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Stability of control systems and graphs of linear systems

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Abstract: New conditions for internal stability of a closed-loop control system are given in terms of the graphs of the multiplication operators induced by the transfer functions of the plant and the controller. These conditions can be given a geometrical interpretation. This relates closed-loop stability to the minimal angle between the graph space associated with the system and the graph space associated with the controller. The maximally stabilizing controller is defined as the controller that maximizes the minimum angle between the graph space associated with the system and the graph space associated with the controller. It is shown that this controller can be calculated as a Nehari extension of the coprime factors of the system.

Keywords: Internal stability; graphs of operators; gap metric; robust control.

1. Introduction

The study of closed-loop stability is one of the central issues in control system design. Different characterizations of closed-loop stability are central to the various design techniques for feedback controllers. The original characterization of closed-loop stability in terms of the position of the closed-loop poles lead to the root-locus and pole-assignment design methodologies. The Nyquist criterion lead to loop-shaping techniques. More recently, closed-loop stability being phrased in terms of the coprime factorizations was central to the development of $\mathscr{H}_\infty$-control theory.

The aim of this paper is to discuss new characterizations of closed loop stability of a control system in terms of the graphs of the multiplication operators induced by the transfer functions of the plant and the controller. It is shown that a condition for internal stability can be phrased in terms of the span of these graphs and the norms of certain Toeplitz and projection operators. These conditions lend themselves to a geometric interpretation of closed-loop stability in terms of the minimum angle between the orthogonal complements of the transposed graphs of the system and the controller. One of the main results shows that a closed-loop system is stable if and only if the gap between the graph associated with the system and the orthogonal complement of the graph associated with the controller is less than one. Part of this paper establishes methods of calculating the angles and the norms of operators that play a role in this study. In particular it is shown that the minimal angle between the orthogonal complements of the graph spaces associated with the system and the controller can be calculated from the $\mathscr{L}_\infty$ norm of a rational function. It is hoped that these characterizations of closed-loop stability will bring about further understanding of control systems. These characterizations link notions of stability to the norm of operators, enabling the use of a operator theoretic methods in the study of control systems.

In this paper it is argued that the minimal angle between the graph spaces associated with the system and controller is a good indicator of the degree of stability of the control system. As an application of the methods which are introduced in this paper a maximally stabilizing controller is constructed for a given system. This controller maximizes the minimal angle between the graph spaces associated with the system

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and controller. It is shown that this controller is identical to the optimal controller of the normalized coprime factor perturbation problem of McFarlane and Glover [8], suggesting geometrical interpretations of coprime factor perturbations. This idea and others are pursued in the PhD Thesis [11].

A number of publications have studied the graphs of systems. In [3,14,6,17] the graphs of systems are used as a means of introducing a metric on the class of unstable systems, and some connections of graphs to robust stability are established. In [13] the graphs of non-linear systems are used to study stabilization problems.

The paper is divided into four main sections: Section 3 introduces the concept of the graph of a linear operator and states some general results concerning these graphs, Section 4 contains the new stability criteria. These are expressed either in terms of the norms of various projection operators or as conditions on the span of the graphs associated with the system and controller. Section 5 aims at giving these conditions a geometrical interpretation. It shows that closed-loop stability can be phrased in terms of the gap between the graph spaces associated with the system and controller or as a condition on the minimal angle between the orthogonal complements of these spaces. It also gives methods to compute the various quantities that are introduced, and finally shows links via skew projections and the commutant lifting theorem to a Bezout equation. In the final section the maximally stabilizing controller is defined as the controller that maximizes the minimum angle between the graph space of the system and the controller. It is shown that this controller can be calculated as a Nehari extension of the coprime factors of the system.

For the sake of simplicity all systems are assumed to be linear and finite dimensional. Most of the results in this paper could be extended to nonrational transfer functions.

The paper is based on an earlier departmental report [10]. After submission of this paper a stability condition equivalent to Theorem 4.5 (S7) appeared in [1].

2. Notation

The notation used throughout this paper is standard in the control literature, [4]. For a matrix $M \in \mathbb{R}^{p\times m}$ or $\mathbb{C}^{p\times m}$, M^{T} denotes its transpose, M^* denotes its conjugate transpose, $\sigma_{\max}(M)$ denotes its maximum singular value, σ_i its i-th singular value and $\sigma_{\min}(M)$ its minimum singular value.

The Hardy spaces $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$, consist of all p vector valued functions square-integrable on the imaginary axis with analytic continuation into the right and left half-plane respectively. The Hilbert space $\mathscr{L}_2^p$ is given by $\mathscr{L}_2^p = \mathscr{H}_2^p \oplus (\mathscr{H}_2^p)^\perp$, and the orthogonal projections P_+ and P_- map $\mathscr{L}_2^p$ onto $\mathscr{H}_2^p$ and $(\mathscr{H}_2^p)^\perp$ respectively. The norm of a function $f \in \mathscr{H}_2^p$ is denoted $\| f \|_2$. The Hardy space $\mathscr{H}_\infty^{p\times m}$ consists of all $p\times m$ bounded functions on the imaginary axis with analytic continuation in the right-half plane and is a subspace of $\mathscr{L}_\infty^{p\times m}$ of all $p\times m$ bounded functions on the imaginary axis. Clearly these functions all have finite $\mathscr{L}_\infty$-norm defined by $\| G \|_\infty := \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}[G(j\omega)]$ and a minimum value on the imaginary axis defined by $\tau(G) := \operatorname{ess\,inf}_{\omega\in\mathbb{R}} \sigma_{\min}[G(j\omega)]$. For a system G, G^* denotes its complex conjugate transpose, i.e. $G(s)^* = \overline{G(-s)}^{\mathrm{T}}$. The symbol $\mathscr{R}\mathscr{H}_2^p$ denotes the subspace of $\mathscr{H}_2^p$ containing the real rational functions, similar definitions apply to the other spaces.

The domain and range of an operator Z is denoted by $\mathscr{D}(Z)$ and $\mathscr{R}(Z)$ respectively. The orthogonal projection operator onto a closed space, $\mathscr{A}$ of $\mathscr{L}_2^p$ is denoted by $P_{\mathscr{A}}$. Given a $p\times m$ symbol G the multiplication operator $M_G : \mathscr{D}(M_G) \to \mathscr{H}_2^m$ is defined by $f \mapsto Gf$. If $G \in \mathscr{L}_\infty^{p\times m}$ the Laurent operator $L_G : \mathscr{L}_2^m \to \mathscr{L}_2^p$, the Hankel operator $H_G : \mathscr{H}_2^m \to (\mathscr{H}_2^p)^\perp$ and the Toeplitz operator $T_G : \mathscr{H}_2^m \to \mathscr{H}_2^p$ with symbol G are defined by $f \mapsto Gf$, $f \mapsto P_{(\mathscr{H}_2^p)^\perp} Gf$ and $f \mapsto P_{\mathscr{H}_2^p} Gf$ respectively.

