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A Note on a System Theoretic Approach to a Conjecture by Peller–Khrushchev: The General Case

RAIMUND OBER

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK

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Based on the construction of infinite-dimensional balanced realizations, an alternative solution to the following inverse spectral problem is presented. Given a decreasing sequence of positive numbers $(\sigma_n)_{n\geq 1}$ (i.e. $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq 0$), does there exist a Hankel operator whose sequence of singular values is $(\sigma_n)_{n\geq 1}$? This paper is an extension of a previously published paper in which the same approach was taken in the case of a monotonically decreasing sequence $(\sigma_n)_{n\geq 1}$.

1. Introduction

MOTIVATED by important problems in prediction theory for Gaussian processes and approximation theory for rational functions [3, 5], Peller and Khrushchev have pointed out the significance of inverse spectral problems for Hankel operators. In this context, they conjectured that, given any nonincreasing sequence $(\sigma_n)_{n\geq 1}$ of positive numbers with $\lim_{n\to\infty} \sigma_n = 0$, there exists a Hankel operator Γ whose singular values $(\sigma_n(\Gamma))_{n\geq 1}$ satisfy $\sigma_n(\Gamma) = \sigma_n$ for all $n \in \mathbb{N}$.

Using delicate function theoretic and functional analytic arguments, Treil & Vasyunin [8] proved this conjecture also for the case when $\lim_{n\to\infty} \sigma_n \neq 0$.

A solution for the case of a strictly monotonically decreasing sequence $(\sigma_n)_{n\geq 1}$, which was based on a system theoretic construction, was published in [7]. A recent result by Arendt & Batty [1] on the asymptotic stability of strongly continuous semigroups of operators allows us to extend to the general case a construction of finite-rank Hankel operators with possibly repeated singular values as implicitly given in [6]. It is therefore now possible to give a complete solution of the problem using balanced infinite-dimensional state-space systems. While this approach allows us to construct Hankel operators with prescribed nonzero singular values and thereby to prove the conjecture as stated in [3], it is at the moment not clear how to deal with the kernel of the Hankel operator within our framework. It is well known that the kernel of a Hankel operator is either zero or infinite dimensional [3]. The method given in [8] not only solves the conjecture but also allows us to assign the dimension of the kernel of the Hankel operator to be either zero or infinite dimensional.

The construction will be given in terms of integral Hankel operators defined as follows:

$$\Gamma : \mathrm{L}^2([0,\infty)) \to \mathrm{L}^2([0,\infty)) : u(t) \mapsto \int_0^\infty h(t+s)u(t)\,\mathrm{d}t,$$

where $h(t) \in \mathrm{L}^2([0,\infty))$ is chosen such that Γ is bounded.

Recall that the singular values $\sigma_n(A)$ $(n \geqslant 0)$ of a Hilbert space operator A are defined as follows: $\sigma_n(A)$ is the nth point from the right of the spectrum of the modulus $|A| = (A^*A)^{\frac{1}{2}}$, whose eigenvalues are counted with multiplicities and the first point of the continuous spectrum is an eigenvalue of infinite multiplicity.

2. Balanced state-space systems and finite-rank Hankel operators

The construction of a Hankel operator with a desired set of singular values is based on a connection between state-space systems and Hankel operators. We shall briefly review this connection for the particular case of balanced realizations.

DEFINITION 2.1 (see [4]) Let $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$ be such that (A, b, c) is a minimal and asymptotically stable continuous-time system. Then (A, b, c) is called balanced if, for

$$W_c = \int_0^\infty e^{At}bb^{\mathrm{T}}e^{tA^{\mathrm{T}}}\,dt, \qquad W_o = \int_0^\infty e^{tA^{\mathrm{T}}}c^{\mathrm{T}}ce^{tA}\,dt,$$

we have $W_c = W_o =: \Sigma =: \operatorname{diag}(\sigma_1, ..., \sigma_n)$.

The following characterization of balanced systems is of particular importance.

PROPOSITION 2.2 (see [4]) *Let $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$ be such that (A, b, c) is a minimal and asymptotically stable continuous-time system. Then (A, b, c) is balanced if and only if there is a diagonal matrix Σ with positive diagonal entries which satisfies the Lyapunov equations*

$$A\Sigma + \Sigma A^{\mathrm{T}} = -bb^{\mathrm{T}}, \qquad A^{\mathrm{T}}\Sigma + \Sigma A = -c^{\mathrm{T}}c.$$

Moreover, if (A, b, c) is balanced then $\Sigma = W_c = W_o$.

