Topological Aspects of Robust Control

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The paper discusses topologies that naturally arise in the study of robust control problems. A new 'robust topology' is defined. It is shown that this topology is Hausdorff.

1 Introduction

Robustness issues in control theory play an important role because each model of a real process has inherent uncertainties that have to be taken into account in any controller design. An underlying issue of robust control is the question of when two plants are close to one another from the point of view of robustness. The most important aspect of robustness is that of robust stability which is addressed in this paper. We are going to introduce a 'robust topology' in order to describe what we mean by two systems being close from the point of view of robust stability.

The purpose of the paper is to give a precise definition of this topology and to study some of its elementary properties. We are also going to compare this topology with the so called graph topology ([5]).

2 Definition of robust topology

In this section we are going to define a topology on sets of plants in such a way that we are able to study robustness issues with the help of this topology. By robustness we mean that if a plant P is stabilized by a controller K then all plants in a sufficiently small neighbourhood of P are also stabilized by the controller K.

Let now $\mathcal{P}$ be a set of plants and $\mathcal{K}$ a set of controllers, where we assume that for each plant $P_0 \in \mathcal{P}$ there is a controller $K_0 \in \mathcal{K}$ that stabilizes P_0.

The *robust topology* $\tau_{\mathcal{K}}(\mathcal{P})$ on the set of plants $\mathcal{P}$ which is induced by the set of controllers $\mathcal{K}$ is defined through the following prebase. For each controller $K \in \mathcal{K}$ let $\mathcal{P}(K)$ be the set of plants in $\mathcal{P}$ that are stabilized by the controller K. The family of sets $(\mathcal{P}(K))_{K\in\mathcal{K}}$ forms a prebase for the robust topology $\tau_{\mathcal{K}}(\mathcal{P})$ on the set $\mathcal{P}$.

Note that in a dual fashion we can define a topology on the set of controllers $\mathcal{K}$. Here we define a prebase on $\mathcal{K}$ by $(\mathcal{K}(P))_{P\in\mathcal{P}}$, where $\mathcal{K}(P)$ is the set of controllers in $\mathcal{K}$ that stabilize the plant $P \in \mathcal{P}$. We have to assume that for each controller $K \in \mathcal{K}$ there is at least one plant $P \in \mathcal{P}$ that is stabilized by K.

In order to be more concrete we are now going to recall one particular way of defining closed loop stability (see e.g. [5]). Let P be a multivariable transfer function of a continuous time finite dimensional system. By a right (left) coprime factorization of P we mean a factorization $P = ND^{-1}$ ($P = \tilde{D}^{-1}\tilde{N}$), where D, N ($\tilde{D}, \tilde{N}$) are asymptotically stable rational transfer functions, D ($\tilde{D}$) being square and invertible, such that the Bezout equation

$$XN + YD = I, \qquad (\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I),$$

has a solution X, Y ($\tilde{X}, \tilde{Y}$), with X, Y ($\tilde{X}, \tilde{Y}$) being asymptotically stable proper transfer functions.

Consider the right coprime factorization $K = N_K D_K^{-1}$ (left coprime factorization $K = \tilde{D}_K^{-1}\tilde{N}_K$). Then K (internally) stabilizes P if and only if

$$\tilde{N}_K N + \tilde{D}_K D \qquad (\tilde{N}N_K + \tilde{D}D_K)$$

is unimodular, i.e. is invertible with asymptotically stable proper inverse (for precise definition of internal stability and result see [5]). Of particular importance in our context is that, given a continuous time plant with rational transfer function P of all (proper) rational controllers $\mathcal{K}(P)$ that stabilize P can be obtained via the so-called Youla parametrization

$$\begin{aligned}\mathcal{K}_p(P) &= \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}) | R \in M(S), \|Y - R\tilde{N}\| \neq 0\} \\ &= \{(\tilde{X} + DS)(\tilde{Y} - NS)^{-1} | S \in M(S), \|\tilde{Y} - NS\| \neq 0\}\end{aligned}$$

where $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$ is a left respectively right coprime factorization with $X, Y, \tilde{X}, \tilde{Y} \in M(S)$ solving the corresponding Bezout equations

$$XN + YD = I \qquad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I.$$

The symbol $M(S)$ stands for the set of asymptotically stable proper rational transfer function matrices of appropriate dimensions. By asymptotic stability we here mean asymptotic stability as a continuous-time system. Essentially the same results hold for more general notions of stability (see

