $((z-1)/(z+1)I-A_c)^{-1}B_c$. Let $u\in U$, $x\in D(A_c^*)$; then

$$\begin{aligned}\left(\frac{z-1}{z+1}I-A_c\right)^{-1}B_c(u)[x]&=B_c(u)\left[\left(\frac{\bar z-1}{\bar z+1}I-A_c^*\right)^{-1}x\right]\\&=(z+1)B_c(u)[(\bar z(I-A_c^*)-(I+A_c^*))^{-1}x]\\&=(z+1)B_c(u)[(\bar zI-A_d^*)^{-1}(I-A_c^*)^{-1}x].\end{aligned}$$

But using the fact that $(I-A_c^*)=2(I+A_d^*)^{-1}$ we obtain,

$$\begin{aligned}B_c(u)[x]&=\langle(I-A_c)^{-1}B_c(u),(I-A_c^*)x\rangle\\&=\sqrt{2}\langle B_d(u),(I-A_d^*)^{-1}x\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\frac{z-1}{z+1}I-A_c\right)^{-1}B_c(u)[x]\\&\quad=(z+1)B_c(u)[(\bar zI-A_d^*)^{-1}(I-A_c^*)^{-1}x]\\&\quad=\sqrt{2}(z+1)\langle B_d(u),(I+A_d^*)^{-1}(\bar zI-A_d^*)^{-1}(I-A_c^*)^{-1}x\rangle\\&\quad=\sqrt{2}\langle B_d(u),(\bar zI-A_d^*)^{-1}(I-A_c^*)^{-1}x\rangle\\&\qquad+\sqrt{2}\langle B_d(u),(I+A_d^*)^{-1}(I-A_c^*)^{-1}x\rangle\\&\quad=\sqrt{2}\langle B_d(u),(\bar zI-A_d^*)^{-1}(I-A_c^*)^{-1}x\rangle+\frac{1}{\sqrt{2}}\langle B_d(u),x\rangle\end{aligned}$$

where the second but last equation uses Lemma 4.2, noting that $(I-A_c^*)^{-1}x\in D(A_c^*)=D((I+A_d^*)^{-1})$. Thus

$$\left(\frac{z-1}{z+1}I-A_c\right)^{-1}B_c(u)=\sqrt{2}(I-A_c)^{-1}(zI-A_d)^{-1}B_d(u)+\frac{1}{\sqrt{2}}B_d(u)\in X.$$

Note that by Lemma 2.6 $(I-A_c)^{-1}X\subseteq D(A_c)\subseteq D(C_c)$ and hence

$$\sqrt{2}(I-A_c)^{-1}(zI-A_d)^{-1}B_dU\subseteq D(C_c).$$

Since $B_dU\subset D(C_c)$ we can apply C_c, and we obtain

$$\begin{aligned} C_c\left(\frac{z-1}{z+1}I-A_c\right)^{-1}B_c+D_c &= \sqrt{2}\,C_c(I-A_c)^{-1}(zI-A_d)^{-1}B_d+\frac{1}{\sqrt{2}}C_cB_d+D_c \\ &= C_d(zI-A_d)^{-1}B_d+C_c(I-A_c)^{-1}B_c+D_c \\ &= C_d(zI-A_d)^{-1}B_d+D_d. \end{aligned}$$

Thus (A_d, B_d, C_d, D_d) is a realization of $G_d(z)$.

We are now in a position to prove (v) of Definition 2.1, i.e., that $\lim_{\lambda>1,\lambda\to 1} C_d(\lambda I+A_d)^{-1}B_d$ exists in the norm topology. By the admissibility of (A_c, B_c, C_c, D_c), we have that

$$\begin{aligned} \lim_{\substack{\lambda\to 1\\ \lambda>1}} C_d(\lambda I+A_d)^{-1}B_d &= -\lim_{\substack{\mu<-1\\ \mu\to -1}} G_d(\mu)+D_d = -\lim_{\substack{\mu<-1\\ \mu\to -1}} G_c\left(\frac{\mu-1}{\mu+1}\right)+D_d \\ &= C_c(I-A_c)^{-1}B_c, \end{aligned}$$

which implies the result. □

Combining the previous two theorems, we can show the following corollary, whose proof is straightforward.

COROLLARY 4.4. *The map* $T: D_X^{U,Y}\to C_X^{U,Y}$ *is a bijection with inverse* $T^{-1}: C_X^{U,Y}\to D_X^{U,Y}$.

Remark 4.5. The following identities that have been used in the above proofs are worthwhile noting for later use:

$$\tfrac{1}{2}(I-A_c)=(I+A_d)^{-1} \quad \text{and} \quad (I-A_c)^{-1}=\tfrac{1}{2}(I+A_d).$$

5. Unitary state-space transformations. In this section we will discuss briefly the effect of a unitary transformation $V: X_1\to X_2$ of the state space on state-space systems. This discussion will be important in § 8 where we will show that a (par-) balanced realization is unique up to a unitary state-space transformation. The first two propositions show that such an operation is well defined and does not change the transfer function. The last result shows that unitarily equivalent systems are carried over by the map $T: D_X^{U,Y}\to C_X^{U,Y}$ and its inverse.

We first consider unitary state-space transformations for admissible discrete-time systems.

PROPOSITION 5.1. *Let* $(A_d, B_d, C_d, D_d)\in D_{X_1}^{U,Y}$. *If* X_2 *is another Hilbert space and* $V: X_1\to X_2$ *is a unitary operator, then*

(1) $(VA_dV^*, VB_d, C_dV^*, D_d)\in D_{X_2}^{U,Y}$.

(2) *If* (A_d, B_d, C_d, D_d) *is a state space realization of the transfer function*

$$G_d(s): \mathbb{C}\backslash\bar{\mathbf{D}}\to L(U, Y),$$

then the $(VA_dV^*, VB_d, C_dV^*, D_d)$ *is a state-space realization of the same transfer function.*

Proof. The proof is straightforward. □

The following proposition, whose proof is straightforward, gives the analogous result for continuous-time systems.

PROPOSITION 5.2. *Let* $((A_c, D(A_c)), B_c, C_c, D_c) \in C_{X_1}^{U,Y}$. *If* X_2 *is another Hilbert space and* $V: X_1 \to X_2$ *is a unitary operator, then*

(1) $((VA_cV^*, VD(A_c)), VB_c, (C_cV^*, VD(C_c)), D_c) \in C_{X_2}^{U,Y}$, *where*

$$(VB_c): U \to ((VD(A_c^*))^{(\prime)}, \|\cdot\|')$$

is given by

$$(VB_c)(u)[x] := B_c(u)[V^*x]$$

$u \in U$, $x \in VD(A_c^*)$.

(2) *If* (A_c, B_c, C_c, D_c) *is a state-space realization of the transfer function*

$$G_c(s): \mathrm{RHP} \to L(U, Y),$$

then $(VA_cV^*, VB_c, C_cV^*, D_c)$ *realizes the same transfer function.*

The following definition introduces the standard notation of unitary equivalence of state-space systems. Note that by the previous two propositions, unitarily equivalent systems have the same transfer function.

DEFINITION 5.3. Two systems $(A_c^i, B_c^i, C_c^i, D_c^i) \in C_{X_i}^{U,Y}$, $i = 1, 2$, are called unitarily equivalent, if there exists a unitary operator $V: X_1 \to X_2$ such that

$$(A_c^2, B_c^2, C_c^2, D_c^2) = (VA_c^1V^*, VB_c^1, C_c^1V^*, D_c^1).$$

An equivalent definition applies to admissible discrete-time systems.

