Proceedings of the 29th Conference
on Decision and Control
Honolulu, Hawaii • December 1990

Robust Stabilization in the presence of Coprime Factor Perturbations

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Abstract

McFarlane and Glover in [3] proposed a method of robust controller design to coprime factor perturbations. In common with all $\mathcal{H}_\infty$ design methods, their bound on the size of allowable perturbations is restrictive, in that there exist perturbations outside this bound that do not destabilize the closed loop system. This paper studies perturbations in certain 'key' directions whose sizes are larger than the robustness margin but do not destabilize the plant.

1 Introduction

The problem of robustly stabilizing a closed loop system in the presence of uncertainty has received a considerable amount of attention in recent years. In particular, the $\mathcal{H}_\infty$ approach to optimal control system analysis and design has provided some promising results in the area of robust stabilization of plants with unstructured uncertainties. This paper will examine uncertainty in the nominal system modelled by additive perturbations on the coprime factors of the system. It will demonstrate that the bound on the admissible uncertainty in McFarlane and Glover [3] is restrictive in that there exit perturbations of larger size than this bound which are still stabilized. Section 2 reviews the coprime factor robust stabilization problem and Section 3 outlines the results of this paper. All systems will be assumed to be SISO, linear, finite dimensional and time-invariant.

Lack of space only allows us to give a brief summary of the results. The details will appear elsewhere.

2 Preliminaries

This section introduces the robust stabilization problem first suggested by Vidyasagar [5]. It is formulated in terms of unstructured additive perturbations on the normalized coprime factors of the nominal system, g. Let the nominal scalar transfer function g, have normalized c.f. (n,m) such that $g = nm^{-1}$ and $n^*n + m^*m = 1$. Then any other scalar transfer function can be written in the form,

$$g_\Delta = (n+\Delta_n)(m+\Delta_m)^{-1} \tag{1}$$

where $\Delta_n, \Delta_m \in \mathcal{H}_\infty$ are stable transfer functions. It is possible to define various families of systems by placing restrictions on the allowable perturbations Δ_n, Δ_m. The normalized robust stabilization problem considered here is to internally stabilize the nominal system, g, with normalized c.f. (n,m) and the family of systems $\mathcal{G}_\epsilon$, defined by

$$\mathcal{G}_\epsilon := \{(n+\Delta_n)(m+\Delta_m)^{-1} : \begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix} \in \mathcal{H}_\infty^{2\times 1}; \left\|\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}\right\|_\infty < \epsilon\} \tag{2}$$

using a dynamic feedback controller k. The maximum stability margin is then defined as the largest possible number, ϵ_{max}, such that there exists a controller k internally stabilizing all systems in the family $\mathcal{G}_{\epsilon_{max}}$.

This problem can also be stated in the general $\mathcal{H}_\infty$ framework [1]. Define the standard $\mathcal{H}_\infty$ plant

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} 0 & m^{-1} \\ 1 & g \end{bmatrix} & \begin{matrix} m^{-1} \\ g \end{matrix} \end{bmatrix}$$

and define the linear fractional transformations

$$\begin{aligned} &\mathcal{F}_u(P, \begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}) \\ &:= P_{22} + P_{21}\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}(I - P_{11}\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix})^{-1}P_{12} \\ &= (n+\Delta_n)(m+\Delta_m)^{-1} \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_l(P,k) &:= P_{11} + P_{12}k(I-P_{22}k)^{-1}P_{21} \\ &= m^{-1}(1-kg)^{-1}\begin{bmatrix} k & 1 \end{bmatrix} \end{aligned} \tag{3}$$

The problem fits into the standard $\mathcal{H}_\infty$ framework. Using the small-gain theorem to show that the feedback system $\mathcal{F}_l(\mathcal{F}_u(P, \begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}), k)$ is stable for all $\Delta_n, \Delta_m \in \mathcal{H}_\infty$ where $\left\|\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}\right\|_\infty < \epsilon$ if and only if $\mathcal{F}_l(P,k) \in \mathcal{H}_\infty$ and $\|\mathcal{F}_l(P,k)\|_\infty < \epsilon^{-1}$, we obtain

Lemma 2.1 ([5]) (i) *The maximum stability margin,*

$$\epsilon_{max}^{-1} = \inf_{k \text{ stabilizing}} \|\mathcal{F}_l(P,k)\|_\infty \tag{4}$$

where the infimum is taken over all stabilizing controllers.