[5]). It has been shown in ([5], p. 115) that all strictly proper controllers $\mathcal{K}_{sp}(P)$ of a plant $\mathcal{P}$ are given by the same linear fractional map where the parameter space $M(S)$ is replaced by the ideal of all strictly proper functions in $M(S)$, and X and $\tilde{X}$ are chosen to be strictly proper. The set of strictly proper functions in $M(S)$ we denote by $M_{sp}(S)$.

Dually there is an analogous parametrization of the set of rational plants that are stabilized by a fixed rational controller.

In the next section we are going to use these results to show that the robust topology on the set of rational plants with rational controllers is in fact Hausdorff.

3 The robust topology is Hausdorff

In the previous section we have defined in an abstract way a topology on the set of plants with rational transfer functions. In order to be able to deal with such a topology in a satisfactory way we would hope that the topology is Hausdorff, i.e. that for each two points in the set we can find neighbourhoods which do not intersect. We will now show that this is indeed the case if the set of plants is $\mathcal{P} = M(\mathbb{R}(s))$ and the set of controllers is $\mathcal{K}_p = M(\mathbb{R}(s))$ or $\mathcal{K}_{sp} = M_{sp}(\mathbb{R}(s))$, where $M(\mathbb{R}(s))$ ($M_{sp}(\mathbb{R}(s))$ stands for the set of proper (strictly proper) rational transfer function matrices with fixed input and output dimensions.

We first need some preliminaries concerning the simultaneous stabilizability of two plants by a single controller. Let P_i, $i = 1, 2$ be two plants. By a *doubly coprime factorization* of P_i, $i = 1, 2$ we mean a choice of right and left coprime factorizations (N_i, D_i) and $(\tilde{N}_i, \tilde{D}_i)$, of P_i, $i = 1, 2$, together with matrices $X_i, Y_i, \tilde{X}_i, \tilde{Y}_i$, $i = 1, 2$, in $M(S)$ such that

$$L_iR_i := \begin{bmatrix} Y_i & X_i \\ -\tilde{N}_i & \tilde{D}_i \end{bmatrix} \begin{bmatrix} D_i & -\tilde{X}_i \\ N_i & \tilde{Y}_i \end{bmatrix} = I, \qquad i = 1, 2.$$

By abuse of language we will say 'let $L_iR_i = I$ be a doubly coprime factorization of P_i, $i = 1, 2$', as well as 'let $L_iR_i = I$ be a doubly coprime factorization of C_i', where $C_i = \tilde{X}_i\tilde{Y}_i^{-1} = Y_i^{-1}X_i$, $i = 1, 2$. We also set

$$L_1R_2 = \begin{bmatrix} Y_1D_2 + X_1N_2 & -Y_1\tilde{X}_2 + X_1\tilde{Y}_2 \\ -\tilde{N}_1D_2 + \tilde{D}_1N_2 & \tilde{N}_1\tilde{X}_2 + \tilde{D}_1\tilde{Y}_2 \end{bmatrix} =: \begin{bmatrix} B_1 & \Delta_C \\ \Delta_P & B_2 \end{bmatrix}$$

We need to quote the following criterion on the simultaneous stabilizability of two plants.