We will now show that the transformation $T: D_X^{U,Y} \to C_X^{U,Y}$ and its inverse preserve the unitary equivalence of systems.

PROPOSITION 5.4. *Let* $(A_d^i, B_d^i, C_d^i, D_d^i) \in D_{X_i}^{U,Y}$, $i = 1, 2$. *Let* $(A_c^i, B_c^i, C_c^i, D_c^i) = T((A_d^i, B_d^i, C_d^i, D_d^i))$, $i = 1, 2$, *be the associated continuous-time systems.*

Then, $(A_c^1, B_c^1, C_c^1, D_c^1)$ *and* $(A_c^2, B_c^2, C_c^2, D_c^2)$ *are unitarily equivalent if and only if* $(A_d^1, B_d^1, C_d^1, D_d^1)$ *and* $(A_d^2, B_d^2, C_d^2, D_d^2)$ *are unitarily equivalent.*

Proof. Assume $(A_d^1, B_d^1, C_d^1, D_d^1)$ and $(A_d^2, B_d^2, C_d^2, D_d^2)$ are unitarily equivalent, i.e., there exists a unitary operator $V: X_1 \to X_2$ such that $(A_d^2, B_d^2, C_d^2, D_d^2) = (VA_d^1V^*, VB_d^1, C_d^1V^*, D_d^1)$.

Since $A_d^2 = VA_d^1V^*$ we have

$$A_c^2 = (I + A_d^2)^{-1}(A_d^2 - I) = (I + VA_d^1V^*)^{-1}(VA_d^1V^* - I) = VA_c^1V^*$$

with $D(A_c^2) = VD(A_c^1)$.

Let $u \in U$, $x_2 \in D((A_c^2)^*)$; then

$$\begin{aligned} B_c^2(u)[x_2] &= \sqrt{2}\langle B_d^2(u), (I + (A_d^2)^*)^{-1}x_2\rangle = \sqrt{2}\langle B_d^1(u), (I + (A_d^1)^*)^{-1}V^*x_2\rangle \\ &= B_c^1(u)[V^*x_2] = [VB_c^1](u)[x_2] \end{aligned}$$

and hence $B_c^2 = VB_c^1$.

$C_c^2 = C_c^1V^*$ since for $x \in D(C_c^2)$, we have

$$C_c^2x = \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} \sqrt{2}\, C_d^2(\lambda I + A_d^2)^{-1}x = \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} \sqrt{2}\, C_d^1(\lambda I + A_d^1)^{-1}V^*x = C_c^1V^*x.$$

The fact that $D_c^1 = D_c^2$ follows, since two unitarily equivalent systems have the same transfer function and thus

$$D_c^2 = D_d^2 - \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} C_d^2(\lambda I + A_d^2)^{-1}B_d^2 = D_d^1 - \lim_{\substack{\lambda \to 1 \\ \lambda > 1}} C_d^1(\lambda I + A_d^1)^{-1}B_d^1 = D_c^1.$$

The converse follows similarly. □

6. Dual systems. If (A, B, C, D) is a finite-dimensional linear system, then (A^T, C^T, B^T, D^T) is called the dual system of (A, B, C, D). It is well known that properties of a system are closely related to those of its dual systems. To examine the reachability operator of an infinite-dimensional system via the observability operator of its dual system, we will now define what we mean by the dual system of an admissible system.

We first consider discrete-time systems.

DEFINITION 6.1. Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$; then the dual system $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ of (A_d, B_d, C_d, D_d) is given by

$$\tilde{A}_d := A_d^*: X \to X, \qquad \tilde{B}_c := C_d^*: Y \to X,$$
$$\tilde{C}_d := B_d^*: X \to U, \qquad \tilde{D}_d := D_d^*: Y \to U.$$

The following lemma shows that the dual system of an admissible system is admissible and shows how the transfer function of a system is related to the transfer function of its dual system.

LEMMA 6.2. *The dual system $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ of an admissible discrete-time system (A_d, B_d, C_d, D_d) in $D_X^{U,Y}$ is an admissible system in $D_X^{Y,U}$.*

If the discrete-time transfer function $G(s): \mathbb{C}\backslash\bar{\mathbf{D}} \to \mathcal{L}(U, Y)$ has an admissible realization (A_d, B_d, C_d, D_d), then the dual system $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ is a realization of the transfer function $\tilde{G}(s): \mathbb{C}\backslash\bar{\mathbf{D}} \to \mathcal{L}(Y, U)$, $s \mapsto \tilde{G}(s) := (G(\bar{s}))^*$, i.e., for all $s \in \mathbb{C}\backslash\bar{\mathbf{D}}$,

$$\tilde{G}(s) = (G(\bar{s}))^* = \tilde{C}_d(sI - \tilde{A}_d)^{-1}\tilde{B}_d + \tilde{D}_d.$$

Proof. We must check (i)–(v) in Definition 2.1. To show (i) note that since $\|A_d^*\| = \|A_d\|$, we have that A_d^* is a contraction. Thus we only have to show that $-1 \notin \sigma_p(A_d^*)$. Assume there exists $x \in X$ such that $A_d^* x = -x$; then

$$0 \leqq \|A_d x + x\|^2 = \|A_d x\|^2 + 2\,\mathrm{Re}\,\langle x, A_d^* x\rangle + \|x\|^2$$
$$= \|A_d x\|^2 - 2\|x\|^2 + \|x\|^2 = \|A_d x\| - \|x\|^2 \leqq 0.$$

Thus $\|A_d x + x\|^2 = 0$ and hence $A_c x = -x$, which is a contraction to $-1 \notin \sigma_p(A_d)$.

The remaining parts of the lemma are straightforward to check. □

Next, we are going to define the dual system of an admissible continuous-time system.

DEFINITION 6.3. Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$. Then the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ of (A_c, B_c, C_c, D_c) is given by

$(\tilde{A}_c, D(\tilde{A}_c)) = (A_c^*, D(A_c^*))$, the adjoint operator of $(A_c, D(A_c))$;

$\tilde{B}_c: Y \to D(A_c)^{(\prime)}$, $y \mapsto \tilde{B}_c(y)[\cdot] := \langle y, C_c(\cdot)\rangle$;

$\tilde{C}_c: D(\tilde{C}_c) \to U$, $D(\tilde{C}_c) = D(\tilde{A}_c) + (I - \tilde{A}_c)^{-1}\tilde{B}_c Y$, where $\tilde{C}_0 x_0$ is defined by

$$\langle u\tilde{C}_c x_0\rangle = B_c(u)[x_0], \quad \text{for } x_0 \in D(A_c^*),\ u \in U,$$

and by

$$\langle \tilde{C}_c x_0 u\rangle = \langle y_0, C_c(I - A_c)^{-1}B_c u\rangle, \quad \text{for } x_0 = (I - \tilde{A}_c)^{-1}\tilde{B}_c y_0,\ y_0 \in Y,\ u \in U;$$

$\tilde{D}_c := D_c^*: Y \to U$.

The following lemma is the continuous-time equivalent of Lemma 6.2.