3. Graph spaces and their related projections

First the notion of the graph of an operator acting on a Hilbert space is introduced. This is the space of all ordered input, output pairs that can be generated by the operator. In [2] the orthogonal projections onto the graph of this operator, and onto its orthogonal complement are expressed in terms of the original operator. Since the application of these results to control problems is of interest here, these results are specialized and simplified to the multiplication operator, with symbol G, which represents the transfer function of a system.

Consider two Hilbert spaces X, Y and a closed linear operator $A: X \to Y$ then,

Definition 3.1. The graph $\mathscr{G}(A)$ of the operator $A: X \to Y$ with domain $\mathscr{D}(A)$ is the totality of all ordered pairs $\{(Ax, x);\ x \in \mathscr{D}(A)\}$ considered as a linear subspace of the Hilbert space $Y \times X$ with the naturally defined inner product.

A notation which will be used later is that of the *transposed graph* $\mathscr{G}^{\mathrm{T}}(A)$ of an operator A. If $\mathscr{G}(A)$ is the graph of the operator A then $\mathscr{G}^{\mathrm{T}}(A) := \{(x, y);\ (y, x) \in \mathscr{G}(A)\}$. The rest of the discussion will be devoted to the multiplication operator with symbol G, a $p \times m$ transfer function; that is the operator $M_G: \mathscr{H}_2^m \to \mathscr{H}_2^p$, where $M_G: f \mapsto Gf$. Clearly the domain $\mathscr{D}(M_G)$ is not the whole space $\mathscr{H}_2^m$ if G is not in $\mathscr{H}_\infty$.

An important object in the study of the graph of the operator M_G will be the so-called coprime factorization of the function G.

Definition 3.2. The pair (M, N), where $M, N \in \mathscr{R}\mathscr{H}_\infty$ constitutes a *right coprime factorization* (r.c.f.) of G (similarly, the pair $(\tilde{N}, \tilde{M})$ where $\tilde{N}, \tilde{M} \in \mathscr{R}\mathscr{H}_\infty$, is a *left coprime factorization* (l.c.f) of G) if

(a) M $(\tilde{M})$ is square and $\det(M(\infty)) \neq 0$ $(\det(\tilde{M}(\infty)) \neq 0)$.

(b) $G = NM^{-1}$ $(G = \tilde{M}^{-1}\tilde{N})$.

(c) N and M are *right coprime*, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathscr{R}\mathscr{H}_\infty$ such that $\tilde{X}N - \tilde{Y}M = I$ ($\tilde{N}$ and $\tilde{M}$ are *left coprime*, i.e. there exist $X, Y \in \mathscr{R}\mathscr{H}_\infty$ such that $\tilde{N}X - \tilde{M}Y = I$).

In Proposition 3.5 an expression for the graph of M_G will be given in terms of the coprime factors of G, but first the following technical lemmas will be stated.

Lemma 3.3. *Let (N, M) be a right coprime factorization of a $p \times m$ transfer function G and $(\tilde{N}, \tilde{M})$ a left coprime factorization. If (U, V) is a right coprime factorization of a $m \times p$ transfer function K then we have the following identities for the operators acting between $\mathscr{H}_2^p$ and $\mathscr{H}_2^m$:*

(a) $T_{[N^* \;\; M^*]} T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} = T^*_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} T^*_{[\tilde{M} \;\; -\tilde{N}]} = 0,$

(b) $T_{[V^* \;\; U^*]} T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} = T^*_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} T^*_{[\tilde{M} \;\; -\tilde{N}]} = T_{(\tilde{M}V - \tilde{N}U)^*}.$

Proof. This follows from noticing that if the $p \times m$ transfer function A is such that $A^* \in \mathscr{R}\mathscr{H}_\infty$, then given any $x \in \mathscr{H}_2^m$ and $y \in [\mathscr{H}_2^m]^\perp$,

$$P_+A(x+y) = P_+Ax + P_+Ay = P_+Ax = P_+AP_+(x+y)$$

as $Ay \in [\mathscr{H}_2^p]^\perp$. To show (a) also use the fact that $M^*\tilde{N}^* - N^*\tilde{M}^* = 0$. $\square$

Lemma 3.4 (see e.g. [17]). *If the $p \times m$ transfer function G has r.c.f. (N, M) and l.c.f. $(\tilde{N}, \tilde{M})$, then the operators*

$$T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} : \mathscr{H}_2^p \to \mathscr{H}_2^p, \qquad T_{[N^* \;\; M^*]} T^*_{[N^* \;\; M^*]} : \mathscr{H}_2^m \to \mathscr{H}_2^m$$

have bounded inverses.

It is now possible to state a number of results concerning the graph of the multiplication operator with symbol G and the orthogonal projections onto these graph spaces.

Proposition 3.5. *For the transfer function G, with right coprime factorization (N, M) and left coprime factorization $(\tilde{N}, \tilde{M})$ we have*

(a) $\mathscr{G}(M_G) = \mathscr{R}(T^*_{[N^* \;\; M^*]})$, $\quad \mathscr{G}(M_G)^\perp = \mathscr{R}\left(T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\right)$,

(b) $\operatorname{Ker}(T^*_{[N^* \;\; M^*]}) = \{0\}$, $\quad \operatorname{Ker}\left(T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\right) = \{0\}$.

Denote by $P_{\mathscr{G}(M_G)} : \mathscr{H}_2^p \times \mathscr{H}_2^m \to \mathscr{H}_2^p \times \mathscr{H}_2^m$ the orthogonal projection onto the closed subspace $\mathscr{G}(M_G)$, and by $P_{G(M_G)^\perp} : \mathscr{H}_2^p \times \mathscr{H}_2^m \to \mathscr{H}_2^p \times \mathscr{H}_2^m$ the orthogonal projection onto its orthogonal complement. Then

$$P_{\mathscr{G}(M_G)} = T^*_{[N^* \;\; M^*]}\left[T_{[N^* \;\; M^*]} T^*_{[N^* \;\; M^*]}\right]^{-1} T_{[N^* \;\; M^*]},$$

$$P_{\mathscr{G}(M_G)^\perp} = T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\left[T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\right]^{-1} T^*_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}.$$

Proof. See Vidyasagar [14] and Cordes and Labrousse [2]. □

Remark 3.6. The previous result can easily be adapted to give expressions for the transposed graph rather than the graph of a multiplication operator.

4. Graphs of linear systems and stability theorems

A basic result (see e.g. [14]) is that checking for closed-loop stability of a control system is equivalent to checking that there exit coprime factors of the system and controller transfer functions that satisfy the Bezout identity. This result is translated in this section into the framework that was developed in the previous section. It is then possible to obtain new criteria for the stability of a closed-loop system.