Theorem 1. *([5]) Given two plants P_i, $i = 1, 2$, together with doubly coprime factorizations $L_iR_i = I$, $i = 1, 2$, then they are simultaneously stabilizable if and only if one of the following equivalent conditions holds:*

1. *There exists a* $M \in M(S)$ *such that* $B_1 + M\Delta_P$ *is unimodular.*

2. *At all real blocking zeros of* Δ_P *in the extended right half plane* $\mathbb{C}_{+e}$, $\det B_1$ *has the same sign. (A blocking zero of* Δ_P *is a value of the independent variable* s *at which all entries of the matrix are zero.)*

The following Proposition contains a result for the dual situation, i.e. given controllers C_i, $i = 1, 2$, does there exist a plant that is stabilized by both controllers.

Proposition 2. *Let* C_i, $i = 1, 2$, *be two controllers together with doubly coprime factorizations* $L_i R_i = I$, $i = 1, 2$. *Then* C_1 *and* C_2 *simultaneously stabilize a plant if and only if one of the following two equivalent conditions holds:*

1. *There exists a* $M \in M(S)$ *such that* $B_1 + \Delta_C M$ *is unimodular.*

2. *At all real blocking zeros of* Δ_C *in the extended right half plane* $\mathbb{C}_{+e}$, *the sign of* $\det B_1$ *remains constant.*

PROOF. We only prove *1.* The equivalence of *1.* and *2.* follows from the Remark on p. 125 and Corollary 6 on p. 118 in [5].

Assume first that C_1 and C_2 are both strictly proper. Then

$$\mathcal{P}(C_i) = \{(N_i + \tilde{Y}_i H)(D_i - \tilde{X}_i H)^{-1} | H \in M(S)\}, \qquad i = 1, 2.$$

Hence C_0 and C_1 simultaneously stabilize a plant if and only if there exist $H_1, H_2 \in M(S)$ such that

$$(N_1 + \tilde{Y}_1 H_1)(D_1 - \tilde{X}_1 H_1)^{-1} = (N_2 + \tilde{Y}_2 H_2)(D_2 - \tilde{X}_2 H_2)^{-1}.$$

Since both sides of the expression give a right coprime factorization of the same plant P, there exists a unimodular matrix U (see [5]) such that

$$(N_2 + \tilde{Y}_2 H_2) = (N_1 + \tilde{Y}_1 H_1)U, \qquad (D_2 - \tilde{X}_2 H_2) = (D_1 - \tilde{X}_1 H_1)U$$

We rewrite these expressions to obtain

$$\begin{bmatrix} D_2 & -\tilde{X}_2 \\ N_2 & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} I \\ H_2 \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{X}_1 \\ N_1 & \tilde{Y}_1 \end{bmatrix} \begin{bmatrix} I \\ H_1 \end{bmatrix} U$$

which is the case if and only if

$$\begin{bmatrix} Y_1 & X_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \begin{bmatrix} D_2 & -\tilde{X}_2 \\ N_2 & \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} I \\ H_2 \end{bmatrix} = \begin{bmatrix} I \\ H_1 \end{bmatrix} U$$

But this implies that

$$\begin{bmatrix} B_1 & \Delta_C \\ \Delta_P & B_2 \end{bmatrix} \begin{bmatrix} I \\ H_2 \end{bmatrix} = \begin{bmatrix} I \\ H_1 \end{bmatrix} U \tag{*}$$

Thus C_1 and C_2 simultaneously stabilize a plant if and only if there exist $H_1, H_2 \in M(S)$ and a unimodular U such that the above equation holds.

Assume that (*) holds for suitable H_1, H_2, U. Then $B_1 + \Delta_C H_2 = U$. Conversely suppose that $B_1 + \Delta_C H_2 = U$ is unimodular for a suitable H_2. Then set $H_1 = (\Delta_P + B_2 H_2)U^{-1}$ and check that (*) holds.

To remove the assumption that C_1 and C_2 are strictly proper argue as in [5], p. 126–127. Q.E.D.

Before proving the main result of this section we need to establish the following technical Lemma (see Lemma 21, p. 96 in [5]for a similar result).