LEMMA 6.4. *The dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ of an admissible continuous-time system (A_c, B_c, C_c, D_c) is admissible.*

If the continuous-time transfer function $G(s): \mathrm{RHP} \to \mathcal{L}(U, Y)$ has an admissible realization (A_c, B_c, C_c, D_c), then the dual system $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a realization of the transfer function $\tilde{G}(s) := (G(\bar{s}))^*$, i.e., for all $s \in \mathrm{RHP}$,

$$\tilde{G}(s) = (G(\bar{s}))^* = \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c + \tilde{D}_c.$$

Proof. We must show (i)-(vi) of Definition 2.4. Of these, (i)-(iv) and (vi) are straightforward.

Before we show (v) of Definition 2.4 we show the last statement of the lemma. Let $u \in U$, $y \in Y$, and consider for $G(s) = C_c(sI - A_c)^{-1}B_c + D_c$,

$$\langle y, G(s)u\rangle = \langle y, (C_c(sI - A_c)^{-1}B_c + D_c)u\rangle.$$

Using the resolvent identity we have that

$$\begin{aligned}\langle y, G(s)u\rangle &= \langle y, C_c(I - A_c)^{-1}B_c u\rangle + \langle y, D_c u\rangle + (1-\bar{s})\langle y, C_c(I - A_c)^{-1}(sI - A_c)^{-1}B_c u\rangle\\ &= \langle \tilde{C}_c(I - \tilde{A}_c)^{-1}\tilde{B}_c y, u\rangle + \langle \tilde{D}_c y, u\rangle + (1-\bar{s})\langle y, C_c(I - A_c)^{-1}(sI - A_c)^{-1}B_c u\rangle.\end{aligned}$$

Note that for $x \in X$ we have

$$\langle y, C_c(I - A_c)^{-1}x\rangle = \tilde{B}_c(y)[(I - A_c)^{-1}x] = \langle (I - A_c^*)^{-1}\tilde{B}_c y, x\rangle.$$

Using this identity, we now obtain that

$$\begin{aligned}\langle y, C_c(I - A_c)^{-1}(sI - A_c)^{-1}B_c u\rangle &= \langle (I - A_c^*)^{-1}\tilde{B}_c y, (sI - A_c)^{-1}B_c u\rangle\\ &= \overline{\langle (sI - A_c)^{-1}B_c u, (I - A_c^*)^{-1}\tilde{B}_c y\rangle}\\ &= \overline{B_c(u)[(\bar{s}I - A_c^*)^{-1}(I - A_c^*)^{-1}\tilde{B}_c y]}\\ &= \overline{\langle u, \tilde{C}_c(\bar{s}I - A_c^*)^{-1}(I - A_c^*)^{-1}\tilde{B}_c y\rangle}\\ &= \langle \tilde{C}_c(\bar{s}I - \tilde{A}_c)^{-1}(I - \tilde{A}_c)^{-1}\tilde{B}_c y, u\rangle.\end{aligned}$$

Summarizing and again applying the resolvent identity, we have

$$\begin{aligned}\langle y, G(s)u\rangle &= \langle \tilde{C}_c(I - \tilde{A}_c)^{-1}\tilde{B}_c y, u\rangle + \langle \tilde{D}_c y, u\rangle + (1-\bar{s})\langle \tilde{C}_c(\bar{s}I - \tilde{A}_c)^{-1}(I - \tilde{A}_c)^{-1}\tilde{B}_c y, u\rangle\\ &= \langle (\tilde{C}_c(\bar{s}I - \tilde{A}_c)^{-1}\tilde{B}_c + \tilde{D}_c)y, u\rangle\\ &= \langle (G(s))^* y, u\rangle.\end{aligned}$$

Hence $(G(\bar{s}))^* = \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c + \tilde{D}_c$ for all $s \in \text{RHP}$. Now (v) of Definition 2.4 follows, since

$$\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}} \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c = \lim_{\substack{s\in\mathbf{R}\\ s\to\infty}} (C_c(sI - A_c)^{-1}B_c)^* = \left(\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}} C_c(sI - A_c)^{-1}B_c\right)^* = 0. \qquad \square$$

We will now show that the notion of duality of two systems is carried over between discrete- and continuous-time systems by the transformation T.

PROPOSITION 6.5. *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ and define $(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$. Let $(A_d^1, B_d^1, C_d^1, D_d^1) \in D_X^{U,Y}$ be another discrete-time system and let $(A_c^1, B_c^1, C_c^1, D_c^1) := T((A_d^1, B_d^1, C_d^1, D_d^1))$ be its corresponding admissible continuous-time system. Then,*

$$(A_d^1, B_d^1, C_d^1, D_d^1) \text{ is the dual system of } (A_d, B_d, C_d, D_d)$$

if and only if

$$(A_c^1, B_c^1, C_c^1, D_c^1) \text{ is the dual system of } (A_c, B_c, C_c, D_c).$$

Proof. Assume $(A_d^1, B_d^1, C_d^1, D_d^1)$ is the dual system of (A_d, B_d, C_d, D_d). Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ be the dual system of (A_c, B_c, C_c, D_c). Then $A_d^1 = \tilde{A}_d = A_d^*$ implies that

$$A_c^1 = (I + A_d^1)^{-1}(A_d^1 - I) = ((A_d - I)(I + A_d)^{-1})^* = A_c^* = \tilde{A}_c.$$

For a justification of the operations with the adjoints, see Weidmann [18, p. 74]. The identity $B_d^1 = \tilde{B}_d = C_d^*$ implies for $y \in Y$, $x \in D(A_c)$, that

$$\begin{aligned} B_c^1(y)[x] &= \sqrt{2}\langle B_d^1 y, (I+(A_d^1)^*)^{-1}x\rangle \\ &= \sqrt{2}\langle y, C_d(I+A_d)^{-1}x\rangle \\ &= \langle y, C_c x\rangle \\ &= \tilde{B}_c(y)[x] \end{aligned}$$

and hence $B_c^1 = \tilde{B}_c$.

To show that $C_c^1 = \tilde{C}_c$, we must consider two cases.

(i) For $x \in D(A_c^1)$ we have $C_c^1 x = \tilde{C}_c x$, since for $u \in U$,

$$\begin{aligned} \langle u, C_c^1 x\rangle = \sqrt{2}\langle u, C_d^1(I+A_d^1)^{-1}x\rangle &= \sqrt{2}\langle B_d u, (I+A_d^*)^{-1}x\rangle \\ &= B_c(u)[x] = \langle u, \tilde{C}_c x\rangle. \end{aligned}$$

(ii) Note that for $y_0 \in Y$, $x_0 := (I-A_c^1)^{-1}B_c^1 y_0 = (I-\tilde{A}_c)^{-1}\tilde{B}_c y_0$, since for $x \in D(A_c)$, we have

$$\begin{aligned} &(I-A_c^1)^{-1}B_c^1(y_0)[x] \\ &= \langle (I-A_c^1)^{-1}B_c^1(y_0), x\rangle = \frac{1}{\sqrt{2}}\langle B_d^1 y_0, x\rangle = \frac{1}{\sqrt{2}}\langle y_0, C_d x\rangle \\ &= \langle y_0, C_c(I-A_c)^{-1}x\rangle = \tilde{B}_c(y_0)[(I-A_c)^{-1}x] \\ &= (I-\tilde{A}_c)^{-1}\tilde{B}_c(y_0)[x]. \end{aligned}$$

Then for $u \in U$,

$$\begin{aligned} \langle C_c^1 x_0, u\rangle &= \lim_{\substack{\lambda\to 1\\ \lambda>1}} \langle C_d^1(\lambda I + A_d^1)^{-1}B_d^1 y_0, u\rangle = \lim_{\substack{\lambda\to 1\\ \lambda>1}} \langle y_0, C_d(\lambda I + A_d)^{-1}B_d u\rangle \\ &= \langle y_0, C_c(I-A_c)^{-1}B_c u\rangle = \langle \tilde{C}_c x_1, u\rangle \\ &= \langle \tilde{C}_c x_0, u\rangle \end{aligned}$$

where the last equality follows since $x_1 := (I-\tilde{A}_c)^{-1}\tilde{B}_c y_0 = x_0$ and hence $C_c^1 = \tilde{C}_c$.