In the first definition the notion of closed-loop stability, or more precisely internal stability, is recalled. Given a $p \times m$ system and a $m \times p$ controller with transfer functions G and K respectively:

Definition 4.1. The pair (G, K) is called *internally stable* if and only if

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - GK)^{-1} & -(I - GK)^{-1}G \\ -K(I - GK)^{-1} & (I - KG)^{-1} \end{bmatrix} \in \mathscr{H}_\infty^{(p+m)\times(p+m)}.$$

In the next proposition some standard criteria for closed-loop stability are reviewed.

Proposition 4.2 (see e.g. [14]). *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$ respectively and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$. Then the following statements are equivalent:*

(S0) *The pair (G, K) is internally stable.*

(S1) $\begin{bmatrix} V & N \\ U & M \end{bmatrix}^{-1} \in \mathscr{R}\mathscr{H}_\infty$.

(S2) $\begin{bmatrix} -\tilde{M} & \tilde{N} \\ \tilde{U} & -\tilde{V} \end{bmatrix}^{-1} \in \mathscr{R}\mathscr{H}_\infty$.

(S3) $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathscr{R}\mathscr{H}_\infty$.

(S4) $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathscr{R}\mathscr{H}_\infty$.

The following theorem proves that a closed-loop system is internally stable if and only if the graph of the system and the transposed graph of the controller span the whole space of signals. This result is stated in terms of the coprime factors; similar results appear in [13] in the context of stabilization of non-linear systems. An independent proof is given here.

Theorem 4.3. *Given the assumptions of Proposition* 4.2, *the following statements are equivalent*:

(S0) *The pair* (G, K) *is internally stable.*

(S5) $$\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m + \begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p = \begin{bmatrix} \mathscr{H}_2^p \\ \mathscr{H}_2^m \end{bmatrix}.$$

Proof. (S0) $\Rightarrow$ (S5): This follows immediately since by assumption the matrix $[\begin{smallmatrix} N & V \\ M & U \end{smallmatrix}]$ is invertible in $\mathscr{R}\mathscr{H}_\infty$. To show the reverse direction first note that if $[\tilde{U}\ \ \tilde{V}]$ is a left coprime factorization of K then (S5) implies

$$M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} \mathscr{H}_2^p \\ \mathscr{H}_2^m \end{bmatrix}\right) = M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m + \begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right) = M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right).$$

As $[-\tilde{U}, \tilde{V}]$ are left coprime there exist $X, Y \in \mathscr{R}\mathscr{H}_\infty$ such that $\tilde{V}Y - \tilde{U}X = I$. Hence

$$M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right) = M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} \mathscr{H}_2^p \\ \mathscr{H}_2^m \end{bmatrix}\right) \supseteq M_{[-\tilde{U}\ \ \tilde{V}]}\left(\begin{bmatrix} X \\ Y \end{bmatrix} \mathscr{H}_2^m\right) = \mathscr{H}_2^m.$$

This implies $(\tilde{V}M - \tilde{U}N)\mathscr{H}_2^m = \mathscr{H}_2^m$ and therefore $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathscr{R}\mathscr{H}_\infty$ [4, Lemma 1, p. 120], implying internal stability. □

This theorem is the basis for our later developments since it shows how to state a stability criterion using the language of graphs of linear systems. The following result provides the key to the derivation of further stability criteria from the previous result.

Lemma 4.4 ([9], p. 201). *Let H be a Hilbert space and let A, B be closed subspaces of H. Denote the orthogonal projection operator onto the space A as $P_A : H \to A$ and use analogous notation for the similar operations onto the subspace B. Then the following statements are equivalent*:
(i) $P_B A = B$; (ii) $H = B^\perp + A$; (iii) $\| P_{A^\perp} P_B \| < 1$.

It is now possible to combine Theorem 4.3 and Lemma 4.4 to give the main result of this section. The following two operators $C_1 : \mathscr{H}_2^p \to \mathscr{H}_2^p$ and $C_0 : \mathscr{H}_2^m \to \mathscr{H}_2^m$ are of importance in this development:

$$C_1 : f \mapsto \left[T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \right]^{-1/2} T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \left(T^*_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \right)^{-1/2},$$

$$C_0 : f \mapsto \left(T_{[N*\ \ M*]} T^*_{[N*\ \ M*]} \right)^{-1/2} T_{[N*\ \ M*]} T^*_{[V*\ \ U*]} \left(T_{[V*\ \ U*]} T^*_{[V*\ \ U*]} \right)^{-1/2}$$

Theorem 4.5. *Given the assumptions of Proposition* 4.2, *the following statements are equivalent*:

(S0) *The pair* (G, K) *is internally stable.*

(S6) $$P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathscr{H}_2^p\right)^\perp}\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right) = \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right)^\perp.$$

(S7) $$\left\| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \mathscr{H}_2^m\right)^\perp} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathscr{H}_2^p\right)^\perp} \right\| < 1.$$

(S8) $\| C_1 \| < 1.$

Moreover,

$$\|C_1\| = \left\| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp} \right\|.$$

Proof. (S0) $\Leftrightarrow$ (S6) and (S0) $\Leftrightarrow$ (S7) follow immediately from Theorem 4.3 and Lemma 4.4.

(S7) $\Leftrightarrow$ (S8): By applying Proposition 3.5 and noticing that the operator $\mathcal{H}_2^p \to \mathcal{H}_2^p \times \mathcal{H}_2^m$,

$$f \mapsto T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \left[T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \right]^{-1/2} f,$$

and the operator $\mathcal{H}_2^m \to \mathcal{H}_2^p \times \mathcal{H}_2^m$,

$$f \mapsto T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \left[T^*_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \right]^{-1/2} f,$$

are isometries, the claim can be established since

$$\begin{aligned}
&\left\| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp} \right\| \\
&= \left\| T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \left[T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \right]^{-1} T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \left[T^*_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \right]^{-1} T^*_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \right\| \\
&= \left\| \left[T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} \right]^{-1/2} T^*_{\left[\begin{smallmatrix} \tilde{M}* \\ -\tilde{N}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \left[T^*_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} T_{\left[\begin{smallmatrix} -\tilde{U}* \\ \tilde{V}* \end{smallmatrix}\right]} \right]^{-1/2} \right\| \\
&= \|C_1\|. \qquad \square
\end{aligned}$$

The final result in this section establishes a connection between the graph space of the system G and the controller K, and the orthogonal complements of these spaces. This result is central to the proof of Theorem 5.7 which is one of the main results of this paper.

