SIAM J. CONTROL AND OPTIMIZATION
Vol. 29, No. 6, pp. 1251–1287, November 1991


# BALANCED PARAMETRIZATION OF CLASSES OF LINEAR SYSTEMS*

RAIMUND OBER†

**Abstract.** Canonical forms and parametrizations are presented for several sets of minimal systems of given dimension: asymptotically stable systems, allpass systems, bounded real systems, positive real systems, minimum-phase systems, and the class of all minimal systems. The approach is based on balancing techniques for these classes of systems. Applications of these results to Hankel operators and model reduction are discussed.

**Key words.** canonical form, parametrization, balanced realization, model reduction

**AMS(MOS) subject classifications.** 93B10, 93B20

**1. Introduction and notation.** Canonical forms for linear systems are of importance since they provide a unique state-space representation of linear systems. They therefore play a major role in system identification where a unique parametrization of the systems in the model set is essential. From a more theoretical point of view, canonical forms permit the study of topological and geometric properties of sets of linear systems [13], [15], [24]. For a survey of results and applications of canonical forms, see [15].

A definition of a canonical form is as follows.

DEFINITION 1.1. Let $M$ be a set of minimal state-space systems. Then a map

$$\Gamma : M \mapsto M$$

is called a *canonical form* if

(1) $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := \Gamma((A, B, C, D))$ is *system equivalent* to $(A, B, C, D)$, i.e., there exists $T \in \Re^{n\times n}$, invertible such that

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}, \quad \tilde{D} = D.$$

(2) If $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$ are system equivalent, then

$$\Gamma((A_1, B_1, C_1, D_1)) = \Gamma((A_2, B_2, C_2, D_2)).$$

Various types of canonical forms for linear systems have been introduced and studied (see, e.g., [30], [12], [15]). Most of these canonical forms for multivariable systems are generalizations of the observer or controller canonical form for single-input single-output systems. The canonical forms presented here are based on balanced representations of linear systems. Balanced realizations for asymptotically stable systems were introduced in [19]. Other balancing techniques were then investigated in [7] for stochastic realizations, in [16] for the class of all minimal systems, and in [28] for bounded real systems. The motivation of those authors was to obtain a simple method for the approximation of a system by a lower-dimensional system in the same class.

It is shown in this paper that the balancing technique leads to canonical forms with desirable properties. In § 2 we consider balanced realizations as they were defined in [19] for asymptotically stable systems. The canonical form established for this class of systems is a modification of a similar canonical form presented in [22]. This canonical form has a structure that shows that allpass systems, which are considered in § 3, are

* Received by the editors November 16, 1988, accepted for publication (in revised form) September 10, 1990.

† Department of Engineering, Cambridge University, Trumpington Street, Cambridge CB1 1PZ, United Kingdom. Present address, Center for Engineering Mathematics, The University of Texas at Dallas, Richardson, Texas 75083.

in some sense building blocks of this canonical form. These results are the basis for the derivation of canonical forms of other classes of systems as presented in later sections of the paper. In [27] the canonical form in [22] was used to derive a canonical form for the class of minimal systems in terms of Riccati-balanced coordinates as introduced by [16]. This canonical form is rewritten in § 4 using the canonical form of § 2. Relating the class of bounded real systems to a subclass of multivariable asymptotically stable systems, it is possible to derive a canonical form for bounded real systems as presented in § 5. Via a Moebius-type transformation, the class of bounded real systems is mapped to the class of positive real systems. Since positive real systems are closely related to minimum-phase systems through spectral factorization, the canonical form derived for positive real systems in § 6 can be used in § 7 to derive a canonical form for minimum-phase systems. Section 8 deals with the relationship of the previous results to discrete-time systems. The question of model reduction of systems given in the canonical forms is considered in § 9. Examining the canonical forms, it appears that they have many common structural properties, which are discussed in § 10.

The objective of the paper is to show that for many classes of systems it is possible to derive canonical forms using the idea of balancing. The general principle of balancing is to associate with a particular class of systems a set of Riccati equations that are intrinsically related with the properties of the particular class of systems. The class of asymptotically stable systems is, for example, associated with a set of Lyapunov equations, whereas the set of positive real systems is associated with the positive real Riccati equations. It is then possible to define what a balanced realization means for such a class of systems. A system is called balanced if the solutions to the two associated equations are identical and diagonal. Having defined the notion of a balanced realization, it can be seen that such a realization of a particular system is not unique. One of the aims of this paper is to show that by imposing further constraints on the realization, it is indeed possible to obtain a unique realization, i.e., a canonical form.

The usefulness of canonical forms very much depends on their properties. One of the standard canonical forms, the controller canonical form, is of particular significance since the parameters of the canonical form have an immediate interpretation as the coefficients of the transfer function. There are, however, drawbacks of the controller canonical form especially concerning the resulting parametrization of linear systems. The set of parameters in the controller canonical form that lead to a minimal system is very complicated. This makes it difficult to use this canonical form in cases where it is necessary to have a geometrically well-behaved parameter space, e.g., in some optimization tasks. One of the main advantages of the canonical forms derived here is that it can be shown that the parameter spaces associated with a canonical form have—especially in the case of single-input single-output systems—desirable geometric properties. This is at the expense of having to partition the set of parametrized systems into suitable subsets. This means that even in the case of single-input single-output systems structural parameters must be introduced.

There would be several ways to derive the results presented here. One of them would be to treat each of the classes of systems separately and construct the canonical form from first principles. This approach would be very tedious, especially because of the complexity of the canonical forms in the case of multivariable systems. Instead, we are going to relate the various classes of systems to one another. This allows us to carry the canonical form over from one class of systems to another without having to repeat the basic construction.

Analyzing these canonical forms, it becomes apparent that all share certain structural properties. It is interesting to see that such widely differing classes of transfer functions, such as, for example, minimum-phase systems and bounded real systems, admit a parametrization that has very similar properties. Having a common structure has the advantage of allowing us to deal with the various classes of systems in a unified way. It was possible to exploit this common structure in the study of the connectivity properties of the various classes of systems [26].

For the presentation of our results we will use the common structure of the canonical forms. We will introduce the following notation, which allows us to simplify the statement of the canonical forms especially in the case of multivariable systems. Most of these definitions are, however, not important for the case of single-input single-output systems.

- A matrix $B=(b_{i,j})_{\substack{1\leqq i<k\\ 1\leqq j\leqq l}}$ is called *positive upper triangular* if there exist indices

$$1\leqq t_1<\cdots<t_j<\cdots<t_k\leqq l$$

such that

$$\begin{aligned} &b_{i,t_i}>0 && \text{for } 1\leqq i\leqq k,\\ &b_{i,j}=0 && \text{for } 1\leqq j<t_i \text{ and } 1\leqq i\leqq k,\\ &b_{i,j}\in\mathfrak{R} && \text{otherwise,}\end{aligned}$$

i.e.,

$$B=\begin{pmatrix} 0 & \cdots & 0 & b_{1,t_1} & b_{1,t_1+1} & \cdots & \cdot & \cdot & \cdot & \cdots & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{2,t_2} & b_{2,t_2+1} & \cdots & & & \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{k,t_k} & b_{k,t_k+1}\cdots \end{pmatrix}.$$

- A matrix $A$ is said to be in *r-balanced form*, $1\leqq r\leqq n$, if for

$$A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\qquad A_{11}\in\mathfrak{R}^{r\times r},$$

we have
(1) $A_{11}$ is skew symmetric.
(2) $A_{12}$ and $A_{22}$ are given by the set of indices

$$1=h_1<\cdots<h_i<h_{i+1}<\cdots<h_q\leqq n-r,$$
$$1\leqq g_q<\cdots<g_{i+1}<g_i<\cdots<g_1\leqq r$$

in the following way:

(a) for $A_{12}=(a_{st})_{\substack{1\leqq s\leqq r\\ 1\leqq t\leqq n-r}}$ we have

$$\begin{aligned} &a_{g_i,h_i}>0 && \text{for } 1\leqq i\leqq q,\\ &a_{g_i,t}=0 && \text{for } t>h_i \quad \text{where } 1\leqq i\leqq q,\\ &a_{s,t}=0 && \text{for } t\geqq h_i \text{ and } s>g_i \quad \text{where } 1\leqq i\leqq q,\end{aligned}$$

i.e.,

$$A_{12}=\begin{pmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots & \\ \cdot & \cdot & \cdots & a_{g_2-1,h_2-1} & a_{g_2-1,h_2} & a_{g_2-1,h_2+1} & \cdots \\ \cdot & \cdot & \cdots & a_{g_2,h_2-1} & a_{g_2,h_2} & 0 & \cdots \\ \cdot & \cdot & \cdots & a_{g_2+1,h_2-1} & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ a_{g_1-1,h_1} & a_{g_1-1,h_1+1} & \cdots & a_{g_1-1,h_2-1} & 0 & 0 & \cdots \\ a_{g_1,h_1} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

(b) $A_{22}$ is given by

$$A_{22}=\begin{pmatrix} 0 & \alpha_2 & & & & \\ -\alpha_2 & 0 & \alpha_3 & & & \\ & -\alpha_3 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha_{n-r} \\ & & & & -\alpha_{n-r} & 0 \end{pmatrix},$$

where for $2 \leqq i \leqq n-r$

$$\alpha_i \begin{cases} =0 & \text{if } i=h_s \text{ for some } 1 \leqq s \leqq q, \\ >0 & \text{otherwise.} \end{cases}$$

(3) $A_{21}=-A_{12}^T$.

- Let $A=(a_{ij})_{\substack{1\leqq i\leqq n\\1\leqq j\leqq n}}$ then we denote by

(1) $[A]_l=(\bar{a}_{ij})_{\substack{1\leqq i\leqq n\\1\leqq j\leqq n}}$ the *lower triangular part* of $A$, i.e.,

$$\bar{a}_{ij}=\begin{cases} 0 & \text{for } j \geqq i, \\ a_{ij} & \text{for } j<i. \end{cases}$$

(2) $[A]_d=(\tilde{a}_{ij})_{\substack{1\leqq i\leqq n\\1\leqq j\leqq n}}$ the *diagonal part* of $A$, i.e.,

$$\tilde{a}_{ij}=\begin{cases} 0 & \text{for } j \neq i, \\ a_{ij} & \text{for } j=i. \end{cases}$$

- Let $(A, B, C, D) \in \Re^{n\times n}\times\Re^{n\times m}\times\Re^{p\times n}\times\Re^{p\times m}$ and $n_1, \cdots, n_j, \cdots, n_k, n_j \in \mathcal{N}$, $\sum_{j=1}^k n_j = n$. Then $(A, B, C, D)$ is said to be *partitioned according to*

$n_1, \cdots, n_j, \cdots, n_k$ if

$$A = (A_{ij})_{1 \le i,j \le k}, \qquad A_{ij} \in \Re^{n_i \times n_j},$$

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_j \\ \vdots \\ B_k \end{pmatrix}, \qquad B_j \in \Re^{n_j \times m},$$

$$C = (C_1, \cdots, C_j, \cdots, C_k), \qquad C_j \in \Re^{p \times n_j}.$$

The following notation and abbreviations will be used throughout the paper.

- Classes of transfer functions:
  —$TL_n^{p,m} = \{p \times m$ transfer functions of McMillan degree $n\}$.
  —$TC_n^{p,m} = \{G(s) \in TL_n^{p,m} | G(s)$ has all its poles in the open left halfplane$\}$.
  —$TA_n^m = \{G(s) \in TC_n^{m,m} | G(s)G(-s)^T = \sigma^2 I$, for some $\sigma > 0$, for all $s \in \mathscr{C}\}$.
  —$TP_n^m = \{G(s) \in TC_n^{m,m} | G(\infty) + G(-\infty)^T > 0, G(iw) + G(-iw)^T > 0$ for all $w \in \Re\}$.
  —$TB_n^{p,m} = \{G(s) \in TC_n^{p,m} | G(-\infty)^T G(\infty) < I, I - G(-iw)^T G(iw) > 0$ for all $w \in \Re\}$.
  —$TM_n^m = \{G(s) \in TC_n^{m,m} | G(s)^{-1} \in TC_n^{m,m}\}$.
  —The corresponding sets of discrete-time systems are defined in § 8.
- Classes of state-space systems:
  —The sets of minimal state-space realizations of the transfer functions in $TL_n^{p,m}$, $TC_n^{p,m}$, $TA_n^m$, $TP_n^m$, $TB_n^{p,m}$, and $TM_n^m$ are denoted by $L_n^{p,m}$, $C_n^{p,m}$, $A_n^m$, $P_n^m$, $B_n^{p,m}$, and $M_n^m$.
  —The corresponding sets of discrete time systems are defined in § 8.
- Symbols:
  —diag $(A_1, \cdots, A_k)$ is a block diagonal matrix with $A_1, \cdots, A_k$ as its block diagonal entries.
  —$\hat{I}_n :=$ diag $(+1, -1, +1, -1, \cdots, (-1)^{n+1}) \in \Re^{n \times n}$.
  —$\mathscr{N}$ denotes the set of natural numbers, $\mathscr{C}$ denotes the set of complex numbers, and $\mathscr{R}$ the set of real numbers.

**2. Asymptotically stable systems.** In this section we will review some of the results on the balanced parametrization of asymptotically stable systems as given in [22]. The particular canonical form presented here for multivariable systems is, however, a simplified version of the one introduced in [22]. First, we will define a balanced realization of a minimal and asymptotically stable continuous-time system as introduced by Moore [19].

DEFINITION 2.1. The set of minimal and asymptotically stable systems $(A, B, C, D) \in \Re^{n \times n} \times \Re^{n \times m} \times \Re^{p \times n} \times \Re^{p \times m}$ is denoted by $C_n^{p,m}$. A system $(A, B, C, D) \in C_n^{p,m}$ is called *balanced* if there exists a diagonal matrix $\Sigma = $ diag $(\sigma_1, \cdots, \sigma_j, \cdots, \sigma_n)$ such that

$$A\Sigma + \Sigma A^T = -BB^T, \qquad A^T\Sigma + \Sigma A = -C^T C. \tag{1}$$

The matrix $\Sigma$ is called the gramian of the system $(A, B, C, D)$ and its diagonal entries are said to be the *singular values* of the system.

Moore [19] has shown that each system in $C_n^{p,m}$ has an equivalent system that is balanced. Such a realization, however, is not unique. If $\Sigma =$ diag $(\sigma_1 I_{n(1)}, \cdots, \sigma_j I_{n(j)}, \cdots, \sigma_k I_{n(k)})$, $\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0$, is the gramian of a balanced system $(A, B, C, D)$, then all equivalent balanced systems with singular values ordered according to multiplicities are given by $(QAQ^T, QB, CQ^T, D)$, where $Q = \text{diag}(Q_1, \cdots, Q_j, \cdots, Q_k)$, $Q_j \in \mathfrak{R}^{n_j \times n_j}$, $Q_j^T Q_j = I_{n_j}$, for $1 \leqq j \leqq k$ (see, e.g., [19], [10]). Thus if $\Sigma$ has distinct diagonal entries, $(A, B, C, D)$ is unique up to a state-space transformation by a sign matrix, i.e., a diagonal matrix whose diagonal entries are $\pm 1$. Hence for this case a canonical form can be obtained by constraining the first nonzero entry of each row of the $B$-matrix to be positive [17], [20], [22]. The following theorem gives a canonical form for all systems in $C_n^{p,m}$ in terms of balanced realizations. It thereby shows how to impose further constraints to obtain a unique balanced realization in the general case.

THEOREM 2.1. *The following two statements are equivalent:*

(1) $G(s) \in TC_n^{p,m}$.