Since $D_d^1 = \tilde{D}_d = \tilde{D}_d^*$, we have that

$$D_c^1 = D_d^1 - \lim_{\substack{\lambda\to 1\\ \lambda>1}} C_d^1(\lambda I + A_d^1)^{-1}B_d^1 = D_d^* - \lim_{\substack{\lambda\to 1\\ \lambda>1}} B_d^*(\lambda I + A_d^*)^{-1}C_d^* = D_c^* = \tilde{D}_c.$$

Hence we have that $(A_c^1, B_c^1, C_c^1, D_c^1)$ is the dual system of (A_c, B_c, C_c, D_c).

To show the converse, assume that $(A_c^1, B_c^1, C_c^1, D_c^1)$ is the dual system of (A_c, B_c, C_c, D_c) and let $(\tilde{A}_c, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ be the dual system of (A_d, B_d, C_d, D_d).

Then

$$A_d^1 = (I+A_c^1)(I-A_c^1)^{-1} = (I+A_c^*)(I-A_c^*)^{-1} = A_d^* = \tilde{A}_d$$

where we apply Theorem 4.19 of Weidmann [18] to justify the manipulations with the adjoints.

The fact that $B_d^1 = \tilde{B}_d$ follows from the following identities, where $y \in Y$, $x \in D(A_c)$,

$$\begin{aligned} \langle B_d^1 y, x\rangle &= \sqrt{2}\langle (I-A_c^1)^{-1}B_c^1 y, x\rangle = \sqrt{2}\, B_c^1(y)[(I-(A_c^1)^*)^{-1}x] \\ &= \sqrt{2}\, \tilde{B}_c(y)[(I-A_c)^{-1}x] = \sqrt{2}\langle y, C_c(I-A_c)^{-1}x\rangle = \langle y, C_d x\rangle \\ &= \langle \tilde{B}_d y, x\rangle. \end{aligned}$$

To show that $C_d^1 = \tilde{C}_d = B_d^*$, let $u \in U$, $x \in D(A_c^*)$,

$$\begin{aligned}\langle u, C_d^1 x\rangle &= \sqrt{2}\langle u, C_c^1(I - A_c^1)^{-1}x\rangle = \sqrt{2}\langle u, \tilde{C}_c(I - A_c^*)^{-1}x\rangle \\ &= \sqrt{2}\, B_c(u)[(I - A_c^*)^{-1}x] = \sqrt{2}\langle (I - A_c)^{-1}B_c u, x\rangle = \langle B_d u, x\rangle \\ &= \langle u, \tilde{C}_d x\rangle.\end{aligned}$$

Since also

$$D_d^1 = C_c^1(I - A_c^1)^{-1}B_c^1 + D_c^1 = (C_c(I - A_c)^{-1}B_c + D_c)^* = D_d^* = \tilde{D}_d,$$

we have the result. □

7. Observability and reachability operators. We are now in a position to discuss some of the central objects of this paper. We define the observability operator for admissible systems. The reachability operator of a system is introduced as the dual of the observability operator of its dual systems.

Having defined observability and controllability gramians of admissible systems, we show one of the main theorems of this paper. It states that the observability operator of a discrete-time system is related by a unitary transformation to the observability operator of its corresponding continuous-time system. This result is the main tool in proving that the transformation T maps discrete-time balanced realizations to continuous-time balanced realizations.

We first define the observability and reachability operators for discrete-time systems.

DEFINITION 7.1. Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$; then the operator

$$\begin{aligned}\mathcal{O}_d : D(\mathcal{O}_d) &\to l_Y^2 \\ x &\mapsto (C_d A_d^n x)_{n \geqq 0}\end{aligned}$$

is called the observability operator of the system (A_d, B_d, C_d, D_d), where

$$D(\mathcal{O}_d) = \{x \in X \mid (C_d A_d^n x)_{n \geqq 0} \in l_Y^2\}.$$

If $\mathcal{O}_d$ is bounded and $\ker(\mathcal{O}_d) = \{0\}$, then the system (A_d, B_d, C_d, D_d) is called observable.

Let $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ be the dual system of (A_d, B_d, C_d, D_d). If the observability operator $\tilde{\mathcal{O}}_d$ of $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ is bounded (and hence $D(\tilde{\mathcal{O}}_d) = X$), then the adjoint of $\tilde{\mathcal{O}}_d$ is called the reachability operator $\mathcal{R}_d$ of (A_d, B_d, C_d, D_d), i.e.,

$$\mathcal{R}_d := \tilde{\mathcal{O}}_d^*.$$

If $\mathcal{R}_d$ exists and range $(\mathcal{R}_d)$ is dense in X, the system (A_d, B_d, C_d, D_d) is called reachable.

The analogous definitions for continuous-time systems are now given.

DEFINITION 7.2. Let $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$, then the operator

$$\begin{aligned}\mathcal{O}_c : D(\mathcal{O}_c) &\to L_Y^2([0, \infty[) \\ x &\mapsto C_c e^{tA_c} x\end{aligned}$$

is called the observability operator of the system (A_c, B_c, C_c, D_c), where

$$D(\mathcal{O}_c) = \{x \in X \mid C_c e^{tA_c}x \text{ exists for almost all } t \in [0, \infty[,\ C_c e^{tA_c}x \in L_Y^2([0, \infty[)\}.$$

We say that (A_c, B_c, C_c, D_c) has a bounded observability operator if $D(A_c) \subseteq D(\mathcal{O}_c)$ and $\mathcal{O}_c$ extends to a bounded operator on X. This extension will also be denoted by $\mathcal{O}_c$.

If (A_c, B_c, C_c, D_c) has bounded observability operator $\mathcal{O}_c$ such that $\ker(\mathcal{O}_c)=\{0\}$, then the system (A_c, B_c, C_c, D_c) is called observable.

Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ be the dual system of (A_c, B_c, C_c, D_c). If the observability operator $\tilde{\mathcal{O}}_c$ of $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is a bounded operator on X, the adjoint of $\tilde{\mathcal{O}}_c$ is called the reachability operator $\mathcal{R}_c$ of (A_c, B_c, C_c, D_c), i.e.,

$$\mathcal{R}_c := \tilde{\mathcal{O}}_c^*.$$

If $\mathcal{R}_c$ exists and range $(\mathcal{R}_c)$ is dense in X, the system (A_c, B_c, C_c, D_c) is called reachable.

The notion of reachability and observability gramians as defined below is central in the discussion of balanced realizations in the next section.

DEFINITION 7.3. Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ with bounded reachability operator $\mathcal{R}_d$ and bounded observability operator $\mathcal{O}_d$. Then

$$\mathcal{W}_d := \mathcal{R}_d\mathcal{R}_d^* : X \to X, \qquad \mathcal{M}_d := \mathcal{O}_d^*\mathcal{O}_d : X \to X$$

are called the reachability and the observability gramian, respectively, of the system (A_d, B_d, C_d, D_d). The reachability gramian $\mathcal{W}_c$ and the observability gramian $\mathcal{M}_c$ of a continuous-time system with bounded reachability operator $\mathcal{R}_c$ and observability operator $\mathcal{O}_c$ are similarly defined to be

$$\mathcal{W}_c := \mathcal{R}_c\mathcal{R}_c^* : X \to X, \qquad \mathcal{M}_c := \mathcal{O}_c^*\mathcal{O}_c : X \to X.$$

Before stating the main theorems of this section we present a collection of standard results on Laguerre functions and straightforward modifications thereof. For a reference, see, e.g., Abramowitz and Stegun [1].