Proposition 4.6. *Given the assumptions of Proposition* 4.2, *if*

$$\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p + \begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m = \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix} \tag{1}$$

then

$$\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp + \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)^\perp = \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix} \tag{2}$$

or equivalently if $\|C_1\| < 1$ *then* $\|C_0\| < 1$. *Moreover,*

$$\|C_0\| = \left\| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}_2^m\right)} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)} \right\|.$$

Proof. Showing that (1) implies (2), is by Lemma 4.4 equivalent to showing that internal stability implies

$$P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)^\perp\right) = \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p.$$

Now

$$P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m\right)^{\perp}\right) = T_{\begin{bmatrix} V \\ U \end{bmatrix}}\left(T^*_{\begin{bmatrix} V \\ U \end{bmatrix}}T_{\begin{bmatrix} V \\ U \end{bmatrix}}\right)^{-1}T^*_{\begin{bmatrix} V \\ U \end{bmatrix}}T_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}(\mathscr{H}_2^p)$$

$$= T_{\begin{bmatrix} V \\ U \end{bmatrix}}\left(T^*_{\begin{bmatrix} V \\ U \end{bmatrix}}T_{\begin{bmatrix} V \\ U \end{bmatrix}}\right)^{-1}T_{(V^*\tilde{M}^*-U^*\tilde{N}^*)}(\mathscr{H}_2^p).$$

By Lemma 3.4 the operator $T_{[V^*\ U^*]}T^*_{[V^*\ U^*]}$ has bounded inverse; it therefore remains to show that the operator $T_{(V^*\tilde{M}^*-U^*\tilde{N}^*)}: \mathscr{H}_2^p \to \mathscr{H}_2^p$ is bijective. By the assumption of internal stability the Toeplitz operator with symbol $(\tilde{M}V - \tilde{N}U)$ is bijective. But this implies that its adjoint operator $T_{V^*\tilde{M}^*-U^*\tilde{N}^*}$ is also bijective [16]. The final claim is verified in a similar way to the analogous result in Theorem 4.5. That $\| C_1 \| < 1$ implies that $\| C_0 \| < 1$ now follows from the first part of this proposition together with Lemma 4.4. □

5. Geometric Interpretation

It is possible to place the results in the previous section into a geometrical framework. This will allow the norm of the operator C_1, defined in the previous section, to be related to the norm of the operator C_0.

First it is necessary to recall the notion of the minimal angle between two subspaces of a Hilbert space.

Definition 5.1. The minimal angle, $\theta_{\min} \in [0, \pi]$, between two closed subspaces A and B of a Hilbert space H is given by

$$\cos \theta_{\min}(A, B) = \sup_{u \in A, v \in B} \frac{|\langle u, v \rangle|}{\| u \| \, \| v \|}.$$

The distance between a point u of a Hilbert space and a subspace B of the space H is defined by $\mathrm{dist}(u, B) = \inf_{v \in B} \| u - v \|$. The following proposition summarizes the connection between the cosine of the minimal angle to norms of projections and a distance formula.

Proposition 5.2 (see e.g. [7]). *Given the closed subspaces A, B of a Hilbert space H,*

$$\cos \theta_{\min}(A, B) = \| P_A P_B \| = \| P_B P_A \| = \sup_{u \in B, \|u\| = 1} \mathrm{dist}(u, A^{\perp}).$$

It is now clear that we can rewrite the stability condition of Theorem 4.5 in terms of geometric notions to obtain a new stability criterion.

Theorem 5.3. *Given the assumptions of Proposition* 4.2 *then the following statements are equivalent*:

(S0) *The pair (G, K) is internally stable.*

(S9) $\theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m\right)^{\perp}, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^{\perp}\right) > 0.$

As a next step we are going to define a metric, the so-called *gap metric* on the set of subspaces of a Hilbert space H.

Definition 5.4. The gap between two closed subspaces A and B of a Hilbert space H is defined as

$$\mathrm{gap}(A, B) := \| P_A - P_B \| = \| P_{A^{\perp}} - P_{B^{\perp}} \|.$$

The gap between two subspaces can be related to the notions introduced before.

Theorem 5.5 (see e.g. [16]). *Let A, B be closed subspaces of a Hilbert space H. Then*

$$\operatorname{gap}(A, B) = \max\{ \| P_A P_{B^\perp} \|, \| P_{A^\perp} P_B \| \} = \max\{\cos\theta_{\min}(A, B^\perp), \cos\theta_{\min}(B, A^\perp)\}$$
$$= \max\Big\{ \sup_{u\in A, \|u\|=1} \operatorname{dist}(u, B), \sup_{v\in B, \|v\|=1} \operatorname{dist}(v, A)\Big\}.$$

If the gap between two subspaces is less than one then we have the following simplified situation.

Proposition 5.6 (see e.g. [16]). *Let A, B be closed subspaces of a Hilbert space H and assume that* $\operatorname{gap}(A, B) < 1$. *Then* $\operatorname{gap}(A, B) = \| P_A P_{B^\perp} \| = \| P_{A^\perp} P_B \|$.

The various results can be summarized to obtain another criterion in terms of the gap of two subspaces. Proposition 4.6 is of importance here in showing that the gap between the two relevant spaces can be expressed as the norm of one operator, in contrast to Theorem 5.5 which would suggest that two norms have to be checked.

Theorem 5.7. *Given the assumptions of Proposition* 4.2 *then the following statements are equivalent*:

(S0) *The pair* (G, K) *is internally stable.*

(S10) $\operatorname{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) < 1.$

Moreover,

$$\operatorname{gap}\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) = \left\| P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathscr{H}_2^m\right)^\perp} P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2^p\right)^\perp} \right\|.$$

Proof. Let $A := [N^{\mathrm T}\ M^{\mathrm T}]^{\mathrm T}\mathscr{H}_2^m$, $B := [V^{\mathrm T}\ U^{\mathrm T}]^{\mathrm T}\mathscr{H}_2^p$. If $\operatorname{gap}(A, B^\perp) < 1$ this implies by Theorem 5.5 that $\| P_{A^\perp} P_{B^\perp} \| < 1$ which implies stability. Conversely assume stability and therefore that $\| P_{A^\perp} P_{B^\perp} \| < 1$. Then by Proposition 4.6, $\| P_A P_B \| < 1$ which implies by Theorem 5.5 that $\operatorname{gap}(A, B^\perp) < 1$. The final claim follows immediately as it has been shown that $\| P_{A^\perp} P_{B^\perp} \| < 1$ if and only if $\operatorname{gap}(A, B^\perp) < 1$ and therefore by Proposition 5.6, $\| P_{A^\perp} P_{B^\perp} \| = \operatorname{gap}(A, B^\perp)$. □

In the following corollary the previous theorem is applied to questions of robust control.

Corollary 5.8. *Let (G, K) be an internally stable control system with*

$$\varepsilon := \varepsilon(G, K) := 1 - \operatorname{gap}\big(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm T}(M_K)]^\perp\big).$$

Then the controller K also stabilizes the plant G_Δ if $\operatorname{gap}(\mathscr{G}(M_G), \mathscr{G}(M_{G_\Delta})) < \varepsilon(G, K)$.

Proof. Clearly

$$\operatorname{gap}\big(\mathscr{G}(M_{G_\Delta}), [\mathscr{G}(M_K)]^\perp\big) \le \operatorname{gap}\big(\mathscr{G}(M_{G_\Delta}), \mathscr{G}(M_G)\big) + \operatorname{gap}\big(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm T}(M_K)]^\perp\big)$$
$$< \varepsilon + (1 - \varepsilon) = 1,$$

and hence the perturbed plant is stabilized by the controller K. □

In order to develop a method to calculate the various quantities which have been introduced, recall the definition of a normalized coprime factorization. A right (left) coprime factorization $G = NM^{-1}$ ($G = \tilde M^{-1}\tilde N$) is called *normalized* if $N^*N + M^*M = I$ ($\tilde N\tilde N^* + \tilde M\tilde M^* = I$).