(ii) *k internally stabilizes all $g_\Delta \in \mathcal{G}_\epsilon$ if and only if*

$$\|\mathcal{F}_l(P,k)\|_\infty \le \epsilon^{-1} \tag{5}$$

However this Lemma is restrictive in the sense that given a controller k where $\|\mathcal{F}_l(P,k)\|_\infty = \epsilon^{-1}$, there exist systems $g_\Delta \notin \mathcal{G}_\epsilon$ that are internally stabilized by the controller k.

3 Outline of Results

This section will summarize the results for scalar systems and illustrate these with an example. Given a nominal system, g with coprime factorization (n,m), and a stabilizing controller, k, then there exists, [4], a coprime factorization of the controller, $k = v^{-1}u$ such that the Bezout identity is satisfied

$$mv - nu = 1 \tag{6}$$

Substituting the factorizations into the expression for $\mathcal{F}_l(P,k)$ defined in (3) gives

$$\begin{aligned} \mathcal{F}_l(P,k) &= m^{-1}(1 - v^{-1}unm^{-1})^{-1}\begin{bmatrix} v^{-1}u & 1 \end{bmatrix} \\ &= \begin{bmatrix} u & v \end{bmatrix} \end{aligned} \tag{7}$$

A perturbation $\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}$ is called destabilizing if and only if the controller, k, does not stabilize the system $(n+\Delta_n)(m+\Delta_m)^{-1}$. The feedback system $\mathcal{F}_l(\mathcal{F}_u(P, \begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}), k)$ is stable by the small gain theorem if the open-loop transfer function $\mathcal{F}_l(P,k)\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}$ has norm less than 1. With (7) this is the case if $\|\mathcal{F}_l(P,k)\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}\|_\infty < 1$. But by a Nyquist stability argument,

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g_Δ is not stabilized if there exists a frequency $\omega_o \in \Re$ such that $\mathcal{F}_l(P,k)\left[\begin{smallmatrix}\Delta_n \\ -\Delta_m\end{smallmatrix}\right](j\omega_o) = 1$ or equivalently,

$$\begin{bmatrix} u & v \end{bmatrix}(j\omega_o)\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}(j\omega_o) = 1. \tag{8}$$

Perturbations that satisfy (8) are on the boundary between stability and instability if we also have that $\epsilon\left[\begin{smallmatrix}\Delta_n \\ -\Delta_m\end{smallmatrix}\right]$ is not a destabilizing perturbation for all $\epsilon < 1$. Finally the observation that a perturbation that can be expressed in the form

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix} = \begin{bmatrix} v \\ -u \end{bmatrix} r \qquad r \in \mathcal{RH}_\infty$$

is never destabilizing, suggests a preliminary decomposition of an arbitrary perturbation of the form

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix} = \begin{bmatrix} -n & v \\ m & -u \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \qquad r_1, r_2 \in \mathcal{RH}_\infty.$$

From the condition (8), a perturbation expressed in the above form is stable if $\|r_1\| < 1$, and destabilizing if there exists a frequency ω_o such that $r_1(j\omega_o) = 1$.

We now come to define our 'key perturbations'. These are the minimum norm destabilizing perturbations in particular directions in the space of all perturbations.

Definition 3.1 Given a plant $g = nm^{-1}$ of McMillan degree j with distinct Hankel singular values and a stabilizing controller $k = v^{-1}u$ such that (6) is satisfied , then $\left[\begin{smallmatrix}\Delta_n \\ -\Delta_m\end{smallmatrix}\right] \in \mathcal{RH}_\infty$ is a *key perturbation in the i^{th} direction*, $i = 0, 1, \ldots j-1$, if and only if

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_i := \begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_{i,u} B_i$$

where B_i is the unique finite Blaschke product of degree i chosen such that $\left[\begin{smallmatrix}\Delta_n \\ -\Delta_m\end{smallmatrix}\right]_i \in \mathcal{RH}_\infty$, and where