(2) *$G(s)$ has a realization $(A, B, C, D) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times m} \times \mathfrak{R}^{p \times n} \times \mathfrak{R}^{p \times m}$ given by the parameters:*

$$\begin{array}{llll}
\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0 & & & \\
n_1, \cdots, & n_j, \cdots, & n_k, & n_j \in \mathcal{N}, \sum_{j=1}^k n_j = n; \\
r_1, \cdots, & r_j, \cdots, & r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m, p); \\
U_1, \cdots, & U_j, \cdots, & U_k, & U_j \in \mathfrak{R}^{p \times r_j}, \quad U_j^T U_j = I_{r_j}; \\
\tilde{B}_1, \cdots, & \tilde{B}_j, \cdots, & \tilde{B}_k, & \tilde{B}_j \in \mathfrak{R}^{r_j \times m} \text{ positive upper triangular}; \\
\tilde{A}_1, \cdots, & \tilde{A}_j, \cdots, & \tilde{A}_k, & \tilde{A}_j \in \mathfrak{R}^{n_j \times n_j} \text{ in } r_j\text{-balanced form}; \\
D, & & & D \in \mathfrak{R}^{p \times m};
\end{array}$$

*in the following way.*

*If $(A, B, C, D)$ is partitioned according to $n_1, \cdots, n_j, \cdots, n_k$, then*

(i) $B_j = \begin{pmatrix} \tilde{B}_j \\ 0 \end{pmatrix}, \qquad 1 \leqq j \leqq k,$

(ii) $C_j = (U_j \Delta_j, 0)$ *where* $\Delta_j = (\tilde{B}_j \tilde{B}_j^T)^{1/2}, \qquad 1 \leqq j \leqq k,$

(iii) $A_{jj} = \tilde{A}_j - \dfrac{1}{\sigma_j} [\text{diag}(\Delta_j^2, 0)]_l - \dfrac{1}{2\sigma_j} [\text{diag}(\Delta_j^2, 0)]_d, \qquad 1 \leqq j \leqq k,$

(iv) $A_{ij} = \dfrac{1}{\sigma_i^2 - \sigma_j^2} \text{diag}((\sigma_j \tilde{B}_i \tilde{B}_j^T - \sigma_i \Delta_i U_i^T U_j \Delta_j), 0), \qquad 1 \leqq i, j \leqq k, \quad i \neq j,$

(v) $D \in \mathfrak{R}^{p \times m}$.

*Moreover, $(A, B, C, D)$ as defined in (2) is balanced with gramian*

$$\Sigma = \text{diag}(\sigma_1 I_{n_1}, \cdots, \sigma_j I_{n_j}, \cdots, \sigma_k I_{n_k}).$$

*The map $\Gamma$, which assigns to each system in $C_n^{p,m}$ the realization given in (2), is a canonical form.*

*Proof.* The derivation of the results is similar to the derivation of the canonical form for systems in $C_n^{p,m}$ as given in [22].

(1) $\Rightarrow$ (2) Let $(A, B, C, D) \in C_n^{p,m}$ be a balanced system with gramian $\Sigma = \sigma I_n$. Then it follows from the Lyapunov equations (1) that

$$A + A^T = -\frac{1}{\sigma} BB^T = -\frac{1}{\sigma} C^T C.$$

Since the realization $(A, B, C, D)$ is unique up to a state-space transformation with a unitary matrix, we can assume without loss of generality that

$$B = \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix},$$

where $\tilde{B} \in \mathfrak{R}^{r \times m}$, $r = \text{rank}(B)$, is in positive upper triangular form. Note that this is a unique representation of $B$.

Since $BB^T = C^TC$ there exists a unique $U \in \mathfrak{R}^{p \times r}$, $U^TU = I_r$, such that

$$C = (U(\tilde{B}\tilde{B}^T)^{1/2}, 0).$$

If we write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad A_{11} \in \mathfrak{R}^{r \times r},$$

we have

$$A_{11} + A_{11}^T = -\frac{1}{\sigma} \tilde{B}\tilde{B}^T.$$

Thus the diagonal elements of $A_{11}$ are given by

$$[A_{11}]_d = -\frac{1}{2\sigma} [\tilde{B}\tilde{B}^T]_d.$$

Since

$$A_{11} = -\frac{1}{\sigma} \tilde{B}\tilde{B}^T - A_{11}^T,$$

$A_{11}$ is completely parametrized by the entries of $\tilde{B}$ and those of the upper triangular part of $A_{11}$. The other blocks of $A$, i.e., $A_{12}$, $A_{21}$, and $A_{22}$ are derived as in [22]. This completes the proof for the case of identical singular values. The general case follows as in [22].

(2) $\Rightarrow$ (1) This is a straightforward modification of Theorem 7.1 of [22]. □

The canonical form presented in the previous theorem is determined both by discrete and continuous parameters. Of the discrete parameters, $n_1, \cdots, n_k$ are of particular importance. They indicate the multiplicities of the singular values $\sigma_1, \cdots, \sigma_k$ and determine the partitioning of the state-space systems into blocks of sizes $n_1, \cdots, n_k$. To each such block corresponds the $n_j$-dimensional subsystem $(A_{jj}, B_j, C_j, D)$, $1 \leqq j \leqq k$. Such a subsystem is a system with identical singular values $\sigma_j$. An interesting aspect of the canonical form is that the off-diagonal blocks $A_{ij}$, $i \neq j$, of the $A$ matrix, which interconnect the various subsystems, are completely determined by the parameters of the diagonal subsystems. It therefore becomes clear that each system is made up of building blocks that are systems with identical singular values. The derivation of a canonical form, therefore, essentially reduces to the derivation of a canonical form for systems with identical singular values. The complexity of the canonical form depends crucially on min $(m, p)$, the minimum of the dimensions of the input and output spaces. As a consequence, the canonical form, if specialized to single-input single-output systems, is considerably simplified.

The following corollary states the canonical form for single-input single-output systems.

COROLLARY 2.1. *The following two statements are equivalent:*

(1) $g(s) \in TC_n^{1,1}$.

(2) $g(s)$ *has a realization* $(A, b, c, d) \in \Re^{n\times n} \times \Re^{n\times 1} \times \Re^{1\times n} \times \Re^{1\times 1}$ *given by the parameters*

$$\begin{array}{ll}
\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
s_1, \cdots, s_j, \cdots, s_k, & s_j = \pm 1, \quad 1 \leqq j \leqq k; \\
b_1, \alpha(1)_1, \cdots, \alpha(1)_j, \cdots, \alpha(1)_{n_1-1}, & b_1 > 0, \quad \alpha(1)_j > 0, \quad 1 \leqq j \leqq n_1 - 1; \\
\vdots & \\
b_i, \alpha(i)_1, \cdots, \alpha(i)_j, \cdots, \alpha(i)_{n_i-1}, & b_i > 0, \quad \alpha(i)_j > 0, \quad 1 \leqq j \leqq n_i - 1; \\
\vdots & \\
b_k, \alpha(k)_1, \cdots, \alpha(k)_j, \cdots, \alpha(k)_{n_k-1}, & b_k > 0, \quad \alpha(k)_j > 0, \quad 1 \leqq j \leqq n_k - 1; \\
d, & d \in \Re;
\end{array}$$

*in the following way:*

(i) $b = (\underbrace{b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{b_k, 0, \cdots, 0}_{n_k})^T$,

(ii) $c = (\underbrace{s_1 b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{s_j b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{s_k b_k, 0, \cdots, 0}_{n_k})$,

(iii) *For* $A =: (A_{ij})_{1 \leqq i,j \leqq k}$, $A_{ij} \in \Re^{n_i \times n_j}$, $1 \leqq i, j \leqq k$, *we have*

(a) *block diagonal entries* $A_{jj}$, $1 \leqq j \leqq k$:

$$A_{jj} = \begin{pmatrix} a_{jj} & \alpha(j)_1 & & & & \\ -\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\ & -\alpha(j)_2 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\ & & & & -\alpha(j)_{n_j-1} & 0 \end{pmatrix}$$

*with* $a_{jj} = -b_j^2/2\sigma_j$.

(b) *off-diagonal blocks* $A_{ij}$, $1 \leqq i, j \leqq k$, $i \neq j$:

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{with } a_{ij} = \frac{-b_i b_j}{s_i s_j \sigma_i + \sigma_j},$$

(iv) $d \in \Re$.

*Moreover,* $(A, b, c, d)$ *as defined in* (2) *is balanced with gramian*

$$\Sigma = \operatorname{diag}(\sigma_1 I_{n_1}, \cdots, \sigma_j I_{n_j}, \cdots, \sigma_k I_{n_k}).$$

*The map* $\Gamma$, *which assigns to each system in* $C_n^{1,1}$ *the realization given in* (2), *is a canonical form.*

*Remark* 2.1. The canonical form of Theorem 2.1 is closely related to the canonical form derived in [22]. The main difference between the two canonical forms is in the structure of the $B$-matrix. In [22] a subblock $B_j$ corresponding to a set of repeated singular values is constrained to have orthogonal rows. The implication of this is that the corresponding subblock of $A$ has a desirable symmetry property. Here we do not impose this symmetry on $A$ and hence we can relax the constraints on $B_j$ and obtain a much simplified parametrization of $B$. Another advantage of this canonical form is that the parameters enter freely and are not constrained by the orthogonality assumption on the rows of $B_j$. Note, however, that there are no differences between the two canonical forms for the case of SISO systems and for multivariable systems if the singular values are distinct.

As pointed out above, the canonical form is essentially determined by the diagonal subsystems that are determined by the block partitioning of the system corresponding to the multiplicities of the singular values. It is therefore not surprising that the canonical form reduces considerably in complexity, especially in the multivariable case, if the system has distinct singular values.

The significance of the previous theorem and corollary lies in the fact that not only is it shown that each asymptotically stable system has a unique representation and a canonical form having a certain structure, but possibly of greater importance, it is shown that the converse is also true. If we have given an arbitrary set of parameters that satisfy the stated constraints and if a system is formed from these parameters, then the theorem guarantees that this system is automatically minimal and asymptotically stable. We therefore have a parametrization of the set of asymptotically stable systems of fixed dimension. Equivalently, we can interpret the theorem as providing a parametrization of the set of transfer functions of fixed McMillan degree whose poles are in the left halfplane.

Note that especially in the single-input single-output case the parameter space has a nice geometric structure, since the continuous parameters are only determined by simple inequality constraints.

An important feature of a balanced realization of a linear system is its close relationship to the Hankel operator corresponding to the system. We define an *integral Hankel operator* with kernel $H(t) \in \mathfrak{R}^{p\times m}$, $t \geqq 0$, given by

$$\mathcal{H}: L^2_{\mathfrak{R}^m}([0,\infty[) \to L^2_{\mathfrak{R}^p}([0,\infty[),$$

$$u(t) \mapsto (\mathcal{H}(u))(s) = \int_0^\infty H(t+s)u(t)\,dt.$$

We assume that $\mathcal{H}$ is well defined and a finite rank operator. The singular values $(\sigma_j)_{1\leqq j\leqq n}$ of $\mathcal{H}$ are defined to be the nonzero eigenvalues of $(\mathcal{H}^*\mathcal{H})^{1/2}$ ordered with respect to their magnitude and taking into account their multiplicities. Under these conditions there exist families of orthonormal vectors (Schmidt vectors) $(v_j)_{1\leqq j\leqq n}$ and $(w_j)_{1\leqq j\leqq n}$ in $L^2_{\mathfrak{R}^m}([0,\infty[)$ and $L^2_{\mathfrak{R}^p}([0,\infty[)$, respectively, such that

$$\mathcal{H}v_j = \sigma_j w_j, \quad \mathcal{H}^* w_j = \sigma_j v_j, \quad 1 \leqq j \leqq n.$$

The significance of Hankel operators in a system-theoretic context is that for a system $(A, B, C, D) \in C_n^{p,m}$ the Hankel operator $\mathcal{H}$ with kernel $H(t) := Ce^{tA}B$ can be interpreted as an operator mapping past inputs to future outputs. If $(A, B, C, D)$ is balanced, it can easily be verified that the singular values of the Hankel operator $\mathcal{H}$ equal the singular values of the system $(A, B, C, D)$ [10]. Moreover, $v_j(t) = B^Te^{tA^T}e_j/\sqrt{\sigma_j}$ and $w_j(t) = Ce^{tA}e_j/\sqrt{\sigma_j}$, $1 \leqq j \leqq n$. Since $v_j(0) = B^Te_j/\sqrt{\sigma_j}$ and $w_j(0) = Ce_j/\sqrt{\sigma_j}$, $1 \leqq j \leqq n$, the starting points of the Schmidt vectors are fully determined by the $B$ and $C$ matrices and the singular values. This gives an interpretation of some of the parameters of a system in canonical form in terms of analytical properties of the corresponding Hankel operator.

By standard realization theory we know that there is a one-to-one correspondence between Hankel operators of rank $n$ and asymptotically systems of McMillan degree $n$. The discussion in the previous paragraph implies that finding a canonical form for linear systems in terms of balanced realizations is equivalent to defining a unique basis for the eigenspaces of the nonzero eigenvalues of the operator $\mathcal{H}^*\mathcal{H}$. This observation gives another interpretation of the complexity of the canonical form in the case of repeated singular values.

If we now consider the Hankel operator $\mathcal{H}$ corresponding to a scalar transfer function in $TC_n^{1,1}$, the eigenvectors and Schmidt vectors of $\mathcal{H}$ coincide (up to a sign), since $\mathcal{H}$ is self-adjoint. If, moreover, $(A, b, c, d)$ is in the canonical form of the previous corollary, the eigenvectors, respectively, Schmidt vectors, are given by $v_j(t) = b^T e^{tA^T} e_j/\sqrt{\sigma_j} = \tilde{s}_j w_j(t) = (\tilde{s}_j/\sqrt{\sigma_j}) c e^{tA} e_j$, $1 \leqq j \leqq n$, where we have used that $A = SA^T S$ and $c^T = Sb$ for $S = \operatorname{diag}(\tilde{s}_1, \cdots, \tilde{s}_j, \cdots, \tilde{s}_n) := \operatorname{diag}(s_1 \hat{I}_{n_1}, \cdots, s_j \hat{I}_n, \cdots, s_k \hat{I}_{n_k})$.

The eigenvalue corresponding to $v_j$ is therefore calculated to be $\tilde{s}_j \sigma$, since

$$(\mathcal{H} v_j)(t) = \int_0^\infty H(t+s) v_j(s)\, ds = \frac{1}{\sqrt{\sigma_j}} \int_0^\infty c e^{(t+s)A} b b^T e^{sA^T} e_j\, ds$$

$$= \sigma_j \frac{1}{\sqrt{\sigma_j}} c e^{tA} e_j = \tilde{s}_j \sigma_j v_j(t).$$

This observation allows us to use the canonical form for scalar systems to investigate the dimensions of the eigenspaces of a finite rank Hankel operator.

THEOREM 2.2. *Let $h(t) \in \mathfrak{R}$, $t \geqq 0$, be the kernel of a finite rank Hankel operator $\mathcal{H}$ acting on $L^2_{\mathfrak{R}}([0, \infty[)$. If $\lambda$ is a nonzero eigenvalue of $\mathcal{H}$, then*

$$|\dim(\ker(\lambda I - \mathcal{H})) - \dim(\ker(-\lambda I - \mathcal{H}))| \leqq 1.$$

*Proof.* By standard realization theory, $h(t)$, $t \geqq 0$, has a realization $(A, b, c, d) \in C_n^{1,1}$, where $n = \operatorname{rank}(\mathcal{H})$, i.e., $h(t) = c e^{tA} b$, $t \geqq 0$. We can assume that $(A, b, c, d)$ is in the canonical form of Corollary 2.1. All nonzero eigenvalues of $\mathcal{H}$ are given by the diagonal entries of $S\Sigma = \operatorname{diag}(s_1\sigma_1 \hat{I}_{n_1}, \cdots, s_j\sigma_j \hat{I}_{n_j}, \cdots, s_k\sigma_k \hat{I}_{n_k})$. We know that $|\lambda| = |s_{i_0}\sigma_{i_0}|$ for some $1 \leqq i_0 \leqq k$, and therefore

$$|\dim(\ker(\lambda I - \mathcal{H})) - \dim(\ker(-\lambda I - \mathcal{H}))| = \operatorname{trace}|(s_{i_0} \hat{I}_{n_{i_0}})| \leqq 1. \qquad \square$$

As a simple corollary to this theorem we have that Hankel operators with positive eigenvalues cannot have repeated singular values.

COROLLARY 2.2. *If $\mathcal{H}$ as defined in the theorem has only nonnegative eigenvalues, then the multiplicity of each of the singular values is one.*

Note that single-input single-output systems whose corresponding Hankel operators have only positive nonzero eigenvalues can be characterized as relaxation systems [25].

*Remark* 2.2. Using a canonical form for symmetric multivariable systems, a similar result was obtained in [23] for self-adjoint Hankel operators acting on vector-valued spaces. More precisely, if $\mathcal{H}$ is a finite rank, self-adjoint Hankel operator acting on the space $L^2_{\mathfrak{R}^p}([0, \infty[)$, then

$$|\dim(\ker(\lambda I - \mathcal{H})) - \dim(\ker(-\lambda I - \mathcal{H}))| \leqq p,$$

where $\lambda$ is a nonzero eigenvalue of $\mathcal{H}$.