Proposition 5.9. *Let K be a stabilizing controller for the plant G. If (N, M) are normalized right coprime factors of G and (U, V) are normalized right coprime factors for K, then*

$$\cos\theta_{\min}\left(\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m\right)^\perp, \left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp\right) = \left\|P_{\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m\right)^\perp} P_{\left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp}\right\| = \operatorname{gap}\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp\right)$$
$$= \|N^*V + M^*U\|_\infty.$$

Proof. Since (G, K) is stable we know by Proposition 5.2 and Proposition 5.6 that

$$\cos\theta_{\min}\left(\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m\right)^\perp, \left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp\right)$$
$$= \left\|P_{\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m\right)^\perp} P_{\left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp}\right\| = \operatorname{gap}\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m, \left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)^\perp\right)$$
$$= \left\|P_{\left(\begin{bmatrix}N\\M\end{bmatrix}\mathscr{H}_2^m\right)} P_{\left(\begin{bmatrix}V\\U\end{bmatrix}\mathscr{H}_2^p\right)}\right\| = \|T_{(N^*V+M^*U)}\| = \|N^*V + M^*U\|_\infty. \qquad \square$$

The results of the previous section can also be derived using an approach that is based on the commutant lifting theorem. There is an interesting connection between the angle of orthogonal complements of shift invariant subspaces and the Bezout equation. To state the result it is necessary to define the notion of a skew projection. If $B^\perp \cap A = \{0\}$, then define the skew projection $P_{A\|B^\perp}: A + B^\perp \to A$ as $P_{A\|B^\perp}: u + v \mapsto u,\ u \in A,\ v \in B^\perp$.

Proposition 5.10. (see e.g. [9]). *Given two closed subspaces A and B of a Hilbert space H where $B^\perp \cap A = \{0\}$ and $H = A + B^\perp$,*

$$\sin\theta_{\min}(A, B^\perp) = \|P_{A\|B^\perp}\|^{-1} = \|P_{B\|A^\perp}\|^{-1} = \sin\theta_{\min}(A^\perp, B).$$

The following result is based on the commutant lifting theorem, and can be found for example in [12]. For a similar result see [5].

Proposition 5.11. *Let $\theta_1 \in \mathscr{H}_\infty^{(p+m)\times p}$ and $\theta_2 \in \mathscr{H}_\infty^{(p+m)\times m}$ be inner functions, i.e. $\theta_i^*\theta_i = I$, $i = 1, 2$. Then:*

(a) The angle between $\mathscr{K}_{\theta_1} = \mathscr{H}_2^{(p+m)} \ominus \theta_1\mathscr{H}_2^p$ and $\mathscr{K}_{\theta_2} = \mathscr{H}_2^{(p+m)} \ominus \theta_2\mathscr{H}_2^m$ is nonzero if and only if there exist functions $\psi_1 \in \mathscr{H}_\infty^{p\times(p+m)}$ and $\psi_2 \in \mathscr{H}_\infty^{m\times(p+m)}$ such that $\theta_1\psi_1 + \theta_2\psi_2 = I$.

(b) The norm of the skew projection, $P_{\mathscr{K}_{\theta_2}\|\mathscr{K}_{\theta_1}}$, of $\mathscr{H}_2^{(p+m)}$ onto $\mathscr{K}_{\theta_2}$ with kernel $\mathscr{K}_{\theta_1}$ is equal to the least possible norm of such ψ_1 (or indifferently ψ_2).

The following proposition shows how a Bezout equation of the type that appeared in the previous proposition is related to stability problems of control systems.

Theorem 5.12. *Let (G, K) be a control system and let (N, M) be a normalized right coprime factorization of the $p \times m$ plant G and (U, V) a normalized right coprime factorization of the $m \times p$ controller K. Also let $(\tilde{N}, \tilde{M})$ be a normalized left coprime factorization of G and $(\tilde{U}, \tilde{V})$ a normalized left coprime factorization of K. Then the following statements are equivalent:*

(S0) *The pair (G, K) is internally stable.*

(S11) *The Bezout equation $I = \begin{bmatrix}N\\M\end{bmatrix}\psi_1 + \begin{bmatrix}V\\U\end{bmatrix}\psi_2$ has a solution ψ_1, ψ_2 in $\mathscr{H}_\infty$.*

Moreover if one of the two conditions is satisfied the solution to the Bezout equation is unique and given by

$$\psi_1 = (\tilde{V}M - \tilde{U}N)^{-1}[-\tilde{U} \quad \tilde{V}], \qquad \psi_2 = (\tilde{M}V - \tilde{N}U)^{-1}[\tilde{M} \quad -\tilde{N}]. \tag{3}$$

Proof. Let $\theta_1 = [N^T, M^T]^T$ and $\theta_2 = [V^T, U^T]^T$ and assume that (G, K) is internally stable which implies that the matrix

$$[\theta_1 \quad \theta_2] = \begin{bmatrix} N & V \\ M & U \end{bmatrix}$$

is invertible in $\mathscr{H}_\infty^{(p+m)\times(p+m)}$, with inverse

$$\begin{bmatrix} (\tilde{V}M - \tilde{U}N)^{-1}[-\tilde{U} \quad V] \\ (\tilde{M}V - \tilde{N}U)^{-1}[\tilde{M} \quad -\tilde{N}] \end{bmatrix}.$$

Applying this inverse to the left of the Bezout equation shows that the unique solution to the equation is given by (3).

To show the converse assume that the equation $I = \theta_1\psi_1 + \theta_2\psi_2$ has a solution ψ_1, ψ_2 in $\mathscr{H}_\infty$. Now premultiply this equation with the matrix $\begin{bmatrix} -\tilde{U} & \tilde{V} \\ \tilde{M} & -\tilde{N} \end{bmatrix}$. From this result, $[-\tilde{U} \ \tilde{V}] = (\tilde{V}M - \tilde{U}N)\psi_1$. Since $(\tilde{U}, \tilde{V})$ are left coprime there exists functions (X, Y) of appropriate dimensions in $\mathscr{H}_\infty$ such that

$$I = -\tilde{U}X + \tilde{V}Y = (\tilde{V}M - \tilde{U}N)\psi_1 \begin{bmatrix} X \\ Y \end{bmatrix}.$$

But this equation implies that $(\tilde{V}M - \tilde{U}N)$ is invertible in $\mathscr{H}_\infty$ which implies the internal stability of (G, K). □

Note that combining the previous propositions we obtain an independent proof of Theorem 5.3. As a corollary to the previous two propositions we obtain another way of calculating the minimal angle between the two spaces that are of importance in assessing the stability of a control system.