Given a system (A, b, c) we associate with it an integral Hankel operator Γ with kernel $h(t) = ce^{tA}b$ $(t \geqslant 0)$. The following proposition relates the singular values of the Hankel operator Γ to the controllability Gramian W_c (and the observability Gramian W_o) of a balanced system.

PROPOSITION 2.3 (see e.g. [2]) *Let (A, b, c) be a minimal and asymptotically stable balanced system, with $\Sigma = W_c = W_o = \operatorname{diag}(\sigma_1, \sigma_2, ..., \sigma_n)$, such that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$, then the Hankel operator Γ with kernel $h(t) = ce^{tA}b$ $(t \geqslant 0)$ is of rank n and has singular values $\sigma_1(\Gamma) = \sigma_1$, $\sigma_2(\Gamma) = \sigma_2, ..., \sigma_n(\Gamma) = \sigma_n$.*

It is clear from this proposition that the problem is solved for finite-rank Hankel operators if we are able to construct finite-dimensional balanced state-space systems with prescribed controllability Gramians W_c. This is indeed possible and follows from the following characterization result for rational transfer functions.

THEOREM 2.4 (see [6])

(I) *The following two statements are equivalent.*

(i) *$g(s)$ is a strictly proper real rational transfer function of McMillan degree n whose poles are in the open left half plane.*

(ii) *$g(s)$ has an n-dimensional state-space realization (A, b, c) which is given by the following parameters:*

$$
\begin{array}{rl}
n(1),\dots,n(j),\dots,n(k), & \textit{where } n(j)\in\mathbb{N}\ \ (j=1,\dots,k) \textit{ and } \sum_{j=1}^{k} n(j)=n,\\
s_1,\dots,s_j,\dots,s_k, & \textit{where } s_j=\pm 1\ \ (j=1,\dots,k),\\
\sigma_1,\dots,\sigma_j,\dots,\sigma_k, & \textit{where } \sigma_j\in\mathbb{R}\ \ (j=1,\dots,k) \textit{ and } \sigma_1>\cdots>\sigma_j>\cdots>\sigma_k>0,\\
b_1,\dots,b_j,\dots,b_k, & \textit{where } b_j>0\ \ (j=1,\dots,k),\\
\alpha_1^{(1)},\dots,\alpha_i^{(1)},\dots,\alpha_{n(1)-1}^{(1)}, & \\
\vdots & \\
\alpha_1^{(j)},\dots,\alpha_i^{(j)},\dots,\alpha_{n(j)-1}^{(j)}, & \\
\vdots & \\
\alpha_1^{(k)},\dots,\alpha_i^{(k)},\dots,\alpha_{n(k)-1}^{(k)}, & \textit{where } \alpha_i^{(j)}>0 \textit{ for } i=1,\dots,n(j)-1\ \ (j=1,\dots,k),
\end{array}
$$

in the following way:

(1) $b^{\mathsf{T}} = [\underbrace{b_1, 0,\dots,0}_{n(1)},\dots,\underbrace{b_j, 0,\dots,0}_{n(j)},\dots,\underbrace{b_k, 0,\dots,0}_{n(k)}]$,

(2) $c = [\underbrace{s_1b_1, 0,\dots,0}_{n(1)},\dots,\underbrace{s_jb_j, 0,\dots,0}_{n(j)},\dots,\underbrace{s_kb_k, 0,\dots,0}_{n(k)}]$,

(3) $A =$

$$
\begin{array}{r}
n(1)\left\{\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right.\\
n(2)\left\{\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right.\\
\\
n(k)\left\{\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right.
\end{array}
\begin{bmatrix}
\overbrace{\begin{matrix} a_{11} & \alpha_1^{(1)} & & 0 \\ -\alpha_1^{(1)} & 0 & \alpha_2^{(1)} & \\ & -\alpha_2^{(1)} & 0 & \\ 0 & & \vdots\quad\vdots & \alpha_{n(1)-1}^{(1)} \\ & & -\alpha_{n(1)-1}^{(1)} & 0 \end{matrix}}^{n(1)} &
\overbrace{\begin{matrix} a_{12} & \\ & \\ & 0 \\ & \\ & \end{matrix}}^{n(2)} & \cdots &
\overbrace{\begin{matrix} a_{1k} & \\ & \\ & 0 \\ & \\ & \end{matrix}}^{n(k)} \\
\begin{matrix} a_{21} & \\ & 0 \\ & \\ & \end{matrix} &
\begin{matrix} a_{22} & \alpha_1^{(2)} & & 0 \\ -\alpha_1^{(2)} & 0 & & \\ & 0 & \vdots & \alpha_{n(2)-1}^{(2)} \\ & & -\alpha_{n(2)-1}^{(2)} & 0 \end{matrix} & \cdots &
\begin{matrix} a_{2k} & \\ & 0 \\ & \\ & \end{matrix} \\
\vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\
\begin{matrix} a_{k1} & \\ & \\ & 0 \\ & \end{matrix} &
\begin{matrix} a_{k2} & \\ & \\ & 0 \\ & \end{matrix} & \cdots &
\begin{matrix} a_{kk} & \alpha_1^{(k)} & & 0 \\ -\alpha_1^{(k)} & 0 & & \\ 0 & \vdots & \vdots & \alpha_{n(k)-1}^{(k)} \\ & & -\alpha_{n(k)-1}^{(k)} & 0 \end{matrix}
\end{bmatrix}
$$