Lemma 3. *Let $\mathcal{I}$ be a nontrivial ideal in S, the set of asymptotically stable proper transfer functions. Let $N \in S^{p\times m}$, $Y \in S^{p\times p}$ and $H \in S^{m\times p}$ be such that*

1. $[N, Y]$ *has full row rank* $(= p)$.
2. $\det(Y + NH) = 0$ *in* S.

Then there exists $G \in \mathcal{I}^{m\times p}$ such that $\det(Y + N(H + G)) \neq 0$.

PROOF. First observe that

$$[Y + NH, N] = [Y \;\; N] \begin{bmatrix} I & 0 \\ H & I \end{bmatrix}$$

so that $[Y + NH, N]$ has full rank p too. Select a nonzero minor m_p of $[Y + NH, N]$ obtained by considering a submatrix which minimizes the number of columns of N. Assume that this is achieved by selecting columns $j_1, \ldots, j_k$ of N and omitting columns $i_1, \ldots, i_k$ of $Y + NH$. Choose now $g \in \mathcal{I}$, $g \neq 0$, and put

$$g_{ji} = \begin{cases} g \text{ if } j = j_s \text{ and } i = i_s, \text{ for some } s = 1, \ldots, k \\ 0 \text{ otherwise.} \end{cases}$$

By the Binet-Cauchy formula we obtain

$$\det(Y+N(H+G)) = \det\left([Y + NH, N]\begin{bmatrix} I \\ G \end{bmatrix}\right) = \sum_K [Y+NH, N]_K \begin{bmatrix} I \\ G \end{bmatrix}_K$$

where K varies over all the increasing p-tuples $(\alpha_1, \ldots, \alpha_p)$ with $1 \leq \alpha_s \leq m + p$, $1 \leq s \leq p$. Observe that:

- Every K containing more than k rows of G is such that $\begin{bmatrix} I \\ G \end{bmatrix}_K = 0$.
- Exactly one p-minor of $\begin{bmatrix} I \\ G \end{bmatrix}$ containing k rows of G is nonzero. This is obtained by selecting rows $j_1, \ldots, j_k$ of G and omitting rows $i_1, \ldots, i_k$ of I. If K_0 is the corresponding p-tuple, then

$$\begin{bmatrix} I \\ G \end{bmatrix}_{K_0} = \pm g^k.$$

- Every K containing less than k column indexes of N is such that $[Y + NH, N]_K = 0$.

We therefore conclude that there is exactly one p-tuple K for which

$$[Y + NH, N]_K \begin{bmatrix} I \\ G \end{bmatrix}_K \neq 0,$$

namely,

$$\det(Y + N(H+G)) = [Y + NH, N]_{K_0} \begin{bmatrix} I \\ G \end{bmatrix}_{K_0} = \pm m_p g^k \neq 0.$$

Q.E.D.

We can now prove the main theorem of this paper.

Theorem 4. *Let*

$$\mathcal{K}(P_i) = \left\{ (Y_i - H\tilde{N}_i)^{-1}(X_i + H\tilde{D}_i) \,\|\, |Y_i - H\tilde{N}_i| \neq 0, H \in M(I) \right\},$$

$i = 0, 1$, *where M is either $M(S)$ or $M_{sp}(S)$.*

If $P_0 \neq P_1$ are proper rational transfer functions of two plants, then there exist $C_i \in \mathcal{K}(P_i)$, $i = 0, 1$, such that $\mathcal{P}(C_0) \cap \mathcal{P}(C_1) = \emptyset$.

Proof. For convenience of notation we only prove the result for the case the set of controllers parametrized by $M(S)$. With the obvious modifications the result also holds if the set of parameters is $M_{sp}(S)$.

a.) We first assume that P_0 and P_1 are not simultaneously stabilizable.