PROPOSITION 7.4. *There exists a complete set of orthogonal real-valued functions* $(L_n(t))_{n\geqq 0} \subseteq L^2([0,\infty[)$ *such that*

(i) $1/(1+z)\, e^{t((z-1)/(z+1))} = \sum_{n=0}^{\infty} L_n(t) z^n$ *for* $|z|<1$.

(ii) $\int_0^\infty L_n(t)L_m(t)\,dt = \frac{1}{2}\delta_{nm}$ *for all* n, m.

(iii) $|L_n(t)| \leqq 1$ $t \in [0,\infty[$, *for* $n \geqq 0$.

(iv) $L_n(t) \in L^1([0,\infty[)$ *for* $n \geqq 0$.

(v) *If* Y *is a separable Hilbert space, then the operator*

$$W : l_Y^2 \to L_Y^2([0,\infty[), \qquad (x_n)_{n\geqq 0} \mapsto \sqrt{2} \sum_{n=0}^{\infty} L_n(t) x_n$$

is unitary, with adjoint

$$W^* : L_Y^2([0,\infty[) \to l_Y^2, \qquad f(t) \mapsto \sqrt{2} \left(\int_0^\infty f(t) L_n(t)\,dt \right)_{n\geqq 0}.$$

Now we will state and prove the main theorems of this section. They show that the observability operators of discrete-time systems are related to the observability operators of their corresponding continuous-time systems by a unitary transformation of the input spaces and vice versa. We first consider the case where a discrete-time system is given. Here the connection of its observability operator to the observability operator of its corresponding continuous-time system is investigated.

THEOREM 7.5. *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ *and let* $(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$ *be the corresponding admissible continuous-time system. Then,*

(i) *For* $x \in D((I+A_d)^{-1}) \cap D(\mathcal{O}_d)$, *we have* $x \in D(\mathcal{O}_c)$ *and*

$$\mathcal{O}_c x = W\mathcal{O}_d x$$

where $\mathcal{O}_c$ is the observability operator of (A_c, B_c, C_c, D_c), and W is the unitary operator defined in Proposition 7.4.

(ii) *If $\mathcal{O}_d : X \to l^2_Y$ is bounded, then $\mathcal{O}_c$ extends to a bounded operator given by*

$$\mathcal{O}_c = W\mathcal{O}_d.$$

Proof. (i) Let $x \in D((I+A_d)^{-1}) \cap D(\mathcal{O}_d) = D(A_c) \cap D(\mathcal{O}_d)$. Write

$$F:[0,\infty[\to Y, \qquad t \mapsto F(t) = \sum_{n=0}^{\infty} L_n(t) C_d A_d^n x,$$

$$F_r:[0,\infty[\to Y, \qquad t \mapsto F_r(t) = \sum_{n=0}^{\infty} L_n(t) r^n C_d A_d^n x,$$

$$G:[0,\infty[\to Y, \qquad t \mapsto G(t) = C_c\, e^{tA_c} x,$$

with $0<r<1$. The function $G(t)$, $t \in [0,\infty[$, is well-defined since $e^{tA_c}x \in D(A_c)$ for $x \in D(A_c)$ and hence $e^{tA_c}x \in D(C_c)$.

First note that $F(t)$ is well defined and in $L^2_Y([0,\infty])$, because $(C_d A_d^n x)_{n \ge 0} \in l^2_Y$ and because $(\sqrt{2}\, L_n(t))_{n\ge 0}$ forms an orthonormal basis in $L^2([0,\infty[)$.

Now we are going to show that

$$\sqrt{2}\, F_r(t) \to G(t) \quad \text{pointwise weakly as } r \to 1-0.$$

Using the notation and results of § 3 we have that $\sum_{n=0}^{\infty} L_n(t) r^n A_d^n = \delta_t(rA_d)$, $0<r<1$, since $\delta_t(rz) \in \mathcal{A}$. Hence,

$$F_r(t) = \sum_{n=0}^{\infty} L_n(t) r^n C_d A_d^n x = C_d \left(\sum_{n=0}^{\infty} L_n(t) r^n A_d^n \right)$$

$$= C_d \delta_t(rA_d) = C_d (I + rA_d)^{-1}\, e^{t(I+rA_d)^{-1}(I-rA_d)} x.$$

Weak convergence now follows from Proposition 3.11.

But $F_r(t) \to F(t)$ in $L^2_Y([0,\infty[)$ as $r \to 1-0$, since

$$(r^n C_d A_d^n x)_{n\ge 0} \to (C_d A_d^n x)_{n \ge 0} \quad \text{in } l^2_Y \text{ as } r \to 1-0.$$

We can now show that these two convergence results imply that for all $y \in Y$

$$\langle y, G(t)\rangle_Y = \langle y, \sqrt{2}\, F(t)\rangle_Y$$

almost everywhere for all $t \in [0,\infty[$. For otherwise, there is an $\varepsilon > 0$ and a measurable set $A \subseteq [0,\infty[$ with Lebesgue measure $\lambda(A) = \varepsilon$ such that

$$\langle y, G(t)\rangle_Y - \langle y, \sqrt{2}\, F(t)\rangle_Y > \varepsilon$$

for $t \in A$. Now, clearly there is an r_0 such that for $r \ge r_0$

$$\lambda\{t \in A: \langle y, \sqrt{2}\, F_r(t)\rangle - \langle y, \sqrt{2}\, F(t)\rangle > \varepsilon/2\} < \varepsilon/2,$$

and by Egoroff's Theorem, there is an r_1 such that for $r \ge r_1$

$$\lambda\{t \in A: \langle y, \sqrt{2}\, F_r(t)\rangle - \langle y, G(t)\rangle > \varepsilon/2\} < \varepsilon/2.$$

These three statements together form a contradiction.

Now, since Y is separable we have that

$$C_c\, e^{tA_c} x = \sqrt{2} \sum_{n=0}^{\infty} L_n(t) C_d A_d^n x$$

almost everywhere for $t \in [0,\infty[$. Thus $\mathcal{O}_c(x) = W\mathcal{O}_d(x)$ for $x \in D(A_c)$.

(ii) Since $\mathcal{O}_d$ is bounded, W is unitary, and $D(A_c)$ is dense in X, $\mathcal{O}_c$ extends to a bounded operator on X. □

The corollary to this theorem shows the equivalent result for reachability operators.