where

$$a_{ij} = \frac{-1}{s_is_j\sigma_i + \sigma_j}\, b_ib_j \quad \text{for } 1 \leqslant i,j \leqslant k.$$

(II) *The parameters in* (ii) *are uniquely defined for a transfer function* $g(s)$. *For any* (A, b, c) *defined as in* (ii), *we have*

$$W_c = W_o = \text{diag}\,(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \dots, \sigma_k I_{n(k)}).$$

The theorem shows that each asymptotically stable transfer function of McMillan degree n has a balanced representation (A, b, c) which has a certain structure. Conversely, the matrices (A, b, c) form an asymptotically stable and minimal system if they are defined as in part (ii) for some arbitrary parameters $\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0$, $b_1 > 0$, $b_2 > 0, \dots, b_k > 0$, $\alpha_i^{(j)} > 0$ $(1 \leq i \leq n(j),\ 1 \leq j \leq k)$, and $s_j = \pm 1$ $(1 \leq j \leq k)$. Hence the previous theorem together with Proposition 2.2 tells us that, in order to construct a finite-rank Hankel operator with singular values given by the sequence $(\sigma_j)_{j=1}^k$ each repeated with multiplicity $n(j)$, we only have to choose arbitrary positive parameters $b_1, b_2, \dots, b_k$ and $\alpha_1^{(1)}$, $\alpha_2^{(1)}, \dots, \alpha_{n(1)-1}^{(1)}$; $\alpha_1^{(2)}$, $\alpha_2^{(2)}, \dots, \alpha_{n(2)-1}^{(2)}; \dots$; $\alpha_1^{(k)}$, $\alpha_2^{(k)}, \dots, \alpha_{n(k)-1}^{(k)}$, as well as $s_i = \pm 1$ for $1 \leq i \leq k$. These parameters are then used to define a state-space system (A, b, c) as in (ii) of the previous theorem. Then the integral Hankel operator Γ given by the kernel $h(t) := ce^{tA}b$ $(t \geq 0)$ has singular values $\sigma_1(\Gamma) = \cdots = \sigma_{n(1)}(\Gamma) = \sigma_1$, $\sigma_{n(1)+1}(\Gamma) = \cdots = \sigma_{n(1)+n(2)}(\Gamma) = \sigma_2, \dots$.

In the next section, we will show that, given an infinite sequence of positive decreasing numbers $(\sigma_i)_{i \geq 1}$, we can construct an infinite-dimensional state-space system (A, b, c) in the same way as in Theorem 2.4 such that the corresponding integral Hankel operator has the sequence $(\sigma_i)_{i \geq 1}$ as its set of singular values.

3. Generalization to the infinite-dimensional case

As we have seen in the last section, the key to the solution of our problem for a finite number of nonzero singular values is the construction of a balanced state-space system with a prescribed diagonal controllability (observability) Gramian. Therefore the next proposition is concerned with the extension of this construction to infinite-dimensional state-space systems.

PROPOSITION 3.1 *Given a sequence* $(\sigma_i)_{i=1}^N$ *of positive numbers such that*

$$\sigma_1 > \sigma_2 > \sigma_3 > \cdots > 0,$$

with $N \in \mathbb{N} \cup \{\infty\}$, *and a sequence of strictly positive integers,*

$$(n(i))_{i=1}^N, \quad \text{where we allow } N < \infty \text{ and } n(N) = \infty.$$

(With a slight abuse of notation, no distinction will be made between $N = \infty$ *and* $N < \infty$ *in mathematical expressions.)*