Fix doubly coprime factorizations $L_i R_i = I$ of P_i, $i = 0, 1$. Since P_0 and P_1 are not simultaneously stabilizable, $\Delta_P = (-\tilde{N}_0 D_1 + \tilde{D}_0 N_1)$ has at least two real blocking zeros in $\mathbb{C}_{+e}$, say ξ_1, ξ_2, at which $\det(Y_0 D_1 + X_0 N_1) = \det B_1$ changes sign. In particular, since $\det B_1(\xi_j) \neq 0$, $j = 1, 2$, $B_1(\xi_j)$ is an invertible matrix, $j = 1, 2$. Thus we can find $\bar{H} \in M(S)$ such that

$$\bar{H}(\xi_j) = B_1(\xi_j)^{-1} \Delta_C(\xi_j). \tag{**}$$

If $\det(\tilde{Y}_1 - N_1\bar{H}) = 0$, consider the ideal $\mathcal{I}$ of S consisting of those elements g for which $g(\xi_j) = 0$, $j = 1, 2$. Then by the lemma, there exists $G \in M(\mathcal{I})$, the set of matrices with entries in $\mathcal{I}$, such that $\det(\tilde{Y}_1 - N_1(\bar{H} + G)) \neq 0$. Observe that $(\bar{H} + G)(\xi_j) = \bar{H}(\xi_j)$, $j = 1, 2$. Hence a choice of $\bar{H}$ satisfying (**) can be made in such a way that $\det(\tilde{Y}_1 - N\bar{H}) \neq 0$. On the other hand

$$\mathcal{K}(P_1) = \{(\tilde{X}_1 + D_1H)(\tilde{Y}_1 - N_1H)^{-1} | H \in M(S), \det(\tilde{Y}_1 - NH) \neq 0\}.$$

Thus, with the obvious meaning of the symbols:

$$\begin{aligned}\Delta_C(\bar{H}) &= -Y_0\tilde{X}_1(\bar{H}) + X_0\tilde{Y}_1(\bar{H}) \\ &= -Y_0(\tilde{X}_1 + D_1\bar{H}) + X_0(\tilde{Y}_1 - N_1\bar{H}) \\ &= \Delta_C - B_1\bar{H}\end{aligned}$$

Now,

$$B_1(\xi_j)^{-1}\Delta_C(\bar{H})(\xi_j) = B_1(\xi_j)^{-1}\Delta_C(\xi_j) - \bar{H}(\xi_j) = 0, \qquad j = 1, 2,$$

implies that $\Delta_C(\bar{H})(\xi_j) = 0$, $j = 1, 2$. Thus we have found doubly coprime factorizations $L_0R_0 = I$, $L_1(\bar{H})R_1(\bar{H}) = I$ relative to which Δ_C has two real blocking zeros in $\mathbb{C}_{+e}$ at which $\det B_1$ changes sign. Therefore C_0 and $C_1(\bar{H})$ stabilize no common plant.

b.) Now assume that P_0 and P_1 are simultaneously stabilizable. Choose right and left coprime factorizations $(\tilde{X}, \tilde{Y})$ and (X, Y) of a simultaneously stabilizing controller C of P_i, $i = 0, 1$. Then choose right coprime factorizations (N_i, D_i) and left coprime factorizations $(\tilde{D}_i, \tilde{N}_i)$ of P_i such that

$$\begin{bmatrix} Y & X \\ -\tilde{N}_i & \tilde{D}_i \end{bmatrix} \begin{bmatrix} D_i & -\tilde{X} \\ N_i & \tilde{Y} \end{bmatrix} = I \qquad i = 0, 1.$$

$$\mathcal{K}(P_0) = \{(Y - T\tilde{N}_0)^{-1}(X + T\tilde{D}_0) | T \in M(S), \det(Y - T\tilde{N}_0) \neq 0\}$$

$$\mathcal{K}(P_1) = \{(\tilde{X} + D_1R)(\tilde{Y} - N_1R)^{-1} | R \in M(S), \det(\tilde{Y} - N_1R) \neq 0\}$$

and so with the obvious meaning of the symbols:

$$\begin{aligned}B_1(T) &= Y_0(T)D_1 + X_0(T)N_1 \\ &= (Y - T\tilde{N}_0)D_1 + (X + T\tilde{D}_0)N_1 \\ &= I + T(-\tilde{N}_0D_1 + \tilde{D}_0N_1) \\ &= I + T\Delta_P\end{aligned}$$

$$\begin{aligned}\Delta_C(T, R) &= -Y_0(T)\tilde{X}_1(R) + X_0(T)\tilde{Y}_1(R) \\ &= -(Y - T\tilde{N}_0)(\tilde{X}_1 + D_1R) + (X + T\tilde{D}_0)(\tilde{Y} - N_1R) \\ &= -R + T - T\Delta_PR \\ &= T - (I + T\Delta_P)R \\ &= T - B_1(T)R.\end{aligned}$$

Now, since $P_0 \neq P_1$, $\Delta_P \neq 0$ in $M(S)$. Thus we can find two real ξ_j, $j = 1, 2$, in $\mathbb{C}_{+e}$ at which $\Delta_P(\xi_j) \neq 0$, $j = 1, 2$. Then we can find matrices T_{ξ_j}, $j = 1, 2$, with

$$\det B_1(T_{\xi_1})(\xi_1) = \det(I + T_{\xi_1}\Delta_P(\xi_1)) > 0$$

$$\det B_2(T_{\xi_2})(\xi_2) = \det(I + T_{\xi_2}\Delta_P(\xi_2)) < 0.$$

Finally, we can find $\bar{T} \in M(S)$ such that $\bar{T}(\xi_j) = T_{\xi_j}$, $j = 1, 2$. Applying the lemma we may assume that $\det(Y - \bar{T}\tilde{N}_0) \neq 0$. In particular $B_1(\bar{T})(\xi_j)$ is invertible and so we can find $\bar{R} \in M(S)$ such that

$$\bar{R}(\xi_j) = (B_1(\bar{T})(\xi_j))^{-1}(\bar{T}(\xi_j)), \qquad j = 1, 2,$$

and by the lemma, $\det(\tilde{Y} - N_1\bar{R}) \neq 0$. From

$$(B_1(\bar{T})(\xi_j))^{-1}\Delta(\bar{T}, \bar{R})(\xi_j) = (B_1(\bar{T})(\xi_j))^{-1}\bar{T}(\xi_j) - \bar{R}(\xi_j) = 0, \quad j = 1, 2.$$

it follows $\Delta(\bar{T}, \bar{R})(\xi_j) = 0$, $j = 1, 2$. Hence we have found doubly coprime factorizations $L_0(T)R_0(T)$ and $L_1(T)R_1(T)$ of P_0 and P_1 relative to which Δ_C has two real blocking zeros in $\mathbb{C}_{+e}$ at which $\det B_1$ changes sign. The corresponding controllers stabilize no common plant. Q.E.D.

We immediately have as a corollary that the robust topology as introduced in the previous section is Hausdorff.

Corollary 5. *If the set of plants and the set of controllers are given by $\mathcal{P} = M(\mathbb{R}(s))$ and $\mathcal{K}_p = M(\mathbb{R}(s))$ ($\mathcal{K}_{sp} = M_{sp}(\mathbb{R}(s))$), then $\tau_{\mathcal{K}_p}(\mathcal{P})$ ($\tau_{\mathcal{K}_{sp}}(\mathcal{P})$) is Hausdorff.*

In the same way we have that the dual topologies on the set of controllers are Hausdorff.

The theorem also has an interesting interpretation regarding model reduction. Since for every two plants however 'close' they are there is a controller that will stabilize one plant but not the other it follows that the general rule is confirmed whereby great care must be taken when a controller is designed on the basis of an approximate model. Note that a weaker form of the theorem was shown in [3].

4 Robust topology and graph topology

Vidyasagar defined the so called graph topology on the set of rational transfer functions. Similarly to our work the intention was to define a topology on the set of all plants that describes in topological terms what we normally mean by robustness. In contrast to our definition his approach does not only require robustness of the stability of the closed loop but also robustness of performance. In particular it is required that the closed loop is continuous in the H^∞ topology with respect to perturbations of the plant. (For more precise definitions and results see [5].) This implies that the robust topology defined here is weaker than the graph topology.