The following remark gives further interpretations of the parameters of the canonical form and relates them to important analytical properties of the system.

*Remark* 2.3. If $(A, b, c, d) \in C_n^{1,1}$ is in balanced canonical form with $g(s) = c(sI - A)^{-1}b$ and $h(t) = ce^{tA}b$, $t \geqq 0$, then we have the following properties:

(i) $\|h(t)\|_2^2 := \int_0^\infty h(t)^2\, dt = \sum_{j=1}^k \sigma_j b_j^2$,

(ii) $\|g(s)\|_\infty := \sup_{w \in \mathfrak{R}} |g(iw)| \leqq 2 \sum_{j=1}^k \sigma_j$,

(iii) $g(0) = \int_0^\infty h(t)\, dt = 2 \sum_{j=1}^k s_j \sigma_j (\sum_{i=1}^{n(j)} (-1)^{i+1})$.

Note that if $(A, b, c, d)$ has distinct singular values and $s_j = 1$ or $s_j = -1$, for all $1 \leqq j \leqq n$, then (ii) and (iii) imply that the bound in (ii) is attained, i.e.,

$$\|g(s)\|_\infty = 2 \sum_{j=1}^n \sigma_j.$$

References to these results, which were slightly adapted here to our particular canonical form, can be found in [18] for (i) and (iii), [33] for (iii), and [10] for (ii).

**3. Allpass systems.** In the previous section we have seen that one of the main structural elements of the canonical form for asymptotically stable systems are systems with identical singular values. In this section we give an interpretation of such systems.

It was shown in [10] that each strictly proper system with identical singular values is the strictly proper part of an allpass system. Conversely, each allpass system has identical singular values. We can therefore say that the "building blocks" of the general canonical form are the strictly proper parts of allpass systems.

DEFINITION 3.1. A system $(A, B, C, D) \in C_n^{m,m}$ is called *allpass* if for $G(s) = C(sI - A)^{-1}B + D$ we have

$$G(s)G(-s)^T = \sigma^2 I, \qquad s \in \mathscr{C},$$

for some $\sigma > 0$. We denote by $A_n^m$ the subset of $C_n^{m,m}$ containing all allpass systems. The set of transfer functions of systems in $A_n^m$ is denoted by $TA_n^m$.

*Remark* 3.1. The usual definition of an allpass system is generalized here to the case where $\sigma$ is not necessarily equal to one. Note that in the mathematical literature the transfer functions of allpass systems are referred to as inner functions in the case of $\sigma = 1$.

Allpass systems or inner functions play an important role in many aspects of control theory, circuit theory, and mathematics. It is therefore of interest to have a canonical form and a parametrization of these systems. To obtain such a canonical form and parametrization of allpass systems from the results for systems with identical singular values, it is necessary to impose a relation between the parameters $U$, $\tilde{B}$, $D$, and $\sigma$.

A canonical form for allpass systems is given in the following theorem.

THEOREM 3.1. *The following two statements are equivalent*:

(1) $G(s) \in TA_n^m$.

(2) $G(s)$ *has a realization* $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{m\times n} \times \mathfrak{R}^{m\times m}$ *given by the parameters*: $\sigma > 0$; $r \in \mathcal{N}$, $1 \leqq r \leqq n$; $\tilde{B} \in \mathfrak{R}^{r\times m}$ *is positive upper triangular*; $\tilde{A}$ *is in r-balanced form*; $D \in \mathfrak{R}^{m\times m}$ *is such that* $DD^T = \sigma^2 I$; $U = -D\tilde{B}^T \Delta^{-1}(\sqrt{\sigma})^{-1}$ *with* $\Delta = (\tilde{B}\tilde{B}^T)^{1/2}$; *in the following way*:

(i) $B = \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix}$;

(ii) $C = (U\Delta, 0)$ *where* $\Delta = (\tilde{B}\tilde{B}^T)^{1/2}$;

(iii) $A=\tilde{A}-\frac{1}{\sigma}[\operatorname{diag}(\Delta^2,0)]_l-\frac{1}{2\sigma}[\operatorname{diag}(\Delta^2,0)]_d$;

(iv) $D\in\Re^{m\times m}$ *is such that* $DD^T=\sigma^2 I$.

*Moreover,* $(A, B, C, D)$ *as defined in* (2) *is balanced with gramian* $\Sigma=\sigma I_n$. *The map* $\Gamma_a$, *which assigns to each system in* $A_n^m$ *the realization given in* (2), *is a canonical form.*

*Proof.* The proof is an application of Theorem 5.1 of [10] and Theorem 2.1. □

It follows from the previous theorem that a parametrization of systems with identical singular values immediately leads to a parametrization of allpass systems and vice versa. For SISO systems this canonical form has a particularly simple structure.

COROLLARY 3.1. *The following two statements are equivalent*:

(1) $g(s)\in TA_n^1$.

(2) $g(s)$ *has a realization* $(A, b, c, d)\in\Re^{n\times n}\times\Re^{n\times 1}\times\Re^{1\times n}\times\Re^{1\times 1}$ *given by the parameters*

$$\sigma>0,\quad b>0,\quad \alpha_j>0,\quad 1\leqq j\leqq n-1,\quad s_1=\pm 1;$$

*in the following way*:

(i) $b=(b,0,\cdots,0)^T$;

(ii) $c=(s_1 b,0,\cdots,0)$;

(iii) $$A=\begin{pmatrix} a & \alpha_1 & & & & \\ -\alpha_1 & 0 & \alpha_2 & & & \\ & -\alpha_2 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha_{n-1} \\ & & & & -\alpha_{n-1} & 0 \end{pmatrix} \quad \textit{with } a=\frac{-b^2}{2\sigma};$$

(iv) $d=-s_1\sigma$.

*Moreover,* $(A, b, c, d)$ *as defined in* (2) *is balanced with gramian* $\Sigma=\sigma I_n$. *The map* $\Gamma_a$ *which assigns to each system in* $A_n^1$ *the realization given in* (2), *is a canonical form.* □

*Remark* 3.2. The $A$-matrix in the canonical form for SISO allpass transfer functions is closely related to the so-called Schwarz form for matrices, which has been studied in connection with stability tests for matrices (see, e.g., [4]). The recursive structure of the tridiagonal matrix permits us to give explicit realization algorithms for allpass systems [21]. It also follows easily from this recursive structure that each allpass transfer function $g(s)$ can be written as a continued fraction as follows:

$$g(s)=-s_1\sigma+\cfrac{s_1 b^2}{s+\cfrac{b^2}{2\sigma}+\cfrac{\alpha_1^2}{s+\cfrac{\alpha_2^2}{\ddots\, s+\cfrac{\alpha_{n-1}^2}{s}}}},$$

where $s_1, b, \sigma, \alpha_1,\cdots,\alpha_{n-1}$ are the same parameters as in the canonical form of the corollary.

**4. Minimal systems.** To apply the balancing technique in § 2, a system must be asymptotically stable. For minimal systems that are not necessarily asymptotically stable, Jonckheere and Silverman [16] have introduced a method that is not based on balancing solutions to Lyapunov equations but is instead based on balancing solutions to Riccati equations. Their definition of a Riccati-balanced system was extended in [27] to include a feed-through term.

DEFINITION 4.1. The class of systems $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{p\times n} \times \mathfrak{R}^{p\times m}$, which are minimal, is denoted by $L_n^{p,m}$. A system $(A, B, C, D) \in L_n^{p,m}$ is called *Riccati balanced* if there exists a positive diagonal matrix $\Sigma_l = \mathrm{diag}(q_1, \cdots, q_j, \cdots, q_n)$ such that

$$
(2)\qquad \begin{aligned}
&(A - BS_r^{-1}D^TC)^T\Sigma_l + \Sigma_l(A - BS_r^{-1}D^TC) - \Sigma_l BS_r^{-1}B^T\Sigma_l + C^TR_r^{-1}C = 0,\\
&(A - BS_r^{-1}D^TC)\Sigma_l + \Sigma_l(A - BS_r^{-1}D^TC)^T - \Sigma_l C^TR_r^{-1}C\Sigma_l + BS_r^{-1}B^T = 0,
\end{aligned}
$$

where $S_r = I + D^TD$ and $R_r = I + DD^T$. $\Sigma_l$ is called the *Riccati gramian* of the system.

The following canonical form for systems in $L_n^{p,m}$ is a modification of a canonical form derived in [27]. The canonical form presented here differs in two ways from the canonical form in [27]. The parametrization of the $B$-matrix has been changed analogously to the changes for the canonical form for asymptotically stable systems. Whereas in the present paper the $B$-matrix is chosen as a parameter of the system, in [27] the $C$-matrix serves as a parameter. As in the case of the present canonical form, such a change of parameter can be performed for all the canonical forms presented in this paper. The calculations involved are tedious but not difficult if it is noted that for $D \in \mathfrak{R}^{p\times m}$ we have that $D^T(I + DD^T)^{1/2} = (I + D^TD)^{1/2}D^T$.

THEOREM 4.1. *The following two statements are equivalent*:

(1) $G(s) \in TL_n^{p,m}$.

(2) *$G(s)$ has a realization $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{p\times n} \times \mathfrak{R}^{p\times m}$ given by the parameters*

$$
\begin{array}{ll}
q_1 > \cdots > q_j > \cdots > q_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
r_1, \cdots, r_j, \cdots, r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m, p); \\
U_1, \cdots, U_j, \cdots, U_k, & U_j \in \mathfrak{R}^{p\times r_j}, \quad U_j^TU_j = I_{r_j}; \\
\tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k, & \tilde{B}_j \in \mathfrak{R}^{r_j\times m} \textit{ positive upper triangular}; \\
\tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k, & \tilde{A}_j \in \mathfrak{R}^{n_j\times n_j} \textit{ in } r_j\textit{-balanced form}; \\
D, & D \in \mathfrak{R}^{p\times m};
\end{array}
$$

*in the following way*:

*If $(A, B, C, D)$ is partitioned according to $n_1, \cdots, n_j, \cdots, n_k$, then*

(i) $B_j = \begin{pmatrix} \tilde{B}_jS_r^{1/2} \\ 0 \end{pmatrix}$ *where* $S_r = I + D^TD$, $1 \leqq j \leqq k$;

(ii) $C_j = (R_r^{1/2}U_j\Delta_j, 0)$ *where* $R_r = I + DD^T$, $\Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2}$, $1 \leqq j \leqq k$;

(iii) $$A_{jj} = \tilde{A}_j - \frac{1 - q_j^2}{q_j}[\mathrm{diag}(\Delta_j^2, 0)]_l - \frac{1 - q_j^2}{2q_j}[\mathrm{diag}(\Delta_j^2, 0)]_d + \mathrm{diag}(\tilde{B}_jD^TU_j\Delta_j, 0), \qquad 1 \leqq j \leqq k;$$

(iv) $A_{ij} = \frac{1}{q_i^2 - q_j^2} \operatorname{diag}((q_j(1+q_i^2)\tilde{B}_i\tilde{B}_j^T - q_i(1+q_j^2)\Delta_i U_i^T U_j \Delta_j), 0)$

$+\operatorname{diag}(\tilde{B}_i D^T U_j \Delta_j, 0), \qquad 1 \leqq i, j \leqq k, \quad i \neq j;$

(v) $D \in \mathfrak{R}^{p \times m}$.

*Moreover,* $(A, B, C, D)$ *as defined in* (2), *is Riccati balanced with gramian*

$$\Sigma_l = \operatorname{diag}(q_1 I_{n_1}, \cdots, q_j I_{n_j}, \cdots, q_k I_{n_k}).$$

*The map* $\Gamma_l$, *which assigns to each system in* $L_n^{p,m}$ *the realization given in* (2), *is a canonical form.* □

This canonical form reduces as follows to scalar systems.

COROLLARY 4.1. *The following two statements are equivalent*:

(1) $g(s) \in TL_n^{1,1}$.

(2) $g(s)$ *has a realization* $(A, b, c, d) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times 1} \times \mathfrak{R}^{1 \times n} \times \mathfrak{R}^{1 \times 1}$ *given by the parameters*

$$\begin{array}{ll}
q_1 > \cdots > q_j > \cdots > q_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
s_1, \cdots, s_j, \cdots, s_k, & s_j = \pm 1, \quad 1 \leqq j \leqq k; \\
b_1,, \alpha(1)_1, \cdots, \alpha(1)_j, \cdots, \alpha(1)_{n_1-1}, & b_1 > 0, \quad \alpha(1)_j > 0, \quad 1 \leqq j \leqq n_1 - 1; \\
\vdots & \\
b_i, \alpha(i)_1, \cdots, \alpha(i)_j, \cdots, \alpha(i)_{n_i-1}, & b_i > 0, \quad \alpha(i)_j > 0, \quad 1 \leqq j \leqq n_i - 1; \\
\vdots & \\
b_k, \alpha(k)_1, \cdots, \alpha(k)_j, \cdots, \alpha(k)_{n_k-1}, & b_k > 0, \quad \alpha(k)_j > 0, \quad 1 \leqq j \leqq n_k - 1; \\
d, & d \in \mathfrak{R};
\end{array}$$

*in the following way*:

(i) $b = (\underbrace{b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{b_k, 0, \cdots, 0}_{n_k})^T$,

(ii) $c = (\underbrace{s_1 b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{s_j b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{s_k b_k, 0, \cdots, 0}_{n_k})$,

(iii) *For* $A =: (A_{ij})_{1 \leqq i, j \leqq k}$ *we have*

(a) *block diagonal entries* $A_{jj}, 1 \leqq j \leqq k$:

$$A_{jj} = \begin{pmatrix}
a_{jj} & \alpha(j)_1 & & & & \\
-\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\
& -\alpha(j)_2 & 0 & \cdot & & 0 \\
& & \ddots & \ddots & \ddots & \\
& 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\
& & & & -\alpha(j)_{n_j-1} & 0
\end{pmatrix}$$

*with* $a_{jj} = \frac{-b_j^2}{1+d^2}\left(\frac{1-q_j^2}{2q_j} - s_j d\right)$;

(b) *off-diagonal blocks* $A_{ij}$, $1 \leq i, j \leq k$, $i \neq j$:

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{with } a_{ij} = \frac{-b_i b_j}{1+d^2}\left(\frac{1 - s_i s_j q_i q_j}{s_i s_j q_i + q_j} - s_j d\right);$$

(iv) $d \in \mathfrak{R}$.

*Moreover,* $(A, b, c, d)$ *as defined in* (2) *is Riccati balanced with Riccati gramian*

$$\Sigma_l = \text{diag}\,(q_1 I_{n_1}, \cdots, q_j I_{n_j}, \cdots, q_k I_{n_k}).$$

*The map* $\Gamma_l$*, which assigns to each system in* $L_n^{1,1}$ *the realization given in* (2)*, is a canonical form.*

*Proof.* The corollary follows immediately from the theorem and a straightforward reparametrization of the entries of the $b$-vector. □

It is instructive to note that this canonical form for minimal systems and the corresponding parametrization has essentially the same structure as the canonical form for asymptotically stable minimal systems. In fact, the only difference is the way in which the parameters enter those entries of the $A$-matrix that are functions of the other parameters. In later sections we will find that the same applies to all classes of systems considered in this paper. It should be noted, however, that in general it is not clear how the parameters of an asymptotically stable system that is parametrized using the above canonical form are related to those parameters with which the system is parametrized in the canonical form for asymptotically stable systems.

The way the canonical form for minimal systems was derived in [27] was by relating a minimal system to the state-space realization of its normalized left coprime factorization. If $G(s)$ is the transfer function of a linear system, then the normalized left coprime factorization $[N(s), M(s)]$ of $G(s)$ is defined such that $G(s) = M(s)^{-1}N(s)$, where $N(s)$, $M(s)$ are asymptotically stable with $M(s)$ proper and satisfy $NN^* + MM^* = I$. Such coprime factorizations play an important role in modern control theory [31]. If a state-space realization of $G(s)$ is available, a state-space realization of $[N(s), M(s)]$ can be calculated by solving the Riccati equations corresponding to the state-space realization of $G(s)$. The canonical form for minimal systems can then be derived by calculating the canonical form for asymptotically stable systems of the asymptotically stable coprime factors. Exploiting the state-space formulae that relate a state-space realization of a system to the state-space realization of its coprime factors, it was possible to derive the canonical form and the parametrization result for minimal systems.

In the following sections we will use the same approach to the derivation of canonical forms for other classes of systems. Rather than deriving a canonical form separately for each class of systems, explicit maps will be constructed to relate the state-space realizations of the class of systems to an appropriate subclass of the set of asymptotically stable systems. Thereby the canonical form for asymptotically stable systems can be exploited to derive a canonical form for other classes of systems.