Define a sequence $(b_i)_{i=1}^N$ *such that* $b_i > 0$ *for* $1 \leq i \leq N$ *and* $\sum_{i=1}^N (b_i^2/\sigma_i) < \infty$ (e.g. $b_i = \sigma_i/i$) *and choose for all* $n(i) > 1$ $(1 \leq i \leq N)$ *strictly positive numbers* $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{n(i)-1}^{(i)}$ *such that*

$$\sum_{\substack{i=1 \\ n(i)>1}}^{N} \sum_{j=1}^{n(i)-1} (\alpha_j^{(i)})^2 < \infty.$$

Then set

$$b^{\mathsf{T}} = (\underbrace{b_1, 0, \dots, 0,}_{n(1)}\ \underbrace{b_2, \dots, 0,}_{n(2)}\ \underbrace{b_3, 0, \dots, 0,}_{n(3)} \ \dots)$$

$$c = b^{\mathsf{T}}$$

$A =$

$$
\begin{array}{r}
n(1)\left\{\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right. \\
n(2)\left\{\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right. \\
n(3)\left\{\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right. \\
\\
\end{array}
\left[\begin{array}{cccc}
\overbrace{\begin{array}{ccccc}
a_{11} & \alpha_1^{(1)} & & 0 & \\
-\alpha_1^{(1)} & 0 & \alpha_2^{(1)} & & \\
 & -\alpha_2^{(1)} & 0 & & \\
 & & \vdots & \vdots & \alpha_{n(1)-1}^{(1)} \\
0 & & & -\alpha_{n(1)-1}^{(1)} & 0
\end{array}}^{n(1)}
&
\overbrace{\begin{array}{ccc}
a_{12} & & \\
 & 0 & \\
 & &
\end{array}}^{n(2)}
&
\overbrace{\begin{array}{ccc}
a_{13} & & \\
 & 0 & \\
 & &
\end{array}}^{n(3)}
& \cdots
\\
\begin{array}{ccc}
a_{21} & & \\
 & 0 & \\
 & &
\end{array}
&
\begin{array}{ccccc}
a_{22} & \alpha_1^{(2)} & & 0 & \\
-\alpha_1^{(2)} & 0 & & & \\
 & \vdots & \vdots & & \alpha_{n(2)-1}^{(2)} \\
 & 0 & -\alpha_{n(2)-1}^{(2)} & & 0
\end{array}
&
\begin{array}{ccc}
a_{23} & & \\
 & 0 & \\
 & &
\end{array}
& \cdots
\\
\begin{array}{ccc}
a_{31} & & \\
 & 0 & \\
 & &
\end{array}
&
\begin{array}{ccc}
a_{32} & & \\
 & 0 & \\
 & &
\end{array}
&
\begin{array}{ccccc}
a_{33} & \alpha_1^{(3)} & & 0 & \\
-\alpha_1^{(3)} & 0 & & & \\
 & \vdots & \vdots & & \alpha_{n(3)-1}^{(3)} \\
 & 0 & -\alpha_{n(3)-1}^{(3)} & & 0
\end{array}
& \cdots
\\
 & & & \cdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}\right]
$$

where $a_{ij} = -b_i b_j/(\sigma_i + \sigma_j)$ for $1 \leq i, j \leq N$.
Then

(i) *A is a Hilbert–Schmidt operator on* ℓ^2.

(ii) *With* $S = \operatorname{diag}(\hat{I}_{n(1)}, \hat{I}_{n(2)}, \hat{I}_{n(3)}, \dots)$, *where* $\hat{I}_{n(i)} := \operatorname{diag}(+1, -1, +1, -1, \dots) \in \mathbb{R}^{n(i) \times n(i)}$ *for* $1 \leq i \leq N$, *we have*

$$A^* = SAS.$$

(iii) $\Sigma = \operatorname{diag}(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \sigma_3 I_{n(3)}, \dots)$ *satisfies the following Lyapunov equations on* ℓ^2:

$$A\Sigma + \Sigma A^* = -bb^*, \qquad A^*\Sigma + \Sigma A = -c^*c.$$

(iv) $\sigma_{\mathrm{p}}(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

(v) *The semigroup* $(\mathrm{e}^{tA})_{t \geq 0}$ *generated by A is asymptotically stable, i.e.* $\lim_{t \to \infty} \mathrm{e}^{tA} x = 0$ *for all* $x \in \ell^2$.

(vi) $\int_0^\infty \mathrm{e}^{tA} bb^* \mathrm{e}^{tA^*}\, \mathrm{d}t = \Sigma$ and $\int_0^\infty \mathrm{e}^{tA^*} c^*c\, \mathrm{e}^{tA}\, \mathrm{d}t = \Sigma$, *where these expressions are* *understood to stand for*

$$\int_0^\infty (b^* e^{tA^*}x)(b^* e^{tA^*}y)\,dt = \langle x, \Sigma y\rangle \quad (x,y \in \ell^2),$$

$$\int_0^\infty (ce^{tA}x)(ce^{tA}y)\,dt = \langle x, \Sigma y\rangle \quad (x,y \in \ell^2).$$

Proof.