Example 1. We are going to consider some examples that show that on a set of plants $\mathcal{P}$ the topologies $\tau_{\mathcal{K}_p}(\mathcal{P})$ and $\tau_{\mathcal{K}_{sp}}(\mathcal{P})$ given by two different sets of controllers $\mathcal{K}_p$ and $\mathcal{K}_{sp}$ can be different. They also indicate the differences in definition between the robust topology and the graph topology.

Consider the sequence of plants given by

$$g_i(s) = \frac{i}{s+i}$$

Note that for all $i \in \mathbb{N}$ the transfer function $g_i(s)$ is asymptotically stable and that for all $s \in \mathbb{C}$,

$$\lim_{i\to\infty} g_i(s) = 1$$

Since

$$\|g_i(s) - 1\|_\infty \geq |g_i(\infty) - 1| = 1,$$

the sequence $(g_i(s))_{i\geq 1}$ does not converge to 1 in the H^∞ topology on the right half plane. Since on the set of rational H^∞ functions the graph topology is equivalent to the H^∞ topology we have shown that the sequence does not converge in the graph topology.

Now define the topology $\tau_{\mathcal{K}_p}(\mathcal{P})$, by setting $\mathcal{P} = \mathbb{R}(s)$ and $\mathcal{K}_p = \mathbb{R}(s)$. We can see that the sequence of transfer functions also does not converge in the topology $\tau_{\mathcal{K}_p}(\mathcal{P})$ since the controller

$$k(s) = \frac{-2s + 0.5}{s - 1}$$

stabilizes the limiting system, i.e. the transfer function $g_0(s) = 1$, but it destabilizes $g_i(s)$ for all $i \in \mathbb{N}$.

If we however define the class of controllers $\mathcal{K}_{sp}$ to be strictly proper transfer functions we can show that the sequence $(g_i(s))_{i\geq 1}$ converges to $g_0(s) = 1$ in $\tau_{\mathcal{K}_{sp}}(\mathcal{P})$. Since $g_i(s)$ in H^∞ for all $i \in \mathbb{N}$, a coprime factorization $(n_i(s), d_i(s))$ of $g_i(s)$, $i \in \mathbb{N}$, is given by

$$n_i(s) = g_i(s), \qquad d_i(s) = 1,$$

$i \in \mathbb{N}$. Let $k(s) \in \mathcal{K}_{sp}$ be a strictly proper controller that stabilizes the limiting transfer function $g_0(s) = 1$. Write $n_k(s)d_k(s)^{-1}$ for a coprime factorization of $k(s)$, where $n_k(s)$ is strictly proper. We want to show that $k(s)$ stabilizes $g_i(s)$ for i sufficiently large. This will be the case if

$$g_i(s)n_k(s) + d_k(s)$$

is unimodular for large enough i. Note that

$$n_k(s) + d_k(s)$$

is unimodular since $k(s)$ stabilizes the transfer function $g_0(s) = 1$. Hence if we can show that

$$(g_i(s)n_k(s) + d_k(s)) - (n_k(s) + d_k(s)) = (g_i(s) - 1)n_k(s)$$

converges to zero in H^∞ we have proved the result since the unimodular functions are open in H^∞. But this is a simple consequence of the fact that $(g_i(s))_{i\geq 1}$ is a bounded sequence converging pointwise to $g_0(s) = 1$ and that $n(s)_k$ is strictly proper. $\square$

It might be worthwhile pointing out at this point that it is relatively easy to characterize the graph topology of functions of fixed McMillan degree. It is essentially already shown in [5] that the graph topology on this set is equivalent to the topology on the set which is induced by the convergence of the coefficients of the transfer function. This topology which is highly nontrivial has been extensively studied by many authors (see e.g. [1], [4], [2]).

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