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_{i,u} = \begin{bmatrix} -n & v \\ m & -u \end{bmatrix}\begin{bmatrix} 1 \\ r_2^i \end{bmatrix}$$

such that r_2^i has i poles in the right half plane i.e. is in $\mathcal{RH}_\infty^i$ and is the unique solution to the two block $\mathcal{H}_\infty$ optimization,

$$\inf_{r_2^i \in \mathcal{RH}_\infty^i} \left\| \begin{bmatrix} -n & v \\ m & -u \end{bmatrix}\begin{bmatrix} 1 \\ r_2^i \end{bmatrix} \right\|_\infty =: \alpha_i \tag{9}$$

This definition is easily generalized to the case when g no longer has distinct Hankel singular values. It is clear from the definition of a key perturbation that

$$\begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_i = B_i$$

and hence (8) is satisfied for at least one frequency. It is worth noting that the key perturbation in the $j-1$ direction is a minimum norm destabilizing perturbation, and the perturbation in the zeroth direction perturbs the nominal system to the inverse of the controller, that is to $(n + \Delta_n^0)(m + \Delta_n^0)^{-1} = k^{-1}$. These perturbations can be calculated by the two-block $\mathcal{H}_\infty$ optimization in (9) or with techniques similar to those used in [3], it can be shown to be equivalent to Hankel Norm approximation problem of the normalized coprime fctors of the controller. Hence for the optimal controller, i.e. the controller achieving the bound $\|\mathcal{F}_l(P,k)\|_\infty = \epsilon_{max}^{-1}$, we have that $\alpha_i = \sqrt{1-\sigma_i^2}/\sqrt{1-\sigma_{j-i}^2}$ for $i = 0, 1, 2 \ldots j-2$ and $\alpha_{j-1} = \epsilon_{max}$ where σ_i are the Hankel singular values of $\left[\begin{smallmatrix}n \\ m\end{smallmatrix}\right]$ the normalized coprime factors of the system g.

Example 3.0.1 (Example 3.1.2 [2]) Consider the system

$$g = \frac{\sqrt{3}}{s^2+1}$$

which has a normalized coprime factorization

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{s^2 + \sqrt{2}s + 2}\begin{bmatrix} \sqrt{3} \\ s^2+1 \end{bmatrix}$$

The Hankel singular values of $\left[\begin{smallmatrix}n \\ m\end{smallmatrix}\right]$ are $\sigma_1^2 = \frac{3}{4}$, and $\sigma_2^2 = \frac{2}{3}$ and therefore the optimal robustness margin of this system with respect to coprime factor uncertainty is $\epsilon_{max} = 0.5$. Using the techniques in [3], the optimally robust controller can be derived and is given by

$$k = -\frac{\sqrt{3}s}{s + 2\sqrt{2}}.$$

Its coprime factors that satisfy the Bezout identity (6) are,

$$\begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}s}{s+\sqrt{2}} & \frac{s+2\sqrt{2}}{s+\sqrt{2}} \end{bmatrix}.$$

It achieves the bound $\|\mathcal{F}_l(P,k)\|_\infty = 2$. A minimum norm destabilizing perturbation is simply

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_1 = \begin{bmatrix} -\frac{\sqrt{3}s}{4(s+\sqrt{2})} \\ \frac{s-2\sqrt{2}}{4(s+\sqrt{2})} \end{bmatrix}$$

whose $\mathcal{H}_\infty$-norm is 0.5. The perturbed system with the perturbation above is,

$$g_\Delta = \frac{-\sqrt{3}(s^3 + \sqrt{2}s^2 - 2s - 4\sqrt{2})}{(3s^3 + 5\sqrt{2}s^2 + 6s + 8\sqrt{2})}$$

Finally a destabilizing perturbation of $\mathcal{H}_\infty$-norm equal to $1/\sqrt{3}$ in the zeroth direction is,

$$\begin{bmatrix} \Delta_n \\ -\Delta_m \end{bmatrix}_0 = \begin{bmatrix} \frac{-1}{2\sqrt{3}} \\ \frac{1}{2} \end{bmatrix}$$

and the perturbed system for this perturbation is k^{-1}. A perturbation of the form $\epsilon\left[\begin{smallmatrix}\Delta_n \\ -\Delta_m\end{smallmatrix}\right]_0$, however, is not destabilizing for all $\epsilon < 1$ even though it might have $\mathcal{H}_\infty$-norm greater than $\epsilon_{max} = 0.5$.

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