(i) $A =: [\tilde{a}_{ij}]_{1\le i,j<\infty}$ is Hilbert–Schmidt if $\sum_{j=1}^\infty \sum_{i=1}^\infty \tilde{a}_{ij}^2 < \infty$ [9: theorem 6.22, p. 152]. Since by assumption $\sum_{i=1,n(i)>1}^{N} \sum_{j=1}^{n(i)-1} (\alpha_j^{(i)})^2 < \infty$, we have to show that $\sum_{i,j=1}^N a_{ij}^2 < \infty$. But

$$\sum_{i,j=1}^N \left(\frac{b_i b_j}{\sigma_i + \sigma_j}\right)^2 \le \tfrac{1}{2} \sum_{i,j=1}^N \frac{b_i^2 b_j^2}{\sigma_i \sigma_j} = \tfrac{1}{2}\left(\sum_{i=1}^N \frac{b_i^2}{\sigma_i}\right)\left(\sum_{j=1}^N \frac{b_j^2}{\sigma_j}\right) < \infty.$$

(ii) and (iii) follow immediately from the definition of A.

(iv) We first show that

$$\sigma_p(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}.$$

Let $\lambda \in \mathbb{C}$ and $x \in \ell^2$ $(x \ne 0)$ such that $Ax = \lambda x$. Then, by (iii),

$$\begin{aligned} -\langle c^* cx, x\rangle &= \langle A^*\Sigma x, x\rangle + \langle \Sigma A x, x\rangle \\ &= \langle \Sigma x, Ax\rangle + \langle \Sigma A x, x\rangle \\ &= \bar{\lambda}\langle \Sigma x, x\rangle + \lambda \langle \Sigma x, x\rangle \\ &= 2(\operatorname{Re}\lambda)\langle \Sigma x, x\rangle \end{aligned}$$

so that

$$\operatorname{Re}\lambda = -\frac{\langle c^* cx, x\rangle}{2\langle \Sigma x, x\rangle} \le 0. \qquad (1)$$

To show that each eigenvalue of A is in the open left half plane, assume that $\lambda = iw$ $(w \in \mathbb{R})$. Then, by (1),

$$0 = \operatorname{Re} iw = -\frac{\langle c^* cx, x\rangle}{2\langle \Sigma x, x\rangle} \le 0.$$

Hence $\langle c^* cx, x\rangle = 0$ and thus

$$\langle c^{\mathrm{T}}, x\rangle = \langle b, x\rangle = 0. \qquad (2)$$

Similarly, for each $\tilde{x} \in \ell^2$ $(\tilde{x} \ne 0)$ such that $A^*\tilde{x} = i\tilde{w}\tilde{x}$ for $\tilde{w} \in \mathbb{R}$, we have $\langle c^{\mathrm{T}}, \tilde{x}\rangle = \langle b, \tilde{x}\rangle = 0$. Then

$$A^*\Sigma x + \Sigma A x = -c^* cx.$$

Hence we may take $\tilde{x} = \Sigma x$ to obtain $\langle c^{\mathrm{T}}, \Sigma x\rangle = 0$, so that $A^*\Sigma x = -iw\Sigma x$, and thus

$$(A\Sigma + \Sigma A^*)\Sigma x = A\Sigma^2 x - iw\Sigma^2 x = -bb^*\Sigma x = 0,$$

which implies that $A\Sigma^2 x = iw\Sigma^2 x$. So the eigenspace of A corresponding to $\lambda = iw$ is a closed invariant subspace of Σ^2. But the closed invariant subspaces of Σ^2 are of the form

$$\tilde{E} = \bigoplus_{i=1}^{N} \tilde{E}_i,$$

where $\tilde{E}_i$ $(1 \leq i \leq N)$ is a closed subspace of

$$\overline{\text{span}} \left\{ e_i : \sum_{j=1}^{i-1} n(j) < i \leq \sum_{j=1}^{i} n(j) \right\}.$$

Hence, the eigenspace of A with respect to the eigenvalue iw is of this form, i.e.