**5. Bounded real systems.** An important subclass of asymptotically stable systems is that with transfer functions bounded by one on the imaginary axis. This class of so-called bounded real systems is used to parametrize all stabilizing controllers of a plant such that the closed-loop system satisfies an $H^\infty$ constraint (see, e.g., [11]). In fact, transfer functions of bounded real systems are the real rational functions in the open unit ball of $H^\infty$.

DEFINITION 5.1. Let $(A, B, C, D) \in C_n^{p,m}$ be such that $I - D^T D > 0$; then $(A, B, C, D)$ is called *bounded real* if for $G(s) = C(sI - A)^{-1}B + D$ we have

$$I - G(-iw)^T G(iw) > 0, \qquad w \in \Re.$$

We denote by $B_n^{p,m}$ the subset of $C_n^{p,m}$ containing all bounded real systems. $TB_n^{p,m}$ denotes the set of transfer functions of systems in $B_n^{p,m}$.

*Remark* 5.1. Other authors call an asymptotically stable system bounded real if

$$I - G(-iw)^T G(iw) \geqq 0, \qquad w \in \Re.$$

In this section we will derive a canonical form for bounded real systems. To do this we first must define what we mean by balancing of bounded real systems. With the classes of systems that we considered in the previous sections, we associated certain Lyapunov and Riccati equations. A system was called balanced if the solutions to the corresponding equations were balanced, i.e., equal and diagonal. Bounded real systems can be shown to satisfy the so-called bounded real Riccati equation. We proceed analogously to define a balanced realization for bounded real systems. The following proposition states several well-known results relating bounded real systems to this Riccati equation (see, e.g., [5], [32]).

PROPOSITION 5.1. *Let $(A, B, C, D) \in C_n^{p,m}$ such that $S := I - D^T D > 0$ and $G(s) = C(sI - A)^{-1}B + D$; then*

(1) $I - G(-iw)^T G(iw) \geqq 0$, $w \in \Re$, *if and only if there exists $P = P^T > 0$ such that*

$$A^T P + PA + C^T C + (PB + C^T D) S^{-1} (PB + C^T D)^T = 0. \tag{3}$$

*We call this Riccati equation the bounded real Riccati equation* (BRRE).

(2) *If either of the conditions in* (1) *is satisfied, then $P = P^T > 0$ for any solution to* (3). *There exist solutions $P_{\min}$ and $P_{\max}$ to* (3) *such that for any solution $P = P^T$ we have*

$$0 < P_{\min} \leqq P \leqq P_{\max}.$$

(3) *If $(A, B, C, D)$ is bounded real, i.e., $I - G(-iw)^T G(iw) > 0$, $w \in \Re$, then*

$$0 < P_{\min} < P_{\max}$$

*and $P_{\min}$ is the unique solution to* (3) *such that $A + BS^{-1}(B^T P + D^T C)$ is asymptotically stable.*

(4) $(A, B, C, D)$ *is bounded real if there exists a solution $P = P^T > 0$ to* (3) *such that $A + BS^{-1}(B^T P + D^T C)$ is asymptotically stable. Moreover, $P_{\min}$ is the unique such solution.*

(5) *A system $(A, B, C, D)$ is bounded real if and only if its dual system $(A^T, C^T, B^T, D^T)$ is bounded real. If $(A, B, C, D)$ is bounded real with $P_{\min}$ and $P_{\max}$ the minimal, respectively, maximal solution to* (3), *then $P_{\min}^{-1}$ is the maximal and $P_{\max}^{-1}$ the minimal solution to the BRRE corresponding to $(A^T, C^T, B^T, D^T)$.*

A problem with the bounded real Riccati equation is that there is no unique positive-definite solution. Opdenacker and Jonckheere [28] introduced a balancing technique for bounded real systems by balancing the minimal solution with the inverse of the maximal solution to the BRRE. We will follow their definition since it is possible to derive the desired canonical form and realization result using this particular choice of solution. Note that balancing the minimal solution of the BRRE with the inverse of its maximal solution is the same as balancing the minimal solution with the minimal solution of the dual equation.

DEFINITION 5.2. A system $(A, B, C, D) \in B_n^{p,m}$ is called *bounded real balanced* if

$$P_{\min} = P_{\max}^{-1} = \operatorname{diag}(p_1, \cdots, p_j, \cdots, p_n) =: \Sigma_b,$$

where $P_{\min}$, $P_{\max}$ are the minimal, respectively, maximal solution to the BRRE. $\Sigma_b$ is called the *bounded real gramian* of $(A, B, C, D)$.

*Remark* 5.2. Note that since $P_{\max} > P_{\min}$ for systems in $B_n^{p,m}$ we have that $0 < \Sigma_b < I$.

Before we can derive a canonical form for bounded real systems we need several lemmas. The first of these states standard identities that we will frequently need.

LEMMA 5.1. *If $D \in \Re^{p \times m}$ is such that $S := I - D^T D > 0$ and $R := I - DD^T > 0$, then $S$ and $R$ have the following properties*:

(i) $D^T R^{-1} = S^{-1} D^T$,

(ii) $D^T R^{1/2} = S^{1/2} D^T$.

The canonical form for bounded real systems will be derived by mapping bounded real systems to a certain class of asymptotically stable systems. The significance of this map is that it maps bounded real balanced systems to asymptotically stable systems that are balanced with respect to the corresponding Lyapunov equations. A canonical form for bounded real systems can therefore be derived by bringing the associated asymptotically stable systems to the canonical form of Theorem 2.1. Reversing the map, we can obtain the desired canonical form for bounded real systems.

The following lemma establishes the map between bounded real systems and a subclass of asymptotically stable systems.

LEMMA 5.2. *Let $(A, B, C, D) \in B_n^{p,m}$. If $P_{\min}$ is the minimal and $P_{\max}$ the maximal solution to the* BRRE, *then with $S = I - D^T D$ and $R = I - DD^T$ we have that*

$$(A_c, B_c, C_c, D_c) := \left( A + BS^{-1} D^T C, [BS^{-1/2}, P_{\max}^{-1} C^T R^{-1/2}], \begin{bmatrix} R^{-1/2} C \\ S^{-1/2} B^T P_{\min} \end{bmatrix}, D \right) \in C_n^{p+m, p+m}$$

*and*

$$A_c P_{\max}^{-1} + P_{\max}^{-1} A_c^T = -B_c B_c^T, \tag{4}$$

$$A_c^T P_{\min} + P_{\min} A_c = -C_c^T C_c. \tag{5}$$

*Proof.* First note that $\tilde{P} = \tilde{P}^T > 0$ solves the bounded real Riccati equation

$$A^T P + PA + C^T C + (PB + C^T D) S^{-1} (PB + C^T D)^T = 0$$

if and only if $\tilde{P}^{-1}$ solves the dual equation

$$AP + PA^T + BB^T + (PC^T + BD^T) R^{-1} (PC^T + BD^T)^T = 0.$$

It is now easy to verify that these two Riccati equations can be rewritten as the Lyapunov equations (4), (5) with $P_{\max}^{-1}$ and $P_{\min}$ as their solutions. Since $P_{\max}^{-1} > 0$ and $P_{\min} > 0$ we have that $(A_c, B_c, C_c, D_c)$ is minimal if and only if $A_c$ is asymptotically stable. Since $(A_c, B_c, D_c, C_c)$ satisfies the Lyapunov equations (4), (5) we know that the eigenvalues of $A_c$ are in the closed left halfplane. To show that they are in fact in the open left halfplane, assume there exists $w \in \Re$ and $x \in \mathscr{C}^n$ such that

$$A_c x = iwx.$$

By applying $x$ to the right and $x^*$ to the left of (5), i.e.,

$$x^* A_c^T P_{\min} x + x^* P_{\min} A_c x = 0 = -x^* C_c^T C_c x,$$

we obtain that $Cx=0$ and thus

$$A_c x = (A+BS^{-1}D^TC)x = Ax = iwx.$$

Hence by the PBH test we have that $(A, B, C, D)$ is not observable, which is a contradiction to the assumption. This implies the asymptotic stability of $A_c$ and the minimality of $(A_c, B_c, C_c, D_c)$. □

In the following lemma the inverse of the map of the previous lemma will be investigated.

LEMMA 5.3. *Let* $(A, B, C, D) \in \Re^{n\times n}\times\Re^{n\times m}\times\Re^{p\times n}\times\Re^{p\times m}$ *be such that* $S := I - D^TD > 0$, $R := I - DD^T$, *and* $P = P^T > 0$, *such that* $P^{-1} > P$. *If*

$$(A_c, B_c, C_c, D_c) := \left(A+BS^{-1}D^TC, [BS^{-1/2}, PC^TR^{-1/2}], \begin{bmatrix} R^{-1/2}C \\ S^{-1/2}B^TP \end{bmatrix}, D\right) \in C_n^{p+m,p+m}$$

*with*

$$A_cP + PA_c^T = -B_cB_c^T, \tag{6}$$

$$A_c^TP + PA_c = -C_c^TC_c, \tag{7}$$

*then* $(A, B, C, D) \in C_n^{p,m}$. *Under the same conditions the eigenvalues of* $A + BS^{-1}(B^TP + D^TC)$ *are in the open left halfplane.*

*Proof.* First note that the Lyapunov equation (6) can be rewritten as

$$\begin{aligned} 0 &= (A+BS^{-1}D^TC+BS^{-1}B^TP)P + P(A+BS^{-1}D^TC+BS^{-1}B^TP)^T \\ &\quad +(I-P^2)BS^{-1}B^T(I-P^2) - P^2BS^{-1}B^TP^2 + PC^TR^{-1}CP, \end{aligned} \tag{8}$$

which is equivalent to

$$\begin{aligned} 0 &= (A+BS^{-1}D^TC+BS^{-1}B^TP)^TP^{-1} + P^{-1}(A+BS^{-1}D^TC+BS^{-1}B^TP) \\ &\quad +(P^{-1}-P)BS^{-1}B^T(P^{-1}-P) - PBS^{-1}B^TP + C^TR^{-1}C. \end{aligned} \tag{9}$$

Similarly, (7) can be rewritten as

$$\begin{aligned} 0 &= (A+BS^{-1}D^TC+BS^{-1}B^TP)^TP + P(A+BS^{-1}D^TC+BS^{-1}B^TP) \\ &\quad -PBS^{-1}B^TP + C^TR^{-1}. \end{aligned} \tag{10}$$

Subtracting (10) from (9) we obtain

$$\begin{aligned} 0 &= (A+BS^{-1}D^TC+BS^{-1}B^TP)^T(P^{-1}-P) \\ &\quad +(P^{-1}-P)(A+BS^{-1}D^TC+BS^{-1}B^TP) + (P^{-1}-P)BS^{-1}B^T(P^{-1}-P). \end{aligned} \tag{11}$$

Since $P^{-1}-P>0$ we can pre- and post-multiply this equation by $(P^{-1}-P)^{-1}$, and hence

$$\begin{aligned} 0 &= (A+BS^{-1}D^TC+BS^{-1}B^TP)(P^{-1}-P)^{-1} \\ &\quad +(P^{-1}-P)^{-1}(A+BS^{-1}D^TC+BS^{-1}B^TP)^T + BS^{-1}B^T. \end{aligned} \tag{12}$$

Since $(P^{-1}-P)^{-1}>0$ this Lyapunov equation implies that $A+BS^{-1}D^TC+BS^{-1}B^TP$ is asymptotically stable if and only if $(A+BS^{-1}D^TC+BS^{-1}B^TP, BS^{-1/2})$ is controllable. If we assume that this is not the case, then there exist $\lambda \in \mathscr{C}$ and $x \in \mathscr{C}^n$ such that

$$x^*(A+BS^{-1}D^TC+BS^{-1}B^TP) = \lambda x^*, \qquad x^*BS^{-1/2} = 0.$$

Pre- and post-multiplying (12) by $x^*$ and $x$, respectively, we obtain that $\mathrm{Re}(\lambda) \leqq 0$. Now assume that $\lambda = iw$, $w \in \mathfrak{R}$; then,

$$x^*(A + BS^{-1}D^TC) = iwx^*,$$

which is a contradiction to the asymptotic stability of $A + BS^{-1}D^TC$. Hence we have the controllability of

$$(A + BS^{-1}D^TC + BS^{-1}B^TP, BS^{-1/2})$$

and the asymptotic stability of $A + BS^{-1}D^TC + BS^{-1}B^TP$. An application of the PBH test now implies the controllability of $(A, B)$.

We can show similarly that

$$(A + BS^{-1}D^TC + PC^TR^{-1}C, R^{-1/2}C)$$

is observable and hence that $(A, C)$ is observable. Having shown the minimality of $(A, B, C, D)$ it remains to show that $A$ is asymptotically stable.

Note that the Lyapunov equation (7) can be rewritten as

$$A^TP + PA + C^TC + (B^TP + D^TC)^TS^{-1}(B^TP + D^TC) = 0.$$

Thus $A$ is asymptotically stable if and only if

$$\left(A, \begin{bmatrix} C \\ S^{-1/2}(B^TP + D^TC) \end{bmatrix}\right)$$

is observable. But if this system is not observable we have that $(A, C)$ is not observable, which is a contradiction to the minimality of $(A, B, C, D)$.

We are now in a position to prove the main theorem of this section, which establishes a canonical form for bounded real systems. Note that the main problem in the proof of the theorem will be to show the "parametrization part" of the result, i.e., that a system with a particular structure is indeed bounded real. Due to the particular parametrization of the systems, a solution of the bounded real Riccati equation can be written immediately. To show that the system is bounded real, it is, however, necessary to show that this solution is the minimal solution. In the previous lemma we have already gone some way toward showing this.

THEOREM 5.1. *The following two statements are equivalent*:

(1) $B(s) \in TB_n^{p,m}$.

(2) $B(s)$ *has a realization* $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{p\times n} \times \mathfrak{R}^{p\times m}$ *given by the parameters*:

$$\begin{array}{ll} 1 > p_1 > \cdots > p_j > \cdots > p_k > 0, & \\ n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\ r_1, \cdots, r_j, \cdots, r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m, p); \\ U_1, \cdots, U_j, \cdots, U_k, & U_j \in \mathfrak{R}^{p\times r_j}, \quad U_j^TU_j = I_{r_j}, \\ \tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k, & \tilde{B}_j \in \mathfrak{R}^{r_j\times m} \textit{ positive upper triangular}; \\ \tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k, & \tilde{A}_j \in \mathfrak{R}^{n_j\times n_j} \textit{ in } r_j\textit{-balanced form}; \\ D, & D \in \mathfrak{R}^{p\times m}, \quad I - D^TD > 0; \end{array}$$

*in the following way*:
*If* $(A, B, C, D)$ *is partitioned according to* $n_1, \cdots, n_j, \cdots, n_k$, *then*

(i) $B_j = \begin{pmatrix} \tilde{B}_jS^{1/2} \\ 0 \end{pmatrix}$ *where* $S = I - D^TD$, $1 \leqq j \leqq k$;

(ii) $C_j = (R^{1/2}U_j\Delta_j, 0)$ *where* $R = I - DD^T$, $\Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2}$, $1 \leqq j \leqq k$;

(iii) $$A_{jj} = \tilde{A}_j - \frac{1+p_j^2}{p_j}[\operatorname{diag}(\Delta_j^2, 0)]_l - \frac{1+p_j^2}{2p_j}[\operatorname{diag}(\Delta_j^2, 0)]_d - \operatorname{diag}(\tilde{B}_j D^T U_j \Delta_j, 0), \qquad 1 \leqq j \leqq k;$$

(iv) $$A_{ij} = \frac{1}{p_i^2 - p_j^2}(p_j(1-p_i^2)\operatorname{diag}(\tilde{B}_i\tilde{B}_j^T, 0) - p_i(1-p_j^2)\operatorname{diag}(\Delta_i U_i^T U_j \Delta_j, 0)) - \operatorname{diag}(B_i D^T U_j \Delta_j, 0), \qquad 1 \leqq i, j \leqq k, \quad i \neq j.$$

(v) $D \in \Re^{p\times m}$, $\quad I - D^T D > 0$.