COROLLARY 7.6. *Let* $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ *and let* $(A_c, B_c, C_c, D_c) := T((A_d, B_d, C_d, D_d))$ *be the corresponding admissible continuous-time system.*

Then, if the reachability operator $\mathcal{R}_d$ *of* (A_d, B_d, C_d, D_d) *exists as a bounded operator, the reachability operator* $\mathcal{R}_c$ *of* (A_c, B_c, C_c, D_c) *exists as a bounded operator and is given by*

$$\mathcal{R}_c = \mathcal{R}_d W^*.$$

Proof. Let $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ be the dual system of (A_d, B_d, C_d, D_d). By definition $\mathcal{R}_d = \tilde{\mathcal{O}}_d^*$, where $\tilde{\mathcal{O}}_d$ is the observability operator of $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$. Now consider $T((\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)) =: (\tilde{A}_c, \tilde{B}_d, \tilde{C}_c, \tilde{D}_c)$. By Proposition 6.5 we know that $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ is the dual system of (A_c, B_c, C_c, D_c). But the reachability operator $\mathcal{R}_c$ of (A_c, B_c, C_c, D_c) is given by $\mathcal{R}_c = \tilde{\mathcal{O}}_c^*$, where $\tilde{\mathcal{O}}_c$ is the observability operator of $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$. By the previous theorem $\tilde{\mathcal{O}}_c = W\tilde{\mathcal{O}}_d$ and hence $\mathcal{R}_c = \tilde{\mathcal{O}}_c^* = \tilde{\mathcal{O}}_d^* W^* = \mathcal{R}_d W^*$. □

We now show that if a continuous-time system has a bounded observability operator $\mathcal{O}_c$ then the observability operator of its corresponding discrete time system is given by a unitary transformation of $\mathcal{O}_c$.

THEOREM 7.7. *Let* $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ *and let* $(A_d, B_d, C_d, D_d) = T^{-1}((A_c, B_c, C_c, D_c))$ *be the corresponding admissible discrete time system. Then,*

(i) *For* $x \in D(A_c) \cap D(\mathcal{O}_c)$, *we have* $x \in D(\mathcal{O}_d)$ *and*

$$\mathcal{O}_d x = W^* \mathcal{O}_c x$$

where $\mathcal{O}_d$ *is the observability operator of* (A_d, B_d, C_d, D_d) *and* W^* *is the unitary operator defined in Proposition* 7.4.

(ii) *If* $\mathcal{O}_c$ *is bounded, then* $\mathcal{O}_d$ *extends to a bounded operator on* X *given by*

$$\mathcal{O}_d = W^* \mathcal{O}_c.$$

Proof. (i) Let $x \in D(A_c) \cap D(\mathcal{O}_c)$; then we know that $G(t) := C_c e^{tA_c} x$ exists for all $t \in [0, \infty]$, since $e^{tA_c} x \in D(A_c) \subseteq D(C_c)$, $t \in [0, \infty[$. By assumption $G(t) \in L_Y^2([0, \infty[)$.

Corollary 4.4 implies that

$$G(t) = \sqrt{2}\, C_d (I + A_d)^{-1} e^{t(I+A_d)^{-1}(A_d - I)} x.$$

For $0 < r < 1$, let $G_r(t) = \sqrt{2}\, C_d (I + rA_d)^{-1} e^{t(I+rA_d)^{-1}(rA_d - I)} x$. Since C_d is bounded, we have by Proposition 3.11 that for all $t \in [0, \infty[$,

$$\lim_{r \to 1-0} G_r(t) = G(t) \quad \text{weakly}.$$

Since C_d is bounded and $\delta_t(rz) \in \mathcal{A}$, where δ_t is as defined in § 3, we have that

$$\begin{aligned} G_r(t) &= \sqrt{2}\, C_d (I + rA_d)^{-1} e^{t(I+rA_d)^{-1}(rA_d - I)} x \\ &= \sqrt{2}\, C_d \delta_t(rA_d) x \\ &= \sqrt{2}\, C_d \left(\sum_{n=0}^{\infty} L_n(t) r^n A_d^n x \right) \\ &= \sqrt{2} \sum_{n=0}^{\infty} L_n(t) r^n C_d A_d^n x \in L_Y^2([0, \infty]). \end{aligned}$$

We will now show that there exists $M>0$ such that

$$\|G_r(t)\| \leqq M < \infty \quad \text{for all } 0<r<1, \quad t\in[0,\infty[.$$

Let $t\in[0,\infty[$; then

$$\begin{aligned}\sup_{0<r<1} \|G_r(t)\| &= \sup_{0<r<1} \|\sqrt{2}\, C_d(I+rA_d)^{-1} e^{t(I+rA_d)^{-1}(rA_d-I)}x\| \\ &\leqq \sqrt{2}\,\|C_d\| \sup_{0<r<1} \|(I+rA_d)^{-1} e^{t(I+rA_d)^{-1}(rA_d-I)}x\| \\ &\leqq M<\infty\end{aligned}$$

where the second to last line follows from Proposition 3.11 noting that $D(A_c)=D((I+A_d)^{-1})$.

Thus for $y\in Y$, we have that

$$|\langle G_r(t), y\rangle\sqrt{2}\, L_n(t)| \leqq \sqrt{2}\, M\|y\|\,|L_n(t)|, \qquad t\in[0,\infty[.$$

Since $L_n(t)\in L^1([0,\infty[)$, we can therefore apply the Dominated Convergence Theorem:

$$\lim_{r\to 1-0}\int_0^\infty \langle G_r(t), y\rangle\sqrt{2}\, L_n(t)\, dt = \int_0^\infty \langle G(t), y\rangle\sqrt{2}\, L_n(t)\, dt.$$

But

$$\begin{aligned}\int_0^\infty \langle G_r(t), y\rangle\sqrt{2}\, L_n(t)\, dt &= \int_0^\infty \left\langle \sum_{i=0}^\infty \sqrt{2}\, L_i(t) r^i C_d A_d^i x, y\right\rangle\sqrt{2}\, L_n(t)\, dt \\ &= 2\sum_{i=0}^\infty \left(r^i\langle C_d A_d^i x, y\rangle \int_0^\infty L_i(t)L_n(t)\, dt\right) \\ &= r^n\langle C_d A_d^n x, y\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty \langle G(t), y\rangle\sqrt{2}\, L_n(t)\, dt &= \lim_{r\to 1-0}\int_0^\infty \langle G_r(t), y\rangle\sqrt{2}\, L_n(t)\, dt \\ &= \langle C_d A_d^n x, y\rangle.\end{aligned}$$

Since $G(t)\in L_Y^2([0,\infty[)$, we have an expansion

$$G(t)=\sum_{n=0}^\infty G_n\sqrt{2}\, L_n(t), \qquad G_n\in Y.$$

Thus $\langle G_n, y\rangle = \langle C_d A_d^n x, y\rangle$ for all $y\in Y$ and hence $G_n = C_d A_d^n x$ for $n\geqq 0$. This implies that

$$\mathcal{O}_c(x) = C_c e^{tA_c}x = \sqrt{2}\sum_{n=0}^\infty L_n(t) C_d A_d^n x = W\mathcal{O}_d(x)$$

and hence $\mathcal{O}_d(x) = W^*\mathcal{O}_c(x)$.

(ii) This is a straightforward consequence of (i). □

In the following corollary the corresponding result is established for the reachability operators.