Corollary 5.13. *Assume the notation of Theorem* 5.12. *If the pair* (G, K) *is internally stable then*

$$\sin \theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right)^{\perp}, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right)^{\perp}\right) = \tau(\tilde{M}V - \tilde{N}U) = \tau(\tilde{V}M - \tilde{U}N),$$

where for $F \in \mathscr{L}_\infty$ *we set* $\tau(F) = \operatorname{ess\,inf}\{\sigma_{\min}(F(s)) \mid \operatorname{Re}(s) = 0\}$.

Proof. The unique solution to the Bezout equation is given by (3). Since $(\tilde{U}, \tilde{V})$ and $(\tilde{N}, \tilde{M})$ are normalized the norm of ψ_1 and ψ_2 is given by $\| \psi_1 \| = \| \psi_2 \| = \|(\tilde{M}V - \tilde{N}U)^{-1}\|_\infty = \|(\tilde{V}M - \tilde{U}N)^{-1}\|_\infty$. Therefore we have by Proposition 5.11 that

$$\sin \theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right)^{\perp}, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right)^{\perp}\right) = \left\| P_{\left[\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right]^{\perp} \| \left[\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right]^{\perp}} \right\|^{-1} = \left(\|(\tilde{M}V - \tilde{N}U)^{-1}\|_\infty \right)^{-1}$$

$$= \tau(\tilde{M}V - \tilde{N}U) = \tau(\tilde{V}M - \tilde{U}N). \qquad □$$

Note that the expression for

$$\sin \theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right)^{\perp}, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right)^{\perp}\right)$$

is consistent with the expression for

$$\cos \theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathscr{H}_2^m\right)^{\perp}, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathscr{H}_2^p\right)^{\perp}\right).$$

For if

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} := \begin{bmatrix} M^*U+N^*V \\ -\tilde{N}U+\tilde{M}V \end{bmatrix} = \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix},$$

then

$$Z_1^*Z_1 + Z_2^*Z_2 = [U^* \quad V^*] \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix}^* \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = [U^* \quad V^*] \begin{bmatrix} U \\ V \end{bmatrix} = I.$$

Therefore $\sigma_{\max}^2(Z_1(j\omega)) + \sigma_{\min}^2(Z_2(j\omega)) = 1$, $\omega \in \mathbb{R}$, and hence

$$\sin^2\theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)^\perp, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp\right) + \cos^2\theta_{\min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right)^\perp, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp\right)$$
$$= \tau^2(\tilde{M}V - \tilde{N}U) + \| N^*V + M^*U \|_\infty^2 = \tau^2(Z_2) + \| Z_1 \|_\infty^2 = 1,$$

as would be hoped.

6. Maximally stabilizing controllers

In the previous section it was shown that the closed-loop system is stable if the gap between the graph of the system and the orthogonal complement of the transposed graph of the controller is less than one. This characterization of stability was given a geometrical interpretation, in terms of the angles between these graph subspaces. The gap is therefore a measure of how close the closed-loop system is to instability, and is consequently an indicator of the degree of stability of the closed-loop system. It is therefore desirable to minimize this measure. The maximally stabilizing controller is defined as the controller which minimizes the gap between the orthogonal complement of its transposed graph and the graph of the system, and therefore maximizes the minimum angle between graph spaces.

Definition 6.1. Given a $p \times m$ system G, the optimal minimal angle $\theta_{\min}^{\text{opt}}$ is defined by

$$\cos \theta_{\min}^{\text{opt}} := \inf_K \operatorname{gap}\left(\mathcal{G}(M_G), [\mathcal{G}^{\mathrm{T}}(M_K)]^\perp\right).$$

Further a controller K achieving this infimum is called a maximally stabilizing controller.

It is possible to calculate such a controller by relating the calculation of the gap to an $\mathcal{H}_\infty$ optimization problem. The following technical lemma will be required. The notation of $X_{\mathrm{R}}^{1/2} \in \mathcal{RH}_\infty$ is used to denote the right spectral factor of a transfer function $X = X^*$, X, $X^{-1} \in \mathcal{RL}_\infty$, such that $X = (X_{\mathrm{R}}^{1/2})^* X_{\mathrm{R}}^{1/2}$.

Lemma 6.2. *Given a $p \times m$ transfer function G with normalized r.c.f. (N, M) and l.c.f. $(\tilde{N}, \tilde{M})$, and a controller K with r.c.f. (U, V), (not necessarily normalized). Then if*

$$\varepsilon := \left\|(N^*V + M^*U)(U^*U + V^*V)_{\mathrm{R}}^{1/2}\right\|_\infty,$$

this implies that

$$\|(N^*V + M^*U)(\tilde{M}V - \tilde{N}U)^{-1}\|_\infty = \varepsilon/(1-\varepsilon^2)^{1/2}.$$

Proof. The following identity will be used in the proof:

$$[V^* \quad U^*] \begin{bmatrix} N & \tilde{M}^* \\ M & -\tilde{N}^* \end{bmatrix} \begin{bmatrix} N^* & M^* \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} = U^*U + V^*V$$

which holds as the transfer function $\left[\begin{smallmatrix} N & \tilde{M}^* \\ M & -\tilde{N}^* \end{smallmatrix}\right]$ is all-pass. Therefore,

$$(N^*V+M^*U)^*(N^*V+M^*U)+(\tilde{M}V-\tilde{N}U)^*(\tilde{M}V-\tilde{N}U)=U^*U+V^*V. \tag{4}$$

By dividing the above expression on the right by the right spectral factor of (U^*U+V^*V) and on the left by its complex conjugate, it is shown that

$$\|(N^*V+M^*U)(U^*U+V^*V)_{\mathrm{R}}^{-1/2}\|_\infty=\left(1-\tau^2\left((\tilde{M}V-\tilde{N}U)(U^*U+V^*V)_{\mathrm{R}}^{-1/2}\right)\right)^{1/2}$$
$$=\left(1-\left\|(U^*U+V^*V)_{\mathrm{R}}^{1/2}(\tilde{M}V-\tilde{N}U)^{-1}\right\|_\infty^{-2}\right)^{1/2}.$$

Dividing the same expression on the right by $(\tilde{M}V-\tilde{N}U)^{-1}$ and on the left by its complex conjugate it follows that

$$(\tilde{M}V-\tilde{N}U)^{-*}(N^*V+M^*U)^*(N^*V+M^*U)(\tilde{M}V-\tilde{N}U)^{-1}+1$$
$$=(\tilde{M}V-\tilde{N}U)^{-*}\left((U^*U+V^*V)_{\mathrm{R}}^{1/2}\right)^*(U^*U+V^*V)_{\mathrm{R}}^{1/2}(\tilde{M}V-\tilde{N}U)^{-1}.$$

This implies that

$$\left\|(U^*U+V^*V)_{\mathrm{R}}^{1/2}(\tilde{M}V-\tilde{N}U)^{-1}\right\|_\infty^2=1+\|(N^*V+M^*U)(\tilde{M}V-\tilde{N}U)^{-1}\|_\infty^2.$$

Substituting this into the above equation gives

$$\|(N^*V+M^*U)(U^*U+V^*V)_{\mathrm{R}}^{-1/2}\|_\infty=\left(1-\left(1+\|(N^*V+M^*U)(\tilde{M}V-\tilde{N}U)^{-1}\|_\infty^2\right)^{-1}\right)^{1/2}.$$

The proof is completed by substituting $\|(N^*V+M^*U)(\tilde{M}V-\tilde{N}U)^{-1}\|_\infty=\varepsilon/(1-\varepsilon^2)^{1/2}$ into the above equation. □

The following proposition connects the norm of the product of the orthogonal projections operators to an $\mathscr{H}_\infty$-optimization problem and is the central tool in calculating maximally stabilizing controllers.