*Moreover,* $(A, B, C, D)$ *as defined in* (2) *is bounded real balanced with bounded real gramian*

$$\Sigma_b = \operatorname{diag}(p_1 I_{n_1}, \cdots, p_j I_{n_j}, \cdots, p_k I_{n_k}).$$

*The map* $\Gamma_b$, *which assigns to each system in* $B_n^{p,m}$ *the realization given in* (2), *is a canonical form.*

*Proof.* (1) ⇒ (2) Let $(A, B, C, D) \in B_n^{p,m}$ and let $(A_c, B_c, C_c, D_c) \in C_n^{p+m,p+m}$ be the uniquely defined system in Lemma 5.2. If $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is another system in $B_n^{p,m}$ and $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c) \in C_n^{p+m,p+m}$ is given as in Lemma 5.2, then $(A, B, C, D)$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are system equivalent if and only if $(A_c, B_c, C_c, D_c)$ and $(\tilde{A}_c, \tilde{B}_c, \tilde{C}_c, \tilde{D}_c)$ are system equivalent. Since $(A_c, B_c, C_c, D_c)$ satisfies the Lyapunov equations (4) and (5) this shows that $(A, B, C, D)$ is bounded real balanced with bounded real gramian $\Sigma_b$ if and only if $(A_c, B_c, C_c, D_c)$ is balanced with gramian $\Sigma = \Sigma_b = \operatorname{diag}(p_1 I_{n_1}, \cdots, p_j I_{n_j}, \cdots, p_k I_{n_k}) < 1$. We can therefore assume that $(A_c, B_c, C_c, D_c)$ is in the canonical form of Theorem 2.1.

Partitioning the systems in the usual way and using the notation of Theorem 2.1, we obtain

(13) $$B_{c,j} = (B_j S^{-1/2}, p_j C_j^T R^{-1/2}) = \begin{pmatrix} \tilde{B}_{c,j} \\ 0 \end{pmatrix},$$

with $\tilde{B}_{c,j} \in \Re^{r_j \times (p+m)}$ positive upper triangular and

(14) $$C_{c,j} = \begin{pmatrix} R^{-1/2}C_j \\ p_j S^{-1/2} B_j^T \end{pmatrix} = (U_{c,j}\Delta_{c,j}, 0),$$

where $U_{c,j} \in \Re^{(p+m)\times r_j}$, $U_{c,j}^T U_{c,j} = I_{r_j}$, and $\Delta_{c,j} = (\tilde{B}_{c,j}\tilde{B}_{c,j}^T)^{1/2}$. Since $(A_c, B_c, C_c, D_c)$ is balanced, $B_{c,j}B_{c,j}^T = C_{c,j}^T C_{c,j}$ and therefore we have that

$$B_{c,j}B_{c,j}^T = B_j S^{-1} B_j^T + p_j^2 C_j^T R^{-1} C_j = C_j^T R^{-1} C_j + p_j^2 B_j S^{-1} B_j^T$$

and hence $B_j S^{-1} B_j^T = C_j^T R^{-1} C_j$, which implies that

(15) $$B_{c,j}B_{c,j}^T = (1+p_j^2) B_j S^{-1} B_j^T.$$

Therefore $r_j = \operatorname{rank}(B_{c,j}) = \operatorname{rank}(\tilde{B}_{c,j}) = \operatorname{rank}(B_j S^{-1/2})$, which implies together with (13) that

$$B_j S^{-1/2} = \begin{pmatrix} \tilde{B}_j \\ 0 \end{pmatrix}$$

with $\tilde{B}_j$ positive upper triangular. This shows (i).

The fact that

$$\operatorname{diag}(\tilde{B}_j\tilde{B}_j^T, 0) = B_jS^{-1}B_j^T = C_j^TR^{-1}C_j$$

immediately implies

$$R^{-1/2}C_j = (U_j\Delta_j, 0), \qquad \Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2},$$

for a unique $U_j \in \Re^{p\times r_j}$ such that $U_j^TU_j = I_{r_j}$ and hence (ii).

Since by Lemma 5.2

$$A_{c,jj} = \tilde{A}_{c,j} - \frac{1}{p_j}[B_{c,j}B_{c,j}^T]_l - \frac{1}{2p_j}[B_{c,j}B_{c,j}^T]_d = A_{jj} + B_jS^{-1}D^TC_j,$$

we have using (13), (14), (15)(i), and (15)(ii), and Lemma 5.1,

$$A_{jj} = \tilde{A}_{c,j} - \frac{1+p_j^2}{p_j}[\operatorname{diag}(\Delta_j^2, 0)]_l - \frac{1+p_j^2}{2p_j}[\operatorname{diag}(\Delta_j^2, 0)]_d - \operatorname{diag}(\tilde{B}_jD^TU_j\Delta_j, 0).$$

Hence we obtain (iii) by setting $\tilde{A}_j := \tilde{A}_{c,j}$.

The parametrization of $A_{ij}$ in (iv) follows immediately from the parametrization of $A_{c,ij}$ in Theorem 2.1 as well as from the expressions for $A_{c,ij}$, $B_{c,j}$, and $C_{c,j}$ in Lemma 5.2.

(2) ⇒ (1) Let $(A, B, C, D)$ be parametrized as in (2). To show $(A, B, C, D) \in C_n^{p,m}$ we construct

$$(A_c, B_c, C_c, D_c) := \left(A + BS^{-1}D^TC, [BS^{-1/2}, PC^TR^{-1/2}], \begin{bmatrix} R^{-1/2}C \\ S^{-1/2}B^TP \end{bmatrix}, D\right),$$

where $P = \operatorname{diag}(p_1I_{n_1}, \cdots, p_jI_{n_j}, \cdots, p_kI_{n_k})$, $S = I - D^TD$, and $R = I - DD^T$.

We must show that $(A_c, B_c, C_c, D_c)$ is in balanced canonical form of Theorem 2.1. To do this we partition the systems in the standard way. Then

$$B_{c,j} = \begin{pmatrix} \tilde{B}_j & p_j\Delta_jU_j^T \\ 0 & 0 \end{pmatrix},$$

which is positive upper triangular. For $C_{c,j}$ we have

$$C_{c,j} = \begin{pmatrix} U_j\Delta_j & 0 \\ p_j\tilde{B}_j^T & 0 \end{pmatrix}.$$

Setting

$$\Delta_{c,j}^2 := \tilde{B}_{c,j}\tilde{B}_{c,j}^T = \operatorname{diag}((1+p_j^2)\Delta_j^2, 0)$$

and

$$U_{c,j} := \frac{1}{\sqrt{1+p_j^2}}\begin{pmatrix} U_j & 0 \\ p_j\tilde{B}_j^T\Delta_j^{-1} & 0 \end{pmatrix},$$

we have that

$$C_{c,j} = (U_{c,j}\Delta_{c,j}, 0)$$

with $U_{c,j}^TU_{c,j} = Ir_j$.

That $A_{c,jj}$ and $A_{c,ij}$ are of the required form follows similarly to the derivations in the first part of the proof. Thus $(A_c, B_c, C_c, D_c)$ is parametrized in the balanced canonical form of Theorem 2.1 with gramian $P$ and hence it is in $C_n^{p+m,p+m}$. Since $P^{-1} > P$ Lemma 5.3 implies that $(A, B, C, D) \in C_n^{p,m}$.

To show that $(A, B, C, D)$ is bounded real balanced we must show that $P$ is the minimal and that $P^{-1}$ is the maximal solution to the BRRE. That $(A, B, C, D)$ is bounded real and $P$ is the minimal solution to the BRRE follows since, by construction, $P$ solves the BRRE corresponding to $(A, B, C, D)$ and because of Lemma 5.3 and Proposition 5.1. Repeating the above arguments for the dual system shows that $P$ is also the minimal solution to the dual bounded real Riccati equation. Proposition 5.1 then implies that $P^{-1}$ is the maximal solution to the BRRE. □

Specializing this theorem to the SISO case we obtain the following corollary.

COROLLARY 5.1. *The following two statements are equivalent*:

(1) $b(s) \in TB_n^{1,1}$.

(2) *$b(s)$ has a realization* $(A, b, c, d) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times 1} \times \mathfrak{R}^{1\times n} \times \mathfrak{R}^{1\times 1}$ *given by the parameters*:

$$
\begin{array}{ll}
1 > p_1 > \cdots > p_j > \cdots > p_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
s_1, \cdots, s_j, \cdots, s_k, & s_j = \pm 1, \quad 1 \leqq j \leqq k; \\
b_1, \alpha(1)_1, \cdots, \alpha(1)_j, \cdots, \alpha(1)_{n_1-1}, & b_1 > 0, \quad \alpha(1)_j > 0, \quad 1 \leqq j \leqq n_1 - 1; \\
\vdots & \\
b_i, \alpha(i)_1, \cdots, \alpha(i)_j, \cdots, \alpha(i)_{n_i-1}, & b_i > 0, \quad \alpha(i)_j > 0, \quad 1 \leqq j \leqq n_i - 1; \\
\vdots & \\
b_k, \alpha(k)_1, \cdots, \alpha(k)_j, \cdots, \alpha(k)_{n_k-1}, & b_k > 0, \quad \alpha(k)_j > 0, \quad 1 \leqq j \leqq n_k - 1; \\
d, & d \in \mathfrak{R}, \quad |d| < 1;
\end{array}
$$

*in the following way*:

(i) $$b = (\underbrace{b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{b_k, 0, \cdots, 0}_{n_k})^T,$$

(ii) $$c = (\underbrace{s_1 b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{s_j b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{s_k b_k, 0, \cdots, 0}_{n_k}),$$

(iii) *For* $A =: (A_{ij})_{1 \leqq i,j \leqq k}$ *we have*

(a) *block diagonal entries* $A_{jj}$, $1 \leqq j \leqq k$:

$$
A_{jj} = \begin{pmatrix}
a_{jj} & \alpha(j)_1 & & & & \\
-\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\
& -\alpha(j)_2 & 0 & \cdot & & 0 \\
& & \ddots & \ddots & \ddots & \\
& 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\
& & & & -\alpha(j)_{n_j-1} & 0
\end{pmatrix}
$$

*with* $a_{jj} = \dfrac{-b_j^2}{1-d^2}\left(\dfrac{1+p_j^2}{2p_j} + s_j d\right)$;

(b) *off-diagonal blocks* $A_{ij}$, $1 \leqq i, j \leqq k$, $i \neq j$:

$$
A_{ij} = \begin{pmatrix}
a_{ij} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix} \quad \text{with } a_{ij} = \frac{-b_i b_j}{1-d^2}\left(\frac{1+s_i s_j p_i p_j}{s_i s_j p_i + p_j} + s_j d\right);
$$

(iv) $d \in \mathfrak{R}$, $|d| < 1$.

*Moreover, $(A, b, c, d)$ as defined in (2) is bounded real balanced with bounded real gramian*

$$\Sigma_b = \text{diag}\,(p_1 I_{n_1}, \cdots, p_j I_{n_j}, \cdots, p_k I_{n_k}).$$

*The map $\Gamma_b$, which assigns to each system in $B_n^{1,1}$ the realization given in (2), is a canonical form.*

*Proof.* The corollary follows immediately from the theorem and a straightforward reparametrization of the entries of the $b$-vector. □

If we analyze the canonical form we have just derived, we can see that it again has the same structure as the canonical forms of the previous sections. The only difference between this canonical form and the previous ones is in the way the parameters enter the $A$-matrix and that the parameters $p_1, \cdots, p_k$ are bounded by one.

**6. Positive real systems.** Positive real systems play an important role in many parts of deterministic and stochastic control and system theory. Balanced realizations of positive real systems were introduced in [7] because of their importance in stochastic systems theory (see, e.g., [8], [6]). In this section we will study the question of the parametrization of positive real systems. The approach we take is analogous to the one we have taken in previous sections. We associate a certain type of Riccati equation with positive real systems, the so-called positive real Riccati equations. Balanced realizations for positive real systems are then defined by balancing the minimal solutions of these equations. We could now derive a canonical form for positive real systems in an analogous way to the way the canonical form was derived for asymptotically stable systems in Theorem 2.1. Instead, we use a Moebius transformation to map positive real systems to bounded real systems. In this way we can easily carry to canonical form for bounded real systems over to provide a canonical form and parametrization for positive real systems.

DEFINITION 6.1. A system $(A, B, C, D) \in C_n^{m,m}$ such that $D + D^T > 0$ is called *positive real* if

$$G(iw) + G(-iw)^T > 0, \qquad w \in \Re.$$

We denote by $P_n^m$ the subset of $C_n^{m,m}$ containing all positive real systems. $TP_n^m$ denotes the set of transfer functions of systems in $P_n^m$.

The derivation of a canonical form for systems in $P_n^m$ will be based on the relationship between positive real and bounded real systems obtained by applying a Moebius transformation to the set $TB_n^{m,m}$ which maps $TB_n^{m,m}$ to $TP_n^m$ (see, e.g., [5]):

$$\begin{aligned} M:\quad & TB_n^{m,m} \to TP_n^m \\ & B(s) \mapsto P(s) := (I - B(s))^{-1}(I + B(s)). \end{aligned}$$

$M$ is a bijection with inverse

$$\begin{aligned} M^{-1}:\quad & TP_n^m \to TB_n^{m,m} \\ & P(s) \mapsto B(s) := (P(s) - I)(P(s) + I)^{-1}. \end{aligned}$$

The corresponding state-space formulae are given in the following proposition.

PROPOSITION 6.1. *The map*

$$\begin{aligned} S_{BP}:\quad & B_n^{m,m} \to P_n^m \\ & (A, B, C, D) \mapsto (A + B(I - D)^{-1}C, \sqrt{2}\, B(I - D)^{-1}, \\ & \qquad \sqrt{2}(I - D)^{-1}C, (I - D)^{-1}(I + D)) \end{aligned}$$

*is a bijection with inverse*

$$S_{BP}^{-1}:\quad P_n^m \to B_n^{m,m}$$
$$(A, B, C, D) \mapsto (A - B(I+D)^{-1}C, \sqrt{2}\, B(I+D)^{-1}, \sqrt{2}\,(I+D)^{-1}C, (D-I)(D+I)^{-1}),$$

such that

- $S_{BP}$ *preserves system equivalence.*
- $P = P^T > 0$ *is a solution to the* BRRE *for* $(A, B, C, D) \in B_n^{m,m}$ *if and only if* $P = P^T > 0$ *is a solution to the positive real Riccati equation* (PRRE)

$$\tilde{A}^T P + P\tilde{A} + (\tilde{C} - \tilde{B}^T P)^T (\tilde{D} + \tilde{D}^T)^{-1} (\tilde{C} - \tilde{B}^T P) = 0$$

*for* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := S_{BP}((A, B, C, D))$.

This proposition implies that we can define a balanced positive real system analogously to the bounded real case by balancing the minimal solution to the PRRE with the inverse of its maximal solution. Note that this amounts to balancing the minimal solution of the PRRE with the minimal solution of its dual equation.

DEFINITION 6.2. A system $(A, B, C, D) \in P_n^m$ is called *positive real balanced* if

$$P_{\min} = P_{\max}^{-1} = \operatorname{diag}(p_1, \cdots, p_j, \cdots, p_n) =: \Sigma_p,$$

where $P_{\min}$, $P_{\max}$ is the minimal, respectively, maximal solution to the PRRE. $\Sigma_p$ is called the *positive real gramian* of $(A, B, C, D)$.

As an immediate consequence of the previous proposition, we have that balancing is preserved by the map $S_{BP}$.

COROLLARY 6.1. *A system* $(A, B, C, D) \in B_n^{m,m}$ *is bounded real balanced with bounded real gramian* $\Sigma_b$ *if and only if* $S_{BP}((A, B, C, D))$ *is positive real balanced with positive real gramian* $\Sigma_p = \Sigma_b$.

Proposition 6.1 together with the canonical form for bounded real systems in Theorem 5.1 allows us to immediately derive a canonical form for systems in $P_n^m$.

THEOREM 6.1. *The following two statements are equivalent*:

(1) $P(s) \in TP_n^m$.