COROLLARY 7.8. *Let $(A_c, B_c, C_c, D_c)\in C_X^{U,Y}$ and let $(A_d, B_d, C_d, D_d) := T^{-1}((A_c, B_c, C_c, D_c))$ be the corresponding admissible discrete-time system. Then, if the reachability operator $\mathcal{R}_c$ of (A_c, B_c, C_c, D_c) exists as bounded operator, the reachability operator $\mathcal{R}_d$ of (A_d, B_d, C_d, D_d) exists as a bounded operator and is given by*

$$\mathcal{R}_d = \mathcal{R}_c W.$$

Proof. Let $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ be the dual system of (A_c, B_c, C_c, D_c). By definition, $\mathcal{R}_c = \tilde{\mathcal{O}}_c^*$, where $\tilde{\mathcal{O}}_c$ is the observability operator of $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$. Now consider $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d) = T^{-1}((\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c))$. By Proposition 6.5 we know that $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$ is the dual system of (A_d, B_d, C_d, D_d). But the reachability operator $\mathcal{R}_d$ of (A_d, B_d, C_d, D_d) is given by $\mathcal{R}_d = \tilde{\mathcal{O}}_d^*$, where $\tilde{\mathcal{O}}_d$ is the observability operator of $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d, \tilde{D}_d)$. By the previous theorem $\tilde{\mathcal{O}}_d = W^*\tilde{\mathcal{O}}_c$. Hence $\mathcal{R}_d = \tilde{\mathcal{O}}_d^* = \tilde{\mathcal{O}}_c^* W = \mathcal{R}_c W$. □

The following corollary to the previous two theorems shows that the properties of observability and reachability as well as the observability and reachability gramians are preserved by the transformation T.

COROLLARY 7.9. *Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ and $(A_c, B_c, C_c, D_c) \in C_X^{U,Y}$ such that*

$$(A_c, B_c, C_c, D_c) = T((A_d, B_d, C_d, D_d)).$$

Then,

(1) *(A_c, B_c, C_c, D_c) is observable (reachable) if and only if (A_d, B_d, C_d, D_d) is observable (reachable).*

(2) *If the reachability gramians $\mathcal{W}_c$, $\mathcal{W}_d$ (observability gramians $\mathcal{M}_c$, $\mathcal{M}_d$) of (A_d, B_d, C_d, D_d) and (A_c, B_c, C_c, D_c) are defined, then*

$$\mathcal{W}_c = \mathcal{W}_d \qquad (\mathcal{M}_c = \mathcal{M}_d).$$

8. Balanced realizations. We will now apply the results on infinite-dimensional state-space systems of the previous sections to tackle the problem that motivated this paper, namely, that of the existence of balanced realizations for continuous-time systems. Our results will allow us to deal with a wider range of transfer functions than previous results; for example, we can handle any transfer function that is bounded in the RHP, and with a limit at infinity along the real axis. This allows us to consider nonstrictly proper delay systems with transfer functions such as $G(s)\,e^{-sT}$, where $G(s)$ is a matrix-valued stable rational transfer function. Previous results were unable to deal with, for example, the pure delay system e^{-sT} because the limits $\lim_{w\to+\infty} e^{-iwt}$ and $\lim_{w\to-\infty} e^{-iwt}$ do not exist and, therefore, the corresponding Hankel operator is not compact. Another example of a function we will be able to deal with is $G(s) = \log(1+1/s)$. This function is unusual in that it has a singularity at 0.

The approach taken is to carry over the discrete-time results by Young using the transformation $T: D_X^{U,Y} \to C_X^{U,Y}$. Thus we will first review Young's results before we turn to proving the continuous-time analogue of his discrete-time realization theorem.

The following definition recalls the notion of a balanced system as defined by Moore [10] and the notion of a parbalanced system as introduced by Young [20].

DEFINITION 8.1. Let $(A_d, B_d, C_d, D_d) \in D_X^{U,Y}$ $((A_c, B_c, C_c, D_c) \in C_X^{U,Y})$ be such that the observability gramian $\mathcal{M}_d$ $(\mathcal{M}_c)$ and reachability gramian $\mathcal{W}_d(\mathcal{W}_c)$ exist. Then the system is

(i) Parbalanced, if $\mathcal{M}_d = \mathcal{W}_d (\mathcal{M}_c = \mathcal{W}_c)$;

(ii) Balanced, if it is parbalanced and moreover the gramians are diagonal.

Before we state any results, we introduce some notation. Let $H: \mathbf{D} \to \mathcal{L}(U, Y)$ be analytic. We say that $H \in P_+L^\infty(\mathbf{D}, \mathcal{L}(U, Y))$ if there exists an analytic function $F: \mathbf{D} \to \mathcal{L}(U, Y)$ such that $H + \bar{F}$ is essentially bounded, where $\bar{F}(z) = F(z^{-1})$. Furthermore, if F can be chosen so that $H + \bar{F} \in C(\mathbf{D}, \mathcal{K}(U, Y))$, where $C(\mathbf{D}, \mathcal{K}(U, Y))$ is the set of norm continuous functions on $\partial\mathbf{D}$ with values in the set of compact operators from U to Y, then H is said to be in $P_+C(\mathbf{D}, \mathcal{K}(U, Y))$.

Similarly, if $H:\mathrm{RHP}\to\mathscr{L}(U,Y)$ is analytic, we say that $H\in P_+L^\infty(\mathrm{RHP},\mathscr{L}(U,Y))$ $(P_+C(\mathrm{RHP},\mathscr{K}(U,Y)))$ if there is an analytic function $F:\mathrm{RHP}\to\mathscr{L}(U,Y)(\mathscr{K}(U,Y))$ such that $H+\tilde{F}$ is essentially bounded (extends to a norm continuous function on the imaginary axis such that $\lim_{w\in\mathbf{R},w\to\infty}(H+\tilde{F})(iw)=\lim_{w\in\mathbf{R},w\to\infty}(H+\tilde{F})(-iw)$), where $\tilde{F}(s)=F(-s)$.

Remark 8.2. If $H\in P_+L^\infty(\mathbf{D},\mathscr{L}(U,Y))$, then the Hankel operator with symbol H is bounded by an operator-valued version of Nehari's Theorem, whereas by Hartmann's Theorem it is compact if $H\in P_+C(\mathbf{D},\mathscr{L}(U,Y))$. Note that if U and Y are finite-dimensional, $H\in P_+L^\infty(\mathbf{D},\mathscr{L}(U,Y))$ $(P_+L^\infty(\mathrm{RHP},\mathscr{L}(U,Y)))$ if and only if H is in $BMOA(\partial\mathbf{D})$ $(BMOA(i\mathbb{R}))$ and $H\in P_+C(\mathbf{D},\mathscr{K}(U,Y))$ $(P_+C(\mathrm{RHP},\mathscr{K}(U,Y)))$ if and only if H is in $VMOA(\partial\mathbf{D})$ $(VMOA(i\mathbb{R}))$ (for references, see [15]).

The following theorem by Young [20], gives criteria for a (par-) balanced realization to exist of a discrete-time transfer function.

THEOREM 8.3. *Let $G_d(z):\mathbb{C}\backslash\bar{\mathbf{D}}\to\mathscr{L}(U,Y)$ be analytic with $G_d(\infty)=D_d\in\mathscr{L}(U,Y)$, and write*

$$g(z):=\frac{1}{z}\left(G_d\left(\frac{1}{z}\right)-D_d\right),\qquad z\in\mathbf{D}.$$

(i) *If $g\in P_+L^\infty(\mathbf{D},\mathscr{L}(U,Y))$, then there exists a separable Hilbert space X and a discrete-time state-space realization (A_d,B_d,C_d,D_d) of $G_d(z)$ with state space X, such that*

$$A_d\in\mathscr{L}(X)\quad \textit{is a contraction,}$$

$$B_d\in\mathscr{L}(U,X),\qquad C_d\in\mathscr{L}(X,Y),$$

and (A_d,B_d,C_d,D_d) is reachable and observable with bounded reachability and observability operators, such that (A_d,B_d,C_d,D_d) is parbalanced, i.e., $\mathcal{M}_d=\mathcal{W}_d$. The gramians $\mathcal{M}_d$, $\mathcal{W}_d$ satisfy the Lyapunov equations

$$A_d\mathcal{W}_dA_d^*-\mathcal{W}_d=-B_d^*B_d,\qquad A_d^*\mathcal{M}_dA_d-\mathcal{M}_d=-C_dC_d^*.$$

If $(\bar{A}_d,\bar{B}_d,\bar{C}_d,\bar{D}_d)$ is another parbalanced realization of $G_d(z)$ with state space $\bar{X}$, then (A_d,B_d,C_d,D_d) and $(\bar{A}_d,\bar{B}_d,\bar{C}_d,\bar{D}_d)$ are unitarily equivalent.