$$E = \bigoplus_{i=1}^{N} E_i.$$

We are going to show that $E_i = \varnothing$ for $1 \leq i \leq N$. Assume $E_{i_0} \neq \varnothing$. For ease of notation, take $i_0 = 1$. Let $x \in E_1$ $(x \neq 0)$, i.e. $x^{\mathsf{T}} = [x_1, x_2, \ldots, x_{n(1)}, 0, 0, \ldots]$. By (2),

$$\langle b, x \rangle = b_1 \bar{x}_1 = 0.$$

Hence, $x_1 = 0$ since $b_1 > 1$. Evaluating $Ax = \mathrm{i}wx$ componentwise, we have

$$Ax = \begin{bmatrix} a_{11} & \alpha_1^{(1)} & & & & & a_{12} & & \cdots & \\ -\alpha_1^{(1)} & 0 & \alpha_2^{(1)} & & & & & & & \\ & -\alpha_2^{(1)} & 0 & & & & & & & \\ & & \vdots & \vdots & \alpha_{n(1)-1}^{(1)} & & & & & \\ & & & -\alpha_{n(1)-1}^{(1)} & 0 & & & & & \\ a_{21} & & & & & & a_{22} & \alpha_1^{(2)} & & \\ & & & & & & -\alpha_1^{(2)} & 0 & \alpha_2^{(2)} & \\ & & & & & & & \vdots & \vdots & \vdots \quad \vdots \\ \vdots & & & & & & & & & \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n(1)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^{(1)} x_2 \\ \alpha_2^{(1)} x_3 \\ -\alpha_2^{(1)} x_2 + \alpha_3^{(1)} x_4 \\ \vdots \\ -\alpha_{n(1)-2}^{(1)} x_{n(1)-2} + \alpha_{n(1)-1}^{(1)} x_{n(1)} \\ -\alpha_{n(1)-1}^{(1)} x_{n(1)-1} \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \mathrm{i}w \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n(1)-1} \\ x_{n(1)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

The last equation shows by successive comparison of coordinates that $x = 0$, since $\alpha_i^{(1)} > 0$ for $1 \leq i \leq n(1) - 1$, which is a contradiction.

(v) Since A is a Hilbert–Schmidt operator and hence bounded, A generates a uniformly continuous semigroup of operators $(e^{tA})_{t\geq 0}$ By a theorem of Arendt & Batty [1], a strongly continuous semigroup of contractions $(e^{tB})_{t\geq 0}$ on a reflexive space is asymptotically stable if the spectrum of its generator B is such that $\sigma_p(B)\cap i\mathbb{R}=\emptyset$ and $\sigma(B)\cap i\mathbb{R}$ is countable. We have shown in (iv) that $\sigma_p(A)\cap i\mathbb{R}=\emptyset$. A is a Hilbert–Schmidt operator and hence compact. Thus, the nonzero spectrum of A consists only of eigenvalues, and so by (iv) we have $\sigma(A)\cap i\mathbb{R}\subseteq\{0\}$ and hence countable.

It remains to show that $(e^{tA})_{t\geq 0}$ is a semigroup of contractions or, equivalently, that A is dissipative, i.e. $\operatorname{Re}\langle Ax, x\rangle\leq 0$ for all $x\in\ell^2$. Let $x\in\ell^2$, then $\operatorname{Re}\langle Ax, x\rangle=\frac{1}{2}\langle (A+A^*)x, x\rangle$.

Hence A is dissipative if $A+A^*$ is nonnegative definite. Since A is compact, $A+A^*$ is compact. Moreover, $A+A^*$ is self-adjoint. Hence $A+A^*$ is nonnegative definite if all eigenvalues of $A+A^*$ are nonpositive. We have

$$A\Sigma+\Sigma A^*=-bb^*, \qquad A^*\Sigma+\Sigma A=-c^*c,$$

and so

$$(A+A^*)\Sigma+\Sigma(A+A^*)=-2bb^*.$$

Let $\lambda_0\in\mathbb{R}$ be an eigenvalue of $A+A^*$ with eigenvectors $x_0\neq 0$, then

$$\begin{aligned}0&\geq -2\langle x_0, bb^*x_0\rangle\\&=\langle x_0, (A+A^*)\Sigma x_0\rangle+\langle x_0, \Sigma(A+A^*)x_0\rangle\\&=\langle (A+A^*)x_0, \Sigma x_0\rangle+\langle x_0, \Sigma(A+A^*)x_0\rangle\\&=2\lambda_0\langle x_0, \Sigma x_0\rangle,\end{aligned}$$

and hence $\lambda_0\leq 0$. Thus, $A+A^*$ is negative semidefinite, and hence A is dissipative which implies the result.