(2) $P(s)$ *has a realization* $(A, B, C, D) \in \Re^{n\times n} \times \Re^{n\times m} \times \Re^{m\times n} \times \Re^{m\times m}$ *given by the following parameters*:

| | |
|---|---|
| $1 > p_1 > \cdots > p_j > \cdots > p_k > 0,$ | |
| $n_1, \cdots, n_j, \cdots, n_k,$ | $n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n;$ |
| $r_1, \cdots, r_j, \cdots, r_k,$ | $r_j \in \mathcal{N}, \quad 1 \leq r_j \leq \min(n_j, m);$ |
| $U_1, \cdots, U_j, \cdots, U_k,$ | $U_j \in \Re^{m\times r_j}, U_j^T U_j = I_{r_j};$ |
| $\tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k,$ | $\tilde{B}_j \in \Re^{r_j\times m}$ *positive upper triangular*; |
| $\tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k,$ | $\tilde{A}_j \in \Re^{n_j\times n_j}$ *in* $r_j$*-balanced form*; |
| $D_b,$ | $D_b \in \Re^{m\times m}, \quad I - D_b^T D_b > 0$ |

*in the following way*:

*If* $(A, B, C, D)$ *is partitioned as* $n_1, \cdots, n_j, \cdots, n_k$, *then*,

(i) $B_j = \begin{pmatrix} \sqrt{2}\, \tilde{B}_j S^{1/2} (I - D_b)^{-1} \\ 0 \end{pmatrix}$ *where* $S = I - D_b^T D_b, \quad 1 \leq j \leq k;$

(ii) $C_j = (\sqrt{2}(I - D_b)^{-1} R^{1/2} U_j \Delta_j, 0)$ *where* $R = I - D_b D_b^T, \Delta_j = (\tilde{B}_j \tilde{B}_j^T)^{1/2}, \quad 1 \leq j \leq k;$

$$\text{(iii)}\quad A_{jj}=\tilde{A}_j-\frac{1+p_j^2}{p_j}[\operatorname{diag}(\Delta_j^2,0)]_l-\frac{1+p_j^2}{2p_j}[\operatorname{diag}(\Delta_j^2,0)]_d$$
$$+\operatorname{diag}(\tilde{B}_jS^{-1/2}(I-D_b^T)(I-D_b)^{-1}R^{1/2}U_j\Delta_j,0),\qquad 1\leqq j\leqq k;$$

$$\text{(iv)}\quad A_{ij}=\frac{1}{p_i^2-p_j^2}(p_j(1-p_i^2)\operatorname{diag}(\tilde{B}_i\tilde{B}_j^T,0)-p_i(1-p_j^2)\operatorname{diag}(\Delta_iU_i^TU_j\Delta_j,0))$$
$$+\operatorname{diag}(\tilde{B}_iS^{-1/2}(I-D_b^T)(I-D_b)^{-1}R^{1/2}U_j\Delta_j,0),$$
$$1\leqq i,j\leqq k,\quad i\neq j.$$

$$\text{(v)}\quad D=(I-D_b)^{-1}(I+D_b).$$

*Moreover, $(A, B, C, D)$ as defined (2) is positive real balanced with positive real gramian*

$$\Sigma_p=\operatorname{diag}(p_1I_{n_1},\cdots,p_jI_{n_j},\cdots,p_kI_{n_k}).$$

*The map $\Gamma_p$, which assigns to each system in $P_n^m$ the realization given in (2), is a canonical form.*

COROLLARY 6.2. *The following two statements are equivalent:*

(1) $p(s)\in TP_n^1$.

(2) *$p(s)$ has a realization $(A,b,c,d)\in\mathfrak{R}^{n\times n}\times\mathfrak{R}^{n\times 1}\times\mathfrak{R}^{1\times n}\times\mathfrak{R}^{1\times 1}$ given by the parameters:*

$$\begin{array}{ll}
1>p_1>\cdots>p_j>\cdots>p_k>0 & \\
n_1,\cdots,n_j,\cdots,n_k, & n_j\in\mathcal{N},\quad \sum_{j=1}^k n_j=n;\\
s_1,\cdots,s_j,\cdots,s_k, & s_j=\pm1,\quad 1\leqq j\leqq k;\\
b_1,\alpha(1)_1,\cdots,\alpha(1)_j,\cdots,\alpha(1)_{n_1-1}, & b_1>0,\quad \alpha(1)_j>0,\quad 1\leqq j\leqq n_1-1;\\
\vdots & \\
b_i,\alpha(i)_1,\cdots,\alpha(i)_j,\cdots,\alpha(i)_{n_i-1}, & b_i>0,\quad \alpha(i)_j>0,\quad 1\leqq j\leqq n_i-1;\\
\vdots & \\
b_k,\alpha(k)_1,\cdots,\alpha(k)_j,\cdots,\alpha(k)_{n_k-1}, & b_k>0,\quad \alpha(k)_j>0,\quad 1\leqq j\leqq n_k-1;\\
d, & d\in\mathfrak{R},\quad d>0;
\end{array}$$

*in the following way:*

$$\text{(i)}\quad b=(\underbrace{b_1,0,\cdots,0}_{n_1},\cdots,\underbrace{b_j,0,\cdots,0}_{n_j},\cdots,\underbrace{b_k,0,\cdots,0}_{n_k})^T,$$

$$\text{(ii)}\quad c=(\underbrace{s_1b_1,0,\cdots,0}_{n_1},\cdots,\underbrace{s_jb_j,0,\cdots,0}_{n_j},\cdots,\underbrace{s_kb_k,0,\cdots,0}_{n_k}),$$

(iii) *For $A=:(A_{ij})_{1\leqq i,j\leqq k}$ we have*

(a) *block diagonal entries $A_{jj}$, $1\leqq j\leqq k$:*

$$A_{jj}=\begin{pmatrix}
a_{jj} & \alpha(j)_1 & & & & \\
-\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\
 & -\alpha(j)_2 & 0 & \cdot & & 0\\
 & & \ddots & \ddots & \ddots & \\
 & 0 & & \cdot & 0 & \alpha(j)_{n_j-1}\\
 & & & & -\alpha(j)_{n_j-1} & 0
\end{pmatrix}$$

*with* $a_{jj}=\dfrac{-b_j^2}{4dp_j}(1-s_jp_j)^2;$

(b) *off-diagonal blocks* $A_{ij}$, $1 \leq i, j \leq k$, $i \neq j$:

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{with } a_{ij} = \frac{-b_i b_j}{2d(s_i s_j p_i + p_j)}(1 - s_i p_i)(1 - s_j p_j);$$

(iv) $d \in \Re$, $d > 0$.

*Moreover,* $(A, b, c, d)$ *as defined in* (2) *is positive real balanced with positive real gramian*

$$\Sigma_p = \operatorname{diag}(p_1 I_{n_1}, \cdots, p_j I_{n_j}, \cdots, p_k I_{n_k}).$$

*The map* $\Gamma_p$, *which assigns to each system in* $P_n^1$ *the realization given in* (2), *is a canonical form.*

Note that in the derivation of the corollary from the theorem we have used an obvious reparametrization of the $b$-vector to obtain the usual statement of the result. The previous theorem and corollary show the by-now-expected structure of the canonical form.

**7. Minimum-phase systems.** The last class of systems for which we would like to derive a canonical form is the class of minimum-phase systems. A minimum-phase system is an asymptotically stable system whose inverse system is also asymptotically stable. Minimum-phase systems, therefore, are precisely those systems whose transfer functions are real rational functions in $H^\infty$, which are units, i.e., invertible in $H^\infty$. These systems are of importance in many different areas. In this section, however, we are mainly interested in results concerning minimum-phase systems that are motivated by problems in stochastic system theory, since those results allow us to use the canonical form derived for positive real systems to obtain a canonical form for minimum-phase systems.

DEFINITION 7.1. A system $(A, B, C, D) \in C_n^{m,m}$ such that $D$ is invertible is called *minimum phase* if $A - BD^{-1}C$ has its eigenvalues in the open left halfplane. We denote by $M_n^m$ the subset of $C_n^{m,m}$ containing all minimum-phase systems. $TM_n^m$ denotes the set of transfer functions of systems in $M_n^m$.

The role minimum-phase systems play in the spectral factorization problem is indicated in the following proposition, which summarizes some standard results (see, e.g., [8], [32]).

PROPOSITION 7.1.

(1) *Let* $(A, B, C, D) \in C_n^{m,m}$ *such that* $D + D^T > 0$ *and assume that there exists a solution* $P = P^T > 0$ *to the corresponding* PRRE. *Then the following statements are equivalent*:

  (i) $(A, B, C, D) \in P_n^m$.

  (ii) *The minimal solution* $P_{\min}$ *and the maximal solution* $P_{\max}$ *to the* PRRE *are such that* $0 < P_{\min} < P_{\max}$.

  (iii) *There exists a solution* $P_0$ *to the* PRRE *such that the eigenvalues of*

$$A - B(D + D^T)^{-1}(C - B^T P_0)$$

*are in the open left halfplane.*

*If* (iii) *is satisfied, then* $P_0 = P_{\min}$.

(2) *Let* $P(s) \in TP_n^m$ *with realization* $(A, B, C, D) \in P_n^m$. *Then*

$$P(iw) + P(-iw)^T = M(-iw)^T M(iw), \qquad w \in \Re,$$

*for some* $M(s) \in TM_n^m$. *A minimal realization of* $M(s)$ *is given by*

$$(A, B, D_m^{-T}(C - B^T P_{\min}), D_m),$$

*where* $P_{\min}$ *is the minimal solution to the* PRRE *corresponding to* $(A, B, C, D)$ *and* $D_m \in \Re^{m\times m}$ *is such that* $D_m^T D_m = D + D^T$.

This proposition suggests how we can relate positive real systems to minimum-phase systems. The precise relationship is given in the following proposition where for a minimum-phase system with a given $D$-matrix we uniquely define an associated positive real system.

PROPOSITION 7.2. *For* $D \in \Re^{m\times m}$, *invertible, let*

$$M_{n,D}^m = \{(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in M_n^m \mid \tilde{D} = D\},$$

$$P_{n,D}^m = \{(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in P_n^m \mid \tilde{D} = \tfrac{1}{2}D^T D\},$$

*with* $TM_{n,D}^m$ *and* $TP_{n,D}^m$ *the sets of corresponding transfer functions.*

*The map*

$$S_{PM,D}: \quad P_{n,D}^m \to M_{n,D}^m$$

$$(A_p, B_p, C_p, D_p) \mapsto (A_m, B_m, C_m, D_m) := (A_p, B_p, D^{-T}(C_p - B_p^T P_{\min}), D),$$

*where* $P_{\min}$ *is the minimal solution to the* PRRE *corresponding to* $(A_p, B_p, C_p, D_p)$, *is a bijection. Its inverse is given by*

$$S_{PM,D}^{-1}: \quad M_{n,D}^m \to P_{n,D}^m$$

$$(A_m, B_m, C_m, D_m) \mapsto (A_p, B_p, C_p, D_p) := (A_m, B_m, D^T C_m + B_m^T P, \tfrac{1}{2}D^T D),$$

*where* $P$ *is the solution to the Lyapunov equation*

$$A_m^T P + P A_m = -C_m^T C_m. \tag{16}$$

*Moreover,* $S_{PM,D}$ *preserves system equivalence.*

*Proof.* By Proposition 7.1 the map $S_{PM,D}$ is well defined. To show that $S_{PM,D}^{-1}$ is well defined we first must show that for $(A_m, B_m, C_m, D_m) \in M_{n,D}^m$ the system $(A_p, B_p, C_p, D_p) = S_{PM,D}^{-1}((A_p, B_p, C_p, D_p))$ is in $C_n^{m,m}$.

First note that $P$ solves the PRRE corresponding to $(A_p, B_p, C_p, D_p)$. Let $P(s) = C_p(sI - A_p)^{-1}B_p + D_p$ and $M(s) = C_m(sI - A_m)^{-1}B_m + D_m$; then standard algebraic manipulations show that

$$P(iw) + P(-iw)^T = M(-iw)^T M(iw), \qquad w \in \Re.$$

Since by assumption $M(s)$ has McMillan degree $n$ this implies that $P(s)$ also has McMillan degree $n$. This shows that $(A_p, B_p, C_p, D_p) \in C_n^{m,m}$.

$(A_p, B_p, C_p, D_p) \in P_n^m$ follows from Proposition 7.1 as

$$\tilde{A} := A_p - B_p(D_p + D_p^T)^{-1}(C_p - B_p^T P) = A_m - B_m D_m^{-1} C_m$$

has all its eigenvalues in the open left halfplane. It is now straightforward to verify that $S_{PM,D}$ is in fact a bijection and preserves system equivalence. □

We now define a minimum-phase system to be minimum-phase balanced if its corresponding positive real system is positive real balanced.

DEFINITION 7.2. A system $(A, B, C, D) \in M_n^m$ is called *minimum-phase balanced* if the system $S_{PM,D}^{-1}((A, B, C, D))$ is positive real balanced. The *minimum-phase gramian* $\Sigma_m$ of $(A, B, C, D)$ is defined to be the positive real gramian $\Sigma_p$ of $S_{PM,D}^{-1}((A, B, C, D))$, i.e., $\Sigma_m = \Sigma_p$.

A canonical form for minimum-phase systems can now be derived using Proposition 7.2 to carry the canonical form for positive real systems over to the class of minimum-phase systems.

THEOREM 7.1. *The following two statements are equivalent*:

(1) $M(s) \in TM_n^m$.

(2) $M(s)$ *has a realization* $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{m\times n} \times \mathfrak{R}^{m\times m}$ *given by the parameters*:

$$\begin{array}{ll}
1 > p_1 > \cdots > p_j > \cdots > p_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
r_1, \cdots, r_j, \cdots, r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m); \\
U_1, \cdots, U_j, \cdots, U_k, & U_j \in \mathfrak{R}^{m\times r_j}, \quad U_j^T U_j = I_{r_j}; \\
\tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k, & \tilde{B}_j \in \mathfrak{R}^{r_j\times m} \quad \textit{positive upper triangular}; \\
\tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k, & \tilde{A}_j \in \mathfrak{R}^{n_j\times n_j} \quad \textit{in } r_j\textit{-balanced form}; \\
D, & D \in \mathfrak{R}^{m\times m}, \quad D \textit{ invertible}
\end{array}$$

*in the following way*:

*If* $(A, B, C, D)$ *is partitioned according to* $n_1, \cdots, n_j, \cdots, n_k$, *then*,

(i) $B_j = \begin{pmatrix} \tilde{B}_j(D^TD)^{1/2} \\ 0 \end{pmatrix}, \qquad 1 \leqq j \leqq k;$

(ii) $C_j = D^{-T}(D^TD)^{1/2}(U_j\Delta_j - p_j\tilde{B}_j^T, 0)$ *where* $\Delta_j = (\tilde{B}_j\tilde{B}_j^T)^{1/2}, \quad 1 \leqq j \leqq k;$

(iii) $$A_{jj} = \tilde{A}_j - \frac{1+p_j^2}{p_j}[\operatorname{diag}(\Delta_j^2, 0)]_l - \frac{1+p_j^2}{2p_j}[\operatorname{diag}(\Delta_j^2, 0)]_d + \operatorname{diag}(\tilde{B}_jU_j\Delta_j, 0), \quad 1 \leqq j \leqq k;$$

(iv) $$A_{ij} = \frac{1}{p_i^2 - p_j^2}(p_j(1-p_i^2)\operatorname{diag}(\tilde{B}_i\tilde{B}_j^T, 0) - p_i(1-p_j^2)\operatorname{diag}(\Delta_iU_i^TU_j\Delta_j, 0)) + \operatorname{diag}(\tilde{B}_iU_j\Delta_j, 0), \qquad 1 \leqq i, j \leqq k, \quad i \neq j.$$

(v) $D \in \mathfrak{R}^{m\times m}$, *invertible*.

*Moreover*, $(A, B, C, D)$ *as defined in* (2) *is minimum-phase balanced with minimum-phase gramian*

$$\Sigma_m = \operatorname{diag}(p_1 I_{n_1}, \cdots, p_j I_{n_j}, \cdots, p_k I_{n_k}).$$

*The map* $\Gamma_m$, *which assigns to each system in* $M_n^m$ *the realization given in* (2), *is a canonical form.*

COROLLARY 7.1. *The following two statements are equivalent*:

(1) $m(s) \in TM_n^1$.