(ii) *If, moreover, $g\in P_+C(\mathbf{D},\mathscr{K}(U,Y))$, there is a basis in X with respect to which (A_d,B_d,C_d,D_d) is balanced.*

To show that for a transfer function G_d such that $\lim_{\lambda<-1,\lambda\to-1}G_d(\lambda)\in\mathscr{L}(U,Y)$, the realization given in the previous theorem is, in fact, admissible, we need to show that -1 is not an eigenvalue of A_d.

LEMMA 8.4. *Let (A_d,B_d,C_d,D_d) be a parbalanced realization of a discrete-time transfer function as given in Theorem 8.3; then $-1\notin\sigma_p(A_d)$.*

Proof. Let $\mathcal{M}_d$ be the observability gramian of (A_d,B_d,C_d,D_d); then

$$A_d^*\mathcal{M}_dA_d-\mathcal{M}_d=-C_d^*C_d.$$

Assume $-1\in\sigma_p(A_d)$ with eigenvector $x\neq 0$, then

$$\langle x,A_d^*\mathcal{M}_dA_dx\rangle-\langle x,\mathcal{M}_dx\rangle=-\langle x,C_d^*C_dx\rangle$$

and hence

$$\langle A_dx,\mathcal{M}_dA_dx\rangle-\langle x,\mathcal{M}_dx\rangle=0=-\|C_dx\|^2,$$

which implies that $C_dx=0$. Hence for all $n\geqq 0$, $\mathcal{O}_dx=(C_dA_d^nx)_{n\geqq 0}=(-1)^nC_dx=0$, which is a contradiction to the observability of (A_d,B_d,C_d,D_d). □

We can now apply our results on the transformation T to obtain realization results for continuous-time transfer functions.

THEOREM 8.5. *Let* $G_c:\mathrm{RHP}\to\mathscr{L}(U,Y)$ *be a continuous-time transfer function that is analytic and such that* $\lim_{s\in\mathbf{R},s\to\infty} G_c(s)\in\mathscr{L}(U,Y)$ *exists.*

(i) *If* $G_c\in P_+L^\infty(\mathrm{RHP},\mathscr{L}(U,Y))$, *then there exists a separable Hilbert space* X *and a parbalanced admissible continuous-time state-space realization* (A_c,B_c,C_c,D_c) *of* G_c *with state space* X. *This system is reachable, observable, and has bounded reachability and observability operators.*

If $(\bar{A}_c,\bar{B}_c,\bar{C}_c,\bar{D}_c)$ *is another parbalanced realization of* $G_c(s)$, *then* (A_c,B_c,C_c,D_c) *and* $(\bar{A}_c,\bar{B}_c,\bar{C}_c,\bar{D}_c)$ *are unitarily equivalent.*

(ii) *If, moreover,* $G_c\in P_+C(\mathrm{RHP},\mathscr{K}(U,Y))$, *then there is a basis in* X *with respect to which* (A_c,B_c,C_c,D_c) *is balanced.*

Proof. Let $G_d:\mathbb{C}\backslash\bar{\mathbf{D}}\to\mathscr{L}(\mathrm{U},\mathrm{Y})$ be the associated discrete-time transfer function $G_d(z)=G_c((z-1)/(z+1))$, and write

$$g(z)=\frac{1}{z}\left(G_d\left(\frac{1}{z}\right)-G_c(1)\right),\qquad z\in\mathbf{D}.$$

Then it is easy to see that $G_c\in P_+L^\infty(\mathrm{RHP},\mathscr{L}(U,Y))(P_+C(\mathrm{RHP},\mathscr{K}(U,Y)))$ if and only if $g\in P_+L^\infty(\mathbf{D},\mathscr{L}(U,Y))(P_+C(\mathbf{D},\mathscr{K}(U,Y)))$.

Hence $G_d(z)$ has a parbalanced realization (A_d,B_d,C_d,D_d) that is admissible since A_d is a contraction, such that $-1\notin\sigma_p(A_d)$ by Lemma 8.4 and since

$$\lim_{\substack{\lambda<1\\ \lambda\to 1}} C_d(\lambda I+A_d)^{-1}B_d=-\lim_{\substack{\lambda<-1\\ \lambda\to -1}} G_d(\lambda)+D_d=-\lim_{\substack{s\in\mathbf{R}\\ s\to\infty}} G_c(s)+D_d\in\mathscr{L}(U,Y)$$

exists. Then $(A_c,B_c,C_c,D_c):=T((A_d,B_d,C_d,D_d))\in C_X^{U,Y}$ is a state-space realization of G_c (Theorem 4.1) that is observable and reachable, such that $\mathscr{W}_c=\mathscr{W}_d$ and $\mathscr{M}_c=\mathscr{M}_d$ (Corollary 7.9).

The statement on the uniqueness of the realization follows from Proposition 5.4. □

The following corollary discusses special cases of transfer functions and gives simple criteria for the existence of a parbalanced or balanced realization of a continuous-time transfer function.

COROLLARY 8.6. *Let* $G_c(s):\mathrm{RHP}\to\mathscr{L}(U,Y)$ *be a continuous-time transfer function, such that* $\lim_{s\in\mathbf{R},s\to\infty} G_c(s)\in\mathscr{L}(U,Y)$ *exists and* $G_c(s)$ *is analytic in* RHP.

(i) *If* $G_c(s)$ *is bounded in the* RHP, *i.e.,* $\sup_{s\in\mathrm{RHP}}\|G_c(s)\|<\infty$, *then* $G_c(s)$ *has a parbalanced realization.*

(ii) *If, in particular,* $G_c(s):\mathrm{RHP}\mapsto\mathscr{K}(U,Y)$, *such that* G_c *is bounded in the* RHP *and* $G_c(s)$ *is norm continuous on the imaginary axis including at the points* $+\infty$ *and* $-\infty$, *i.e.,* $w\mapsto G_c(iw)$, $w\in\mathbb{R}$, *is norm continuous and* $\lim_{w\to-\infty} G_c(iw)=\lim_{w\to+\infty} G_c(iw)$, *then* $G_c(s)$ *has a balanced realization.*

As examples to the previous realization results we can consider delay systems. It follows immediately from the previous corollary that the transfer function e^{-sT} of a pure delay with time constant $T>0$, has a parbalanced realization. Note that the limits $\lim_{w\to+\infty} e^{-iwT}$ and $\lim_{w\to-\infty} e^{-iwT}$ do not exist. If $G(s)$ is a matrix-valued strictly proper stable rational transfer function, then the transfer function $G(s)\,e^{-sT}$ of the delayed system has a balanced realization.

Another example of a function that has a parbalanced continuous-time state-space realization is the function $G_c(s)=\log(1+1/s)$, which is well known to be in $BMOA(i\mathbb{R})$. We note that G_c has a singularity at 0.

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