Proposition 6.3. *Suppose the $p\times m$ transfer function G has a* normalized *r.c.f. (N, M) and a* normalized *l.c.f. $(\tilde{N}, \tilde{M})$. Assume that the $m\times p$ transfer function K has a* normalized *r.c.f. (U, V) and a* normalized *l.c.f. $(\tilde{U}, \tilde{V})$ then,*

(a) $$\left\|P_{\left(\left[\begin{smallmatrix}V\\U\end{smallmatrix}\right]\mathscr{H}_2^p\right)^\perp}P_{\left(\left[\begin{smallmatrix}N\\M\end{smallmatrix}\right]\mathscr{H}_2^m\right)^\perp}\right\|=\inf_{Q\in\mathscr{R}\mathscr{H}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde{M}^*\\-\tilde{N}^*\end{bmatrix}-\begin{bmatrix}V\\U\end{bmatrix}Q\right\|_\infty=\inf_{Q\in\mathscr{R}\mathscr{H}_\infty^{m\times m}}\left\|\begin{bmatrix}-\tilde{U}^*\\\tilde{V}^*\end{bmatrix}-\begin{bmatrix}N\\M\end{bmatrix}Q\right\|_\infty.$$

(b) $$\left\|P_{\left(\left[\begin{smallmatrix}V\\U\end{smallmatrix}\right]\mathscr{H}_2^p\right)}P_{\left(\left[\begin{smallmatrix}N\\M\end{smallmatrix}\right]\mathscr{H}_2^m\right)}\right\|=\inf_{Q\in\mathscr{R}\mathscr{H}_\infty^{p\times p}}\left\|\begin{bmatrix}V\\U\end{bmatrix}-\begin{bmatrix}\tilde{M}^*\\-\tilde{N}^*\end{bmatrix}Q\right\|_\infty=\inf_{Q\in\mathscr{R}\mathscr{H}_\infty^{m\times m}}\left\|\begin{bmatrix}N\\M\end{bmatrix}-\begin{bmatrix}-\tilde{U}^*\\\tilde{V}^*\end{bmatrix}Q\right\|_\infty.$$

(c) $$\left\|P_{\left(\left[\begin{smallmatrix}V\\U\end{smallmatrix}\right]\mathscr{H}_2^p\right)}-P_{\left(\left[\begin{smallmatrix}N\\M\end{smallmatrix}\right]\mathscr{H}_2^m\right)^\perp}\right\|=\inf_{Q\in\mathscr{R}\mathscr{H}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde{M}^*\\-\tilde{N}^*\end{bmatrix}-\begin{bmatrix}V\\U\end{bmatrix}Q\right\|_\infty.$$

Proof. (a) The proof of the first identity can be split into two parts. The first is to assume that

$$\left\|P_{\left(\left[\begin{smallmatrix}V\\U\end{smallmatrix}\right]\mathscr{H}_2^p\right)^\perp}P_{\left(\left[\begin{smallmatrix}N\\M\end{smallmatrix}\right]\mathscr{H}_2^m\right)^\perp}\right\|=1$$

and to show that this implies

$$\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty=1.$$

The second part is to assume that

$$\left\|P_{\left(\begin{bmatrix}V\\ U\end{bmatrix}\mathcal{H}_2^p\right)^\perp}P_{\left(\begin{bmatrix}N\\ M\end{bmatrix}\mathcal{H}_2^m\right)^\perp}\right\|<1$$

and to show that under this assumption the first two expressions are equal.

To prove the first part assume

$$\left\|P_{\left(\begin{bmatrix}V\\ U\end{bmatrix}\mathcal{H}_2^p\right)^\perp}P_{\left(\begin{bmatrix}N\\ M\end{bmatrix}\mathcal{H}_2^m\right)^\perp}\right\|=1.$$

Then by the equivalence of (S4) and (S7), it is clear that $(\tilde MV-\tilde NU)^{-1}\in\mathcal{RH}_\infty$. Now

$$\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty\geq\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|[\tilde M\quad -\tilde N]\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty$$
$$=\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\|I-(\tilde MV-\tilde NU)Q\|_\infty=1$$

as $(\tilde MV-\tilde NU)(s_0)v_0=0$ for some $s_0\in\mathbb{C}$, $\mathrm{Re}(s_0)\geq 0$, and $v_0\in\mathbb{C}^p$. This proves the first part.

To prove the second part, assume

$$\sigma:=\left\|P_{\left(\begin{bmatrix}V\\ U\end{bmatrix}\mathcal{H}_2^p\right)^\perp}P_{\left(\begin{bmatrix}N\\ M\end{bmatrix}\mathcal{H}_2^m\right)^\perp}\right\|<1.$$

Then by the equivalence of (S4) and (S7), it is clear that $(\tilde MV-\tilde NU)^{-1}\in\mathcal{RH}_\infty$ and $\sigma=\|M^*U+N^*V\|_\infty$ by Proposition 5.9. Using $Q=(1-\sigma^2)(\tilde MV-\tilde NU)^{-1}$, see [8, Theorem 4.2], then

$$\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty\leq\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-(1-\sigma^2)\begin{bmatrix}V\\ U\end{bmatrix}(\tilde MV-\tilde NU)^{-1}\right\|_\infty$$
$$=\left\|\begin{bmatrix}\tilde M & -\tilde N\\ N^* & M^*\end{bmatrix}\left(\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-(1-\sigma^2)\begin{bmatrix}V\\ U\end{bmatrix}(\tilde MV-\tilde NU)^{-1}\right)\right\|_\infty$$
$$=\left\|\begin{bmatrix}\sigma^2\\ -(1-\sigma^2)(M^*U+N^*V)(\tilde MV-\tilde NU)^{-1}\end{bmatrix}\right\|_\infty$$

As (U, V) is a normalized coprime factorization and $\sigma=\|M^*U+N^*V\|_\infty$, this implies by Lemma 6.2 that $\|(M^*U+N^*V)(\tilde MV-\tilde NU)^{-1}\|_\infty=\sigma(1-\sigma^2)^{-1/2}$. Hence,

$$\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty\leq\left\|\begin{bmatrix}\sigma^2\\ -(1-\sigma^2)(M^*U+N^*V)(\tilde MV-\tilde NU)^{-1}\end{bmatrix}\right\|_\infty$$
$$=\left(\sigma^4+\sigma^2(1-\sigma^2)\right)^{1/2}=\sigma.$$