(2) $m(s)$ *has a realization* $(A, b, c, d) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times 1} \times \mathfrak{R}^{1\times n} \times \mathfrak{R}^{1\times 1}$ *given by the parameters*:

$$\begin{array}{ll}
1 > p_1 > \cdots > p_j > \cdots > p_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
s_1, \cdots, s_j, \cdots, s_k, & s_j = \pm 1, \quad 1 \leqq j \leqq k; \\
b_1, \alpha(1)_1, \cdots, \alpha(1)_j, \cdots, \alpha(1)_{n_1-1}, & b_1 > 0, \quad \alpha(1)_j > 0, \quad 1 \leqq j \leqq n_1 - 1; \\
\vdots & \\
b_i, \alpha(i)_1, \cdots, \alpha(i)_j, \cdots, \alpha(i)_{n_i-1}, & b_i > 0, \quad \alpha(i)_j > 0, \quad 1 \leqq j \leqq n_i - 1; \\
\vdots & \\
b_k, \alpha(k)_1, \cdots, \alpha(k)_j, \cdots, \alpha(k)_{n_k-1}, & b_k > 0, \quad \alpha(k)_j > 0, \quad 1 \leqq j \leqq n_k - 1; \\
d, & d \in \mathfrak{R}, \quad d \neq 0
\end{array}$$

*in the following way:*

(i) $b=(\underbrace{b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{b_k, 0, \cdots, 0}_{n_k})^T$,

(ii) $c=\frac{1}{d}((\underbrace{(s_1-p_1)b_1, 0, \cdots, 0}_{n_1}, \cdots, \underbrace{(s_j-p_j)b_j, 0, \cdots, 0}_{n_j}, \cdots, \underbrace{(s_k-p_k)b_k, 0, \cdots, 0}_{n_k})$,

(iii) *For* $A=: (A_{ij})_{1\leqq i,j\leqq k}$ *we have*

(a) *block diagonal entries* $A_{jj}, 1\leqq j\leqq k$:

$$A_{jj}=\begin{pmatrix} a_{jj} & \alpha(j)_1 & & & & \\ -\alpha(j)_1 & 0 & \alpha(j)_2 & & & \\ & -\alpha(j)_2 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha(j)_{n_j-1} \\ & & & & -\alpha(j)_{n_j-1} & 0 \end{pmatrix}$$

*with* $a_{jj}=\dfrac{-b_j^2}{2d^2p_j}(1-s_jp_j)^2$;

(b) *off-diagonal blocks* $A_{ij}, 1\leqq i,j\leqq k, i\neq j$:

$$A_{ij}=\begin{pmatrix} a_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{with } a_{ij}=\frac{-b_ib_j}{d^2(s_is_jp_i+p_j)}(1-s_ip_i)(1-s_jp_j);$$

(iv) $d\in\Re, d\neq 0$.

*Moreover,* $(A, b, c, d)$ *as defined in* (2) *is minimum-phase balanced with minimum-phase gramian*

$$\Sigma_m=\mathrm{diag}\,(p_1I_{n_1}, \cdots, p_jI_{n_j}, \cdots, p_kI_{n_k}).$$

*The map* $\Gamma_m$, *which assigns to each system in* $M_n^1$ *the realization given in* (2), *is a canonical form.*

While the canonical form we have just derived has a structure that is very similar to the canonical forms derived for the other classes of systems, there is a certain loss of symmetry in this canonical form. This is most apparent in the case of single-input single-output systems. In the previously derived canonical forms, the entries of the $b$-vector and $c$-vector had the same modulus. Moreover, the canonical forms were sign-symmetric (see § 10 for a precise definition). The sign-symmetry is lost in the above canonical form for single-input single-output minimum-phase systems. By standard arguments it can be shown that all single-input single-output minimum-phase systems have a sign-symmetric realization; the canonical form for single-input single-output asymptotically stable systems of Corollary 2.1 restricted to minimum-phase systems is such an example. It is, however, not clear whether it is possible to find a "balancing scheme" for minimum-phase systems such that the corresponding canonical form is sign-symmetric for single-input single-output systems and at the same time leads to a parametrization whose parameter space has the desirable geometric structure of the canonical form presented here.

**8. Discrete-time systems.** The canonical forms derived in the previous sections dealt with continuous-time systems. In this section we will show that the standard technique of bilinearly transforming continuous-time systems to discrete-time systems can be applied to derive canonical forms for various classes of discrete-time systems. We first define the classes of systems that will be considered.

DEFINITION 8.1. Let $(A, B, C, D) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m} \times \mathfrak{R}^{p\times n} \times \mathfrak{R}^{p\times m}$ and $G(z) = C(zI-A)^{-1}B+D$.

(1) If all eigenvalues of $A$ are in the open unit disk, then $(A, B, C, D)$ is called *discrete-time asymptotically stable.* The set of discrete-time asymptotically stable systems in $L_n^{p,m}$ is denoted by $D_n^{p,m}$ with $TD_n^{p,m}$ the corresponding set of transfer functions.

(2) A system $(A, B, C, D) \in D_n^{p,m}$ is called *discrete-time bounded real* if

$$I - G(e^{-i\theta})^T G(e^{i\theta}) > 0, \qquad \theta \in [0, 2\pi].$$

The set of discrete-time bounded real systems in $D_n^{m,m}$ is denoted by $DB_n^{p,m}$ with $TDB_n^{p,m}$ the corresponding set of transfer functions.

(3) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time positive real* if

$$G(e^{-i\theta})^T + G(e^{i\theta}) > 0, \qquad \theta \in [0, 2\pi].$$

The set of discrete-time positive real systems in $D_n^{p,m}$ is denoted by $DP_n^m$ with $TDP_n^m$ the corresponding set of transfer functions.

(4) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time minimum phase* if

$$\tilde{G}(z) := G(z)^{-1} \in TD_n^{m,m}.$$

The set of discrete-time minimum-phase systems in $D_n^{m,m}$ is denoted by $DM_n^m$ with $TDM_n^m$ the corresponding set of transfer functions.

(5) A system $(A, B, C, D) \in D_n^{m,m}$ is called *discrete-time allpass* if for some $\sigma > 0$,

$$G(e^{i\theta}) G(e^{-i\theta})^T = \sigma^2 I, \qquad \theta \in [0, 2\pi].$$

The set of discrete-time allpass systems in $D_n^{m,m}$ is denoted by $DA_n^m$ with $TDA_n^m$ the corresponding set of transfer functions.

The following proposition summarizes some basic results on the bilinear transformation.

PROPOSITION 8.1. *The transformation*

$$TU_n^{p,m}: TC_n^{p,m} \to TD_n^{p,m},$$

$$G_c(s) \mapsto G_d(z) := G_c\left(\frac{z-1}{z+1}\right)$$

*is a bijection with inverse*

$$(TU_n^{p,m})^{-1}: TD_n^{p,m} \to TC_n^{p,m},$$

$$G_d(z) \mapsto G_c(s) := G_d\left(\frac{1+s}{1-s}\right),$$

*which induces a bijection between $TB_n^{p,m}$ and $TDB_n^{p,m}$. If $p = m$ then $TU_n^{m,m}$ induces a bijection between $TA_n^m$ and $TDA_n^m$, $TP_n^m$ and $TDP_n^m$, as well as $TM_n^m$ and $TDM_n^m$.*

This mapping also has a formulation in terms of state-space systems, which is given in the next proposition [3], [10], [22].

PROPOSITION 8.2. *The transformation*

$$SU_n^{p,m}:\quad C_n^{p,m} \to D_n^{p,m},$$
$$(A_c, B_c, C_c, D_c) \mapsto (A_d, B_d, C_d, D_d),$$
$$(A_d, B_d, C_d, D_d) := ((I-A_c)^{-1}(I+A_c), \sqrt{2}(I-A_c)^{-1}B_c, \sqrt{2}\, C_c(I-A_c)^{-1}, D_c + C_c(I-A_c)^{-1}B_c)$$

*is a bijection with inverse*

$$(SU_n^{p,m})^{-1}:\quad D_n^{p,m} \to C_n^{p,m},$$
$$(A_d, B_d, C_d, D_d) \mapsto (A_c, B_c, C_c, D_c),$$
$$(A_c, B_c, C_c, D_c) := ((I+A_d)^{-1}(A_d - I), \sqrt{2}(I+A_d)^{-1}B_d, \sqrt{2}\, C_d(I+A_d)^{-1}, D_d - C_d(I+A_d)^{-1}B_d),$$

*which induces a bijection between $B_n^{p,m}$ and $DB_n^{p,m}$. If $p=m$, then $SU_n^{m,m}$ induces a bijection between $A_n^m$ and $DA_n^m$, $P_n^m$ and $DP_n^m$, as well as $M_n^m$ and $DM_n^m$. The map $SU_n^{p,m}$ preserves system equivalence as well as sign-symmetry of state-space realizations if $p=m$; i.e., for $(A_c, B_c, C_c, D_c) = (SU_n^{m,m})^{-1}((A_d, B_d, C_d, D_d))$, $(A_d, B_d, C_d, D_d) \in D_n^{m,m}$ we have*

$$A_c = SA_c^T S, \qquad B_c = SC_c^T$$

*if and only if*

$$A_d = SA_d^T S, \qquad B_d = SC_d^T,$$

*for some $S = \operatorname{diag}(\pm 1, \cdots, \pm 1)$.*

The previous proposition allows us to carry over the canonical forms for continuous-time asymptotically stable systems to the discrete-time case.

THEOREM 8.1. *If $\Gamma$, $\Gamma_a$, $\Gamma_b$, $\Gamma_p$, and $\Gamma_m$ are the canonical forms for the sets $C_n^{p,m}$, $A_n^m$, $B_n^{p,m}$, $P_n^m$, and $M_n^m$ as defined in the previous sections, then $D\Gamma := SU_n^{p,m}\Gamma(SU_n^{p,m})^{-1}$, $D\Gamma_a := SU_n^{m,m}\Gamma_a(SU_n^{m,m})^{-1}$, $D\Gamma_b := SU_n^{p,m}\Gamma_b(SU_n^{p,m})^{-1}$, $D\Gamma_p := SU_n^{m,m}\Gamma_p(SU_n^{m,m})^{-1}$, and $D\Gamma_m := SU_n^{m,m}\Gamma_m(SU_n^{m,m})^{-1}$ are canonical forms for the sets $D_n^{p,m}$, $DA_n^m$, $DB_n^{p,m}$, $DP_n^m$, and $DM_n^m$.*

*Remark* 8.1. Analogously to the continuous-time case we can introduce balancing techniques for the various classes of discrete-time systems by balancing solutions to the corresponding discrete-time Lyapunov and Riccati equations. Since the map $SU_n^{p,m}$ leaves such solutions invariant (see, e.g., [3], [10], [22]), $SU_n^{p,m}$, in fact, preserves balancing. Hence the canonical forms for discrete-time systems introduced in Theorem 8.1 are therefore in terms of balanced representations.

**9. Model reduction.** One of the main advantages of balanced representations is that they can be used for a very straightforward method of model reduction. Moore [19] has, in fact, introduced balanced realizations for stable linear systems to have an efficient way of performing model reduction. His scheme was based on the following state-space projection method. Consider an $n$-dimensional balanced system $(A, B, C, D)$ and partition it conformally as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (C_1, C_2),$$

such that for $1 \leqq N < n$, $A_{11} \in \mathfrak{R}^{N\times N}$, $B_1 \in \mathfrak{R}^{N\times m}$, and $C_1 \in \mathfrak{R}^{p\times N}$. The principal subsystem $(A_{11}, B_1, C_1, D)$ is then considered to be an approximant of $(A, B, C, D)$.

Pernebo and Silverman [29] were the first to show that the approximant of a balanced system in $C_n^{p,m}$ is again balanced, minimal, and asymptotically stable. Their result, however, assumes that truncation does not occur between states corresponding to repeated singular values. Otherwise the approximant may no longer be asymptotically stable and minimal. Analogous results with the same restriction were shown in [7] for positive real systems, in [16] for Riccati balanced systems, and in [28] for bounded real systems.

We now suggest a model reduction technique that is more suitable to our particular setup than the method discussed above. Rather than working with the state-space systems directly, we perform the model reduction by reducing the parameters corresponding to a certain system.

We have seen that associated with each system is a set of parameters as follows:

$$\begin{array}{ll}
\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
r_1, \cdots, r_j, \cdots, r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m, p); \\
U_1, \cdots, U_j, \cdots, U_k, & U_j \in \Re^{p \times r_j}, \quad U_j^T U_j = I_{r_j}; \\
\tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k, & \tilde{B}_j \in \Re^{r_j \times m} \quad \text{positive upper triangular;} \\
\tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k, & \tilde{A}_j \in \Re^{n_j \times n_j} \quad \text{in } r_j\text{-balanced form;} \\
D, & D \in \Re^{p \times m}
\end{array}$$

A reduced-order system of degree $N$ can now easily be defined by retaining of these parameters only those that correspond to the first $N$ states; i.e., if $j_N$ is such that $n_1 + \cdots + n_{j_N} < N \leqq n_1 + \cdots + n_{jn} + h_{j_N+1}$, then take the following parameters:

$$\begin{array}{ll}
\sigma_1, \cdots, \sigma_{j_N}, \sigma_{j_N+1}, & \\
n_1, \cdots, n_{j_N}, Pn_{j_N+1}, & Pn_{j_N+1} := N - (n_1 + \cdots + n_{j_N}); \\
U_1, \cdots, U_{j_N}, PU_{j_N+1}, PU_{j_N+1}, & \text{the first } \min(r_{j_N+1}, Pn_{j_N+1}) \\
 & \text{columns of } U_{j_N+1}. \\
\tilde{B}_1, \cdots, \tilde{B}_{j_N}, P\tilde{B}_{j_N+1}, & P\tilde{B}_{j_N+1} \text{ the first } \min(r_{j_N+1}, Pn_{j_N+1}) \\
 & \text{rows of } \tilde{B}_{j_N+1}; \\
\tilde{A}_1, \cdots, \tilde{A}_{j_N}, P\tilde{A}_{j_N+1}, P\tilde{A}_{j_N+1}, & \text{the principal submatrix of} \\
 & \tilde{A}_{j_N+1} \text{ of size } Pn_{j_N+1}.
\end{array}$$

By the parametrization results of the previous sections these parameters define a unique reduced-order system that is in the same class of the systems as the original system. We can call this scheme a *parameter projection method.*

We can summarize this method in the following theorem.

THEOREM 9.1. *If* $(A, B, C, D)$ *is a continuous-time system in* $C_n^{p,m}(A_n^m, L_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ *given in the canonical form of Theorem* 2.1 (3.1, 4.1, 5.1, 6.1, 7.1) *with parameters*

$$\begin{array}{ll}
\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0, & \\
n_1, \cdots, n_j, \cdots, n_k, & n_j \in \mathcal{N}, \quad \sum_{j=1}^k n_j = n; \\
r_1, \cdots, r_j, \cdots, r_k, & r_j \in \mathcal{N}, \quad 1 \leqq r_j \leqq \min(n_j, m, p); \\
U_1, \cdots, U_j, \cdots, U_k, & U_j \in \Re^{p \times r_j}, \quad U_j^T U_j = I_{r_j}; \\
\tilde{B}_1, \cdots, \tilde{B}_j, \cdots, \tilde{B}_k, & \tilde{B}_j \in \Re^{r_j \times m} \quad \textit{positive upper triangular;} \\
\tilde{A}_1, \cdots, \tilde{A}_j, \cdots, \tilde{A}_k, & \tilde{A}_j \in \Re^{n_j \times n_j} \quad \textit{in } r_j\textit{-balanced form;} \\
D, & D \in \Re^{p \times m};
\end{array}$$

*then each N-dimensional system, $1 \leqq N < n$, obtained by the balanced parameter space model reduction scheme, i.e., each system given by the parameters*

$$\begin{array}{ll} \sigma_1, \cdots, \sigma_{j_N}, \sigma_{j_N+1}, & \\ n_1, \cdots, n_{j_N}, Pn_{j_N+1}, & Pn_{j_N+1} := N-(n_1+\cdots+n_{j_N}); \\ U_1, \cdots, U_{j_N}, PU_{j_N+1}, & PU_{j_N+1} \text{ the first } \min(r_{j_N+1}, Pn_{j_N+1}) \\ & \text{columns of } U_{j_N+1}; \\ \tilde{B}_1, \cdots, \tilde{B}_{j_N}, P\tilde{B}_{j_N+1}, & P\tilde{B}_{j_N+1} \text{ the first } \min(r_{j_N+1}, Pn_{j_N+1}) \\ & \text{rows of } \tilde{B}_{j_N+1}; \\ \tilde{A}_1, \cdots, \tilde{A}_{j_N}, P\tilde{A}_{j_N+1}, & P\tilde{A}_{j_N+1} \text{ the principal submatrix of} \\ D, & \tilde{A}_{j_N+1} \text{ of size } Pn_{j_N+1}, \end{array}$$

*and parametrized as in Theorem* 2.1 (3.1, 4.1, 5.1, 6.1, 7.1) *is again in canonical form and therefore in the same class of systems, i.e., in* $C_N^{p,m}$ $(A_N^m, L_N^{p,m}, B_n^{p,m}, P_N^m, M_N^m)$.

From this result we can easily obtain a result concerning model reduction using the state-space projection method.