(vi) We first have to show that

$$\int_0^\infty e^{tA}bb^*e^{tA^*}\,dt : \ell^2\to\ell^2$$

is bounded. Consider the operator

$$V : \ell^2\to L^2([0,\infty)) : x\mapsto b^*e^{tA^*}x.$$

We are going to show that V is well defined, i.e. $V\ell^2\subseteq L^2([0,\infty))$ and that V is bounded. If V is bounded this implies the boundedness of $V^*V=\int_0^\infty e^{tA}bb^*e^{tA^*}\,dt$. Define $A(n):=P_nAP_n^*$ and $b(n):=P_nb$ for $1\leq n<\infty$, where, for $n\geq 1$,

$$P_n : \ell^2\to\mathbb{C}^n : [x_i]_{i\geq 1}^{\mathrm{T}}\mapsto[x_1, x_2,\ldots, x_n]^{\mathrm{T}}$$

and let, for $n\geq 1$,

$$V_n : \ell^2\to L^2([0,\infty)) : x\mapsto b(n)^*e^{tA(n)^*}P_nx.$$

Note that V_n $(n\geq 1)$ is well defined since by Theorem 2.4 all the eigenvalues of $A(n)$ are in the open left plane and hence $V_n(x)\in L^2([0,\infty))$ $(n\geq 1, x\in\ell^2)$.

Since A is Hilbert–Schmidt, we have

$$\lim_{n\to\infty} \|A - P_n^* A(n) P_n\| = 0.$$

Hence, for all $x \in \ell^2$, we have

$$\lim_{n\to\infty} \|e^{tA}x - P_n^* e^{tA(n)} P_n x\| = 0$$

uniformly for t in bounded intervals. This, together with the fact that

$$\lim_{t\to\infty} b(n) P_n = b$$

in ℓ^2, implies that, for all $M > 0$,

$$\lim_{n\to\infty} \int_0^M (V_n x)(t)\,dt = \int_0^M (Vx)(t)\,dt. \tag{3}$$

Since, for all $n \geq 1$ and $x \in \ell^2$,

$$\|V_n x\|^2 = \left\langle P_n x, \left(\int_0^\infty e^{tA(n)} b(n) b(n)^* e^{tA(n)^*}\,dt\right) P_n x\right\rangle = \langle \Sigma P_n x, P_n x\rangle \leq \langle \Sigma x, x\rangle,$$

it follows that $(\|V_n\|)_{n\geq 1}$ is bounded. Since the step functions are dense in $L^2([0, \infty))$, it therefore follows that $(V_n)_{n\geq 1}$ is weakly Cauchy [9: theorem 4.26, p. 77] and that $(V_n)_{n\geq 1}$ converges weakly to a bounded operator $\tilde{V}$. But (3) implies that $V = \tilde{V}$. Hence, V is bounded, and therefore

$$V^* V = \int_0^\infty e^{tA} b b^* e^{tA^*}\,dt$$

is bounded, which implies by virtue of (ii) that $\int_0^\infty e^{tA^*} c^* c e^{tA}\,dt$ is bounded.

Next we show that $\int_0^\infty e^{tA^*} c^* c e^{tA}\,dt$ solves $A^* X + XA = -c^* c$. Let $x, y \in \ell^2$; then

$$\begin{aligned}
&\left\langle A^*\left(\int_0^\infty e^{tA^*} c^* c e^{tA}\,dt\right)x, y\right\rangle + \left\langle\left(\int_0^\infty e^{tA^*} c^* c e^{tA}\,dt\right)Ax, y\right\rangle \\
&= \int_0^\infty (\langle A^* e^{tA^*} c^* c e^{tA} x, y\rangle + \langle e^{tA^*} c^* c e^{tA} Ax, y\rangle)\,dt \\
&= \int_0^\infty (\langle c^* c e^{tA} x, e^{tA} Ay\rangle + \langle c^* c e^{tA} Ax, e^{tA} y\rangle)\,dt \\
&= \int_0^\infty \left(\frac{d}{dt}\langle c^* c e^{tA} x, e^{tA} y\rangle\right) dt \\
&= \lim_{s\to\infty} [\langle c^* c e^{tA} x, e^{tA} y\rangle]_0^s \\
&= -\langle c^* c x, y\rangle,
\end{aligned}$$

which shows the claim. Since we know that $\Sigma = \operatorname{diag}(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \ldots)$ solves $A^* X + XA = -c^* c$, we have to show that the solution to $A^* X + XA = -c^* c$ is unique in order to conclude that $\int_0^\infty e^{tA^*}c^*ce^{tA}\,dt = \Sigma$. Assume that X_1 and X_2 are bounded solutions to $A^*X + XA = -c^*c$. Then

$$A^*\Delta + \Delta A = 0,$$

with $\Delta = X_1 - X_2$. Now consider for $x,y \in \ell^2$,

$$\begin{aligned}\frac{d}{dt}\langle \Delta e^{tA}x, e^{tA}y\rangle &= \langle \Delta e^{tA}x, Ae^{tA}y\rangle + \langle \Delta Ae^{tA}x, e^{tA}y\rangle \\ &= \langle (A^*\Delta + \Delta A)e^{tA}x, e^{tA}y\rangle \\ &\equiv 0 \quad \text{on } (0,\infty).\end{aligned}$$

Hence

$$\langle \Delta x, y\rangle = \langle \Delta e^{0A}x, e^{0A}y\rangle = \langle \Delta e^{tA}x, e^{tA}y\rangle \quad \text{for } t \in [0,\infty).$$

By the asymptotic stability of $(e^{tA})_{t\geq 0}$, however, we have

$$\langle \Delta x, y\rangle = \lim_{t\to\infty}\langle \Delta e^{tA}x, e^{tA}y\rangle = 0 \quad \text{for all } x,y \in \ell^2.$$

Since Δ is bounded we have $\Delta = 0$. □

In [7], the same result was proved for the case of a sequence of nonrepeated positive numbers. The stability theorem for strongly continuous semigroups of operators by Arendt & Batty [1] makes it possible to prove the uniqueness of the solution of the infinite-dimensional Lyapunov equation (iii) for a not necessarily self-adjoint infinite-dimensional A-matrix. This allows us to extend the finite-dimensional construction of balanced systems to the infinite-dimensional setting also in the general situation where we are dealing with sequences of possibly repeated positive numbers.

THEOREM 3.2 *Let the sequences* $(\sigma_i)_{i=1}^N$ *and* $(n(i))_{i=1}^N$ *and the state-space system* (A,b,c) *be as in Proposition* 3.1. *Then the Hankel operator* Γ *with kernel* $h(t) := ce^{tA}b$ *is bounded and its singular values are given by the sequence* $(\sigma_i)_{i=1}^N$ *each repeated with multiplicity* $(n(i))_{i=1}^N$.

Proof. In the proof of Proposition 3.1, we introduced the bounded operator V. Note that by the definition of S we have $c^T = Sb$. Hence, we obtain the decomposition $\Gamma = VSV^*$, which shows that Γ is bounded. Consider the family of vectors $(b^*e^{tA^*}e_i)_{i\geq 1}$. By

$$\begin{aligned}(\Gamma(b^*e^{tA^*}e_i))(s) &= \int_0^\infty ce^{(s+t)A}bb^*e^{tA^*}e_i\,dt \\ &= ce^{sA}\Sigma e_i \\ &= \langle S\Sigma e_i, e_i\rangle b^*e^{sA^*}e_i,\end{aligned}$$

it follows that $(b^*e^{tA^*}e_i)_{i\geq 1}$ is a family of eigenvectors, with eigenvalues

$(\langle S\Sigma e_i, e_i \rangle)_{i \geq 1}$, which are orthogonal since

$$\langle b^* e^{tA^*} e_i, b^* e^{tA^*} e_j \rangle_{L^2} = \left\langle e_i, \left(\int_0^\infty e^{tA} b b^* e^{tA^*} \, dt \right) e_j \right\rangle = \langle e_i, \Sigma e_j \rangle$$

for $1 \leq i,j < \infty$.

Let $E := \overline{\text{span}} \{ b^* e^{tA^*} e_i : 1 \leq i < \infty \}$. For $u \in E^\perp$, the orthogonal complement of E, we have, for all $1 \leq i < \infty$,

$$0 = \langle u(t), b^* e^{tA^*} e_i \rangle_{L^2} = \int_0^\infty b^* e^{tA^*} e_i u(t) \, dt = \left\langle e_i, \int_0^\infty e^{tA} b u(t) \, dt \right\rangle.$$

Thus, $\int_0^\infty e^{tA} b u(t) \, dt = 0$, and hence

$$(\Gamma(u))(s) = c e^{sA} \int_0^\infty e^{tA} b u(t) \, dt = 0 \quad \text{for all } s \in [0, \infty).$$

So $E^\perp = \ker \Gamma$. Choosing an orthogonal basis for $\ker \Gamma$, it follows that $L^2([0, \infty))$ is spanned by an orthogonal set of eigenvectors, which implies that $\sigma(\Gamma) = \overline{\sigma_p(\Gamma)}$. Since Γ is self-adjoint, it follows that the singular values of Γ are given by $(|\langle \Sigma S e_i, e_i \rangle|)_{i \geq 1}$ which shows the result. □

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