COROLLARY 9.1. *If* $(A, B, C, D)$ *is a continuous-time system in* $C_n^{p,m}$ $(A_n^m, L_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ *given in the canonical form of Theorem* 2.1 (3.1, 4.1, 5.1, 6.1, 7.1) *then each N-dimensional principal subsystem of* $(A, B, C, D)$ *is again in canonical form and therefore in the same class of systems, i.e., in* $C_N^{p,m}(A_N^m, L_N^{p,m}, B_n^{p,m}, P_N^m, M_N^m)$ *if for some* $0 \leqq j_N \leqq k-1$,

$$n_1+\cdots+n_{j_N}+r_{j_N+1} \leqq N \leqq n_1+\cdots+n_{j_N}+n_{j_N+1},$$

*where we set* $n_0 = 0$.

*Proofs.* The result follows by inspection since the reduced-order system is parametrized by a set of parameters as in Theorem 9.1. □

Because of the particular nature of the nonuniqueness of balanced realizations Corollary 9.1 is, in fact, more general than the results in [29], [7], [16], and [28], where it is assumed that truncation occurs at a point of nonrepeated singular values. We can recover these results immediately in the following corollary.

COROLLARY 9.2. *If* $(A, B, C, D)$ *is a continuous-time system given in* $C_n^{p,m}(L_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ *that is Lyapunov balanced (Riccati balanced, bounded real balanced, positive real balanced, minimum-phase balanced) with gramian* $\Sigma = \operatorname{diag}(\sigma_1 I_{n_1}, \cdots, \sigma_k I_{n_k})$, *then the N-dimensional principal subsystem is in the same class of systems, i.e.,* $C_N^{p,m}(L_N^{p,m}, B_n^{p,m}, P_N^m, M_N^m)$, *if* $N = n_1+n_2+\cdots+n_{j_0}$, *for some* $j_0 = 1, \cdots, k$.

We obtain a very general model reduction result if either the input or the output dimension of the system is one. Then we have that the $r_j$ parameters are also one, and hence the condition Corollary 9.1 is always satisfied.

COROLLARY 9.3. *Assume that* $\min(p, m) = 1$. *If* $(A, B, C, D)$ *is a continuous-time system in* $C_n^{p,m}$ $(A_n^m, L_n^{p,m}, B_n^{p,m}, P_n^m, M_n^m)$ *given in the canonical form of Theorem* 2.1 (3.1, 4.1, 5.1, 6.1, 7.1), *then each N-dimensional principal subsystem of* $(A, B, C, D)$ *is again in canonical form and therefore in the same class of systems, i.e., in* $C_N^{p,m}$ $(A_N^m, L_N^{p,m}, B_n^{p,m}, P_N^m, M_N^m)$.

*Remark* 9.1. Note that for the particular type of canonical forms presented in this paper we cannot, in general, expect to have a model reduction result based on the state-space projection method by which truncation can occur at an arbitrary place. This is due to the fact that we use $(\tilde{B}_j\tilde{B}_j^T)^{1/2}$ as a parameter in the $C$-matrix and that $(\tilde{B}_j\tilde{B}_j^T)^{1/2}$ has no specific structure. In the case of the canonical forms presented in [22] and [27], $(\tilde{B}_j\tilde{B}_j^T)^{1/2}$ was constrained to be diagonal and we have the general model reduction property.

In [1] and [22] a model reduction technique was suggested for balanced discrete-time systems in $D_n^{p,m}$ by carrying a discrete-time system over to a continuous-time system using the map $SU_n^{p,m}$. This corresponding continuous-time system is reduced to a lower-order system which is then mapped back to a discrete-time system using the inverse mapping. Thereby we obtain a lower-order approximant to the discrete-time system. The following corollary shows how it is possible to obtain in this way a discrete-time version of Corollary 9.1.

COROLLARY 9.4. *Let $(A_d, B_d, C_d, D_d)$ be in one of the following classes of discrete-time asymptotically stable systems: $D_n^{p,m}$, $DA_n^m$, $DB_n^{p,m}$, $DP_n^m$, and $DM_n^m$. Assume that $(A_d, B_d, C_d, D_d)$ is given in the corresponding canonical form. Let $(\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c)$ be the N-dimensional principal subsystem of $(A_c, B_c, C_c, D_c) = (SU_n^{p,m})^{-1}((A_d, B_d, C_d, D_d))$; then $(\hat{A}_d, \hat{B}_d, \hat{C}_d, \hat{D}_d) := SU_n^{p,m}((\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c))$ is in the corresponding subclass of N-dimensional systems, if*

$$n_1 + \cdots + n_{j_N} + r_{j_N+1} \leqq N \leqq n_1 + \cdots + n_{j_N} + n_{j_N+1}.$$

Clearly, all the other continuous-time results in this section can be carried over to discrete-time systems in the same way.

**10. Final remarks.** The paper dealt with canonical forms and parametrizations for the following classes of linear systems of fixed dimensions: asymptotically stable systems, allpass systems, the general class of minimal systems, positive real systems, bounded real systems, and minimum-phase systems. All the canonical forms are given in terms of balanced realizations for the particular class of systems. Several aspects of these parametrizations were discussed including model reduction.

It was pointed out that all the canonical forms have a similar structure. Only the way the parameters enter the entries of the system matrices determines whether or not a system belongs to a certain class of systems. We are going to make a few further remarks concerning common properties of the various canonical forms.

**Sign-symmetry and Cauchy index.** With the exception of minimum-phase systems all scalar systems that are given in one of the previously derived canonical forms have the so-called sign-symmetry property, i.e.,

$$A^T = SAS, \qquad b = Sc^T,$$

where $S$ is a diagonal matrix whose diagonal terms are $\pm 1$. In particular, if

$$n_1, \cdots, n_j, \cdots, n_k, \qquad s_1, \cdots, s_j, \cdots, s_k$$

are the usual structural parameters of a scalar system given in one of the canonical forms of Corollaries 2.1, 3.1, 4.1, 5.1, 6.2, 7.1, then the sign-symmetry matrix $S$ is

$$S = \operatorname{diag}(s_1 \hat{I}_{n_1}, \cdots, s_j \hat{I}_{n_j}, \cdots, s_k \hat{I}_{n_k}),$$

where $\hat{I}_{n_j} = \operatorname{diag}(+1, -1, +1, \cdots, (-1)^{n_j+1}) \in \Re^{n_j \times n_j}$. Note that by Proposition 8.2 the canonical form of a discrete-time system is sign symmetric if its corresponding continuous-time system is sign symmetric. An important property of the sign-symmetry matrix of a system is that it can be related to the Cauchy index of its transfer function. The Cauchy index of a rational function is defined as follows.

DEFINITION 10.1. Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients. The Cauchy index $C_{\text{ind}}(g(x))$ of $g(x)=p(x)/q(x)$ is defined as the number of jumps from $-\infty$ to $+\infty$ minus the number of jumps from $+\infty$ to $-\infty$ of $g(x)$ as $x$ varies from $-\infty$ to $+\infty$.

A consequence of a result in [2] is that if a system is sign symmetric with respect to a sign-symmetry matrix $S$, the Cauchy index of its transfer function $g(s)$ is given by

$$C_{\text{ind}}(g(s))=\text{trace}\,(S).$$

In [25] it was shown that systems in $C_n^{1,1}$ with Cauchy index $n$ characterize the so-called relaxation systems. These are systems whose impulse response is a completely monotonic function.

**Geometric aspects of the parameter space.** Another important aspect of the canonical forms presented in this paper is the comparatively simple structure of the parameter space. In many other canonical forms for minimal systems the parameter set at which the systems lose minimality is described by complicated sets of algebraic equations. Here minimality is preserved provided certain parameters are strictly positive. A disadvantage, however, is that even in the case of SISO systems several distinct structures are necessary to parametrize any of the classes of systems considered here.

Let $S_n$ denote any of the following classes of single-input single-output systems: $C_n^{1,1}$, $L_n^{1,1}$, $B_n^{1,1}$, or $P_n^1$. We have shown that each system in $S_n$ has associated with it a unique set of discrete parameters:

$$n_1,\cdots,n_j,\cdots,n_k,\qquad n_j\in\mathcal{N},\quad \textstyle\sum_{j=1}^k n_j=n,$$

$$s_1,\cdots,s_j,\cdots,s_k,\qquad s_j=\pm 1,\quad 1\leqq j\leqq k.$$

If we denote by $S_n(n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k)$ the set of systems in $S_n$ with the discrete parameters $(n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k)$, then we can clearly write the set $S_n$ as the disjoint union of the sets $S_n(n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k)$, i.e.,

$$S_n=\bigcup_{\substack{1\leqq k\leqq n\\ n_1+n_2+\cdots+n_k=n\\ s_1=\pm1,s_2\pm1,\cdots,s_k=\pm1}} S_n(n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k).$$

It follows easily from the structure of the continuous parameters that for each choice of discrete parameters $n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k$ the parameter set of $S_n(n_1,n_2,\cdots,n_k;s_1,s_2,\cdots,s_k)$ is, in fact, diffeomorphic to $\mathfrak{R}^{n+k}\times\mathfrak{R}$. Therefore we have a decomposition of the set $S_n$ into disjoint "cells," each of which are diffeomorphic to a Euclidean space. The term "cell" is used here in a loose sense and not in its strict topological meaning. We will not go any further into a topological investigation of this decomposition. We refer to [9] and [14] where a different decomposition, which originates from a continued fraction expansion of scalar transfer functions, is introduced and investigated. It is, however, interesting to consider the number of cells in our decomposition. The total number of cells of $S_n$ is easily seen by induction to be $2\times 3^{n-1}$. The number of cells of dimension $n+k$ is equal to the number of different choices of $k$ blocks of singular values times the possible choices of signs, which is $2^k$ and therefore gives

$$2^k\binom{n-1}{k-1}.$$

Note that we neglect the parameter that corresponds to the $d$-term, which is not relevant to the present discussion. Another number of interest is the number of cells of fixed dimension corresponding to a certain Cauchy index. These numbers are not so easily determined and we will just give a small table that contains these numbers for small dimensions.

*Number of cells of given dimension and Cauchy index.*

| Order | dim | Cauchy index | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $n+k$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
| 1 | 2 | | | | 1 | | 1 | | | |
| 2 | 3 | | | | | 2 | | | | |
| | 4 | | | 1 | | 2 | | 1 | | |
| 3 | 4 | | | | 1 | | 1 | | | |
| | 5 | | | | 4 | | 4 | | | |
| | 6 | | 1 | | 3 | | 3 | | 1 | |
| 4 | 5 | | | | | 2 | | | | |
| | 6 | | | 2 | | 8 | | 2 | | |
| | 7 | | | 6 | | 12 | | 6 | | |
| | 8 | 1 | | 4 | | 6 | | 4 | | 1 |

It is surprising to see that the different numbers we have obtained coincide with the numbers found in [9] for the cell decomposition of $L_n^{1,1}$, which was derived from continued fractions of the transfer functions. Simple examples show, however, that this decomposition is not identical with the decomposition derived here. We have not considered minimum-phase systems here. Similar results, however, also hold for these systems.

The geometry of the parameter space was used in [26] to study connectivity properties of the various classes of systems. See [24] for the case of asymptotically stable systems.

## REFERENCES

[1] U. M. Al-Saggaf and G. F. Franklin, *An error bound for a discrete reduced order model of a linear multivariable system*, IEEE Trans. Automat. Control, 32 (1987), pp. 815–819.

[2] B. D. O. Anderson, *On the computation of the Cauchy index*, Quart. Appl. Math., 1972, pp. 577–582.

[3] B. D. O. Anderson, K. L. Hitz, and N. D. Diem, *Recursive algorithm for spectral factorization*, IEEE Trans. Circuits and Systems, 21 (1974), pp. 742–750.

[4] B. D. O. Anderson, E. I. Jury, and M. Mansour, *Schwarz matrix properties for continuous and discrete time systems*, Internat. J. Control, 23 (1976), pp. 1–16.

[5] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[6] P. E. Caines, *Linear Stochastic Systems*, John Wiley, New York, 1988.

[7] U. B. Desai and D. Pal, *A transformation approach to stochastic model reduction*, IEEE Trans. Automat. Control, 29 (1984), pp. 1097–1100.

[8] P. L. Faurre, M. Clerget, and F. Germain, *Opérateurs rationnels positifs*, Dunod, Paris, 1979.

[9] P. A. Fuhrmann and P. S. Krishnaprasad, *Towards a cell decomposition for rational functions*, IMA J. Math. Control and Inform., 3 (1986), pp. 137–150.

[10] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their $L^\infty$-error bounds*, Internat. J. Control, 39 (1984), pp. 1115–1193.

[11] K. Glover and J. C. Doyle, *State-space formulae for all stabilizing controllers that satisfy an $H_\infty$-norm bound and relations to risk sensitivity*, Systems Control Lett., 11 (1988), pp. 167–172.

[12] R. P. GUIDORZI, *Invariants and canonical forms for systems: structural and parametric identification*, Automatica, 17 (1981), pp. 117–133.

[13] U. HELMKE, *Zur Topologie des Raumes linearer Kontrollsysteme*, Ph.D. thesis, University of Bremen, Bremen, Germany, 1982.

[14] U. HELMKE, D. HINRICHSEN, AND W. MANTHEY, *A cell decomposition of the space of real Hankels of rank $\leqq n$ and some applications*, Tech. Report 183, Institut für dynamische Systeme, Universität Bremen, Bremen, Germany, 1988.

[15] D. HINRICHSEN, *Canonical forms and parametrization problems in linear systems theory*, in Fourth IMA International Conference on Control Theory, P. A. Cook, ed., Academic Press, New York, 1986.

[16] E. A. JONCKHEERE AND L. M. SILVERMAN, *A new set of invariants for linear systems—application to reduced order compensator design*, IEEE Trans. Automat. Control, 28 (1983), pp. 953–964.

[17] P. T. KABAMBA, *Balanced forms: canonicity and parametrization*, IEEE Trans. Automat. Control, 30 (1985), pp. 1106–1109.

[18] S. S. MAHIL, F. W. FAIRMAN, AND B. S. LEE, *Some integral properties for balanced realizations of scalar systems*, IEEE Trans. Automat. Control, 29 (1984), pp. 181–183.

[19] B. C. MOORE, *Principal component analysis in linear systems: controllability, observability and model reduction*, IEEE Trans. Automat. Control, 26 (1981), pp. 17–32.

[20] R. OBER, *Problems of parametrization of linear systems*, Master's thesis, Engineering Department, Cambridge University, Cambridge, UK, August 1985.

[21] ———, *Asymptotically stable allpass transfer functions: canonical form, parametrization and realization*, in Proc. IFAC World Congress, Munich, 1987.

[22] ———, *Balanced realizations: canonical form, parametrization, model reduction*, Internat. J. Control, 46 (1987), pp. 643–670.

[23] ———, *Balanced realizations for finite and infinite dimensional linear systems*, Ph.D. thesis, Engineering Department, Cambridge University, Cambridge, UK, 1987.

[24] ———, *Topology of the set of asymptotically stable systems*, Internat. J. Control, 46 (1987), pp. 263–280.

[25] ———, *The parametrization of linear systems using balanced realizations: relaxation systems*, in Linear Circuits, Systems and Signal Processing: Theory and Application, C. I. Byrnes, C. F. Martin, and R. E. Saeks, eds., North-Holland, Amsterdam, 1988.

[26] ———, *Connectivity properties of classes of linear systems*, Internat. J. Control, 50 (1989), pp. 2049–2073.

[27] R. OBER AND D. MCFARLANE, *Balanced canonical forms for minimal systems: a normalized coprime factor approach*, Linear Algebra Appl., Special Issue on Linear Systems and Control, 122–124 (1989), pp. 23–64.

[28] P. C. OPDENACKER AND E. A. JONCKHEERE, *A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds*, IEEE Trans. Circuits and Systems, 35 (1988), pp. 184–189.

[29] L. PERNEBO AND L. M. SILVERMAN, *Model reduction via balanced state space representations*, IEEE Trans. Automat. Control, 37 (1982), pp. 382–387.

[30] J. RISSANEN, *Basic of invariants and canonical forms for linear dynamic systems*, Automatica, 10 (1974), pp. 175–182.

[31] M. VIDYASAGAR, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

[32] J. C. WILLEMS, *Least squares stationary optimal control and the algebraic Riccati equation*, IEEE Trans. Automat. Control, 16 (1971), pp. 621–634.

[33] D. A. WILSON AND A. KUMAR, *Symmetry properties of balanced systems*, IEEE Trans. Automat. Control, 28 (1983), pp. 927–929.