Therefore it has been established that

$$\inf_{Q\in\mathcal{RH}_\infty^{p\times p}}\left\|\begin{bmatrix}\tilde M^*\\ -\tilde N^*\end{bmatrix}-\begin{bmatrix}V\\ U\end{bmatrix}Q\right\|_\infty\leq\left\|P_{\left(\begin{bmatrix}V\\ U\end{bmatrix}\mathcal{H}_2^p\right)^\perp}P_{\left(\begin{bmatrix}N\\ M\end{bmatrix}\mathcal{H}_2^m\right)^\perp}\right\| \tag{5}$$

whenever

$$\left\| P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathscr{H}_2^p\right)^\perp} P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \mathscr{H}_2^m\right)^\perp} \right\| < 1. \tag{6}$$

The proof of the reverse inequality is as follows. Let

$$\mu := \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty .$$

It is now argued that $\mu \geq \| M^*V + N^*U \|_\infty$. Establishing this claim completes the proof for by Proposition 5.9,

$$\inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \mu \geq \| M^*V + N^*U \|_\infty = \left\| P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathscr{H}_2^p\right)^\perp} P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \mathscr{H}_2^m\right)^\perp} \right\|$$

under the assumption (6). Now note that

$$\mu = \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} V^* & U^* \\ -\tilde{U} & \tilde{V} \end{bmatrix} \left(\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right) \right\|_\infty$$

$$= \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} V^*\tilde{M} - U^*\tilde{N}^* - Q \\ -(\tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*) \end{bmatrix} \right\|_\infty \geq \| \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^* \|_\infty .$$

To show that $\| \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^* \|_\infty = \| M^*V + N^*V \|_\infty$ construct the all-pass function

$$\begin{bmatrix} N^* & M^* \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} V & -\tilde{U}^* \\ U & \tilde{V}^* \end{bmatrix} = \begin{bmatrix} M^*V + N^*U & M^*\tilde{V}^* - N^*\tilde{U}^* \\ \tilde{M}V - \tilde{N}U & -(\tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^*)^* \end{bmatrix}.$$

Then

$$\| M^*V + N^*U \|_\infty^2 = \| I - (\tilde{M}V - \tilde{N}U)^* (\tilde{M}V - \tilde{N}U) \|_\infty = \| I - (\tilde{M}V - \tilde{N}U)(\tilde{M}V - \tilde{N}U)^* \|_\infty$$
$$= \| \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^* \|_\infty^2 ,$$

as the transfer function $(\tilde{M}V - \tilde{N}U) \in \mathscr{R}\mathscr{H}_\infty^{p\times p}$ is square. This completes the proof of the first equality. The second equality is just the dual of the first and can be proved in an analogous fashion.

(b) These expressions follow immediately, as

$$\inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} Q \right\|_\infty = \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M} & -\tilde{N} \\ N^* & M^* \end{bmatrix} \left(\begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} Q \right) \right\|_\infty$$

$$= \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p\times p}} \left\| \begin{bmatrix} (\tilde{M}V - \tilde{N}U) - Q \\ M^*U + N^*V \end{bmatrix} \right\|_\infty = \| M^*U + N^*V \|_\infty$$

$$= \left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathscr{H}_2^p} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \mathscr{H}_2^m} \right\|$$

from Proposition 5.9 where $Q = \tilde{M}V - \tilde{N}U \in \mathscr{R}\mathscr{H}_\infty$. The second expression in (b) is proved analogously.

(c) This statement is a consequence of (a) and Propositions 5.6 and 4.6. □

It is now possible to answer the question of how to calculate the maximally stabilizing controller. It is also possible to obtain an analytical expression for the optimal minimal angle; see Definition 6.1.

Theorem 6.4. *Suppose the $p \times m$ transfer function G has a* normalized *r.c.f.* (N, M) *and a* normalized *l.c.f.* $(\tilde{N}, \tilde{M})$ *respectively. Then*

$$\cos \theta_{\min}^{\text{opt}} := \inf_K \operatorname{gap}\left(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm{T}}(M_K)]^{\perp}\right) = \sigma_1, \quad \textit{where } \sigma_1 = \left\| H_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right\|.$$

A maximally stabilizing controller exists and each maximally stabilizing controller has a right coprime factorization (U, V) satisfying the extension

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \sigma_1. \tag{7}$$

Proof. In the previous proposition it was shown that for a controller K with right coprime factorization (U, V),

$$\operatorname{gap}\left(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm{T}}(M_K)]^{\perp}\right) = \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty.$$

Since there exists a stabilizing controller for the plant G, the infimum of the above expression over all controllers is less than one. The problem of finding the optimal minimal angle therefore means finding the infimum of the expressions

$$1 > \inf_K \operatorname{gap}\left(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm{T}}(M_K)]^{\perp}\right) = \inf_{\substack{K = UV^{-1} \\ U, V \text{coprime}}} \ \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty.$$

It is claimed that given any $U, V, Q \in \mathscr{R}\mathscr{H}_\infty$ satisfying

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty < 1$$

this implies $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] Q$ is coprime. For

$$1 > \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty \geq \left\| [\tilde{M} \quad -\tilde{N}] \left(\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right) \right\|_\infty = \| I - (\tilde{M}V - \tilde{N}U) Q \|_\infty$$

and therefore $[I - (I - (\tilde{M}V - \tilde{N}U)Q)]^{-1} = ((\tilde{M}V - \tilde{N}U)Q)^{-1} \in \mathscr{H}_\infty$ which implies the coprimeness of $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] Q$. It is thus also clear that the controller $K = (UQ)(VQ)^{-1} = UV^{-1}$. Therefore the two conditions that the controller is stabilizing and that (U, V) are coprime are automatically satisfied. Hence

$$1 > \inf_K \operatorname{gap}\left(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm{T}}(M_K)]^{\perp}\right) = \inf_{U, V \in \mathscr{R}\mathscr{H}_\infty} \ \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

$$= \inf_{U, V \in \mathscr{R}\mathscr{H}_\infty} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V_0 \\ U_0 \end{bmatrix} \right\|_\infty = \left\| H_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right\|$$

by Nehari's Theorem. This shows that $K_0 = U_0 V_0^{-1}$ is a maximally stabilizing controller. Further any maximally stabilizing controller K with normalized coprime factors (U, V) must satisfy

$$\sigma_1 := \left\| H_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} \right\| = \operatorname{gap}\left(\mathscr{G}(M_G), [\mathscr{G}^{\mathrm{T}}(M_K)]^{\perp}\right) = \inf_{Q \in \mathscr{R}\mathscr{H}_\infty^{p \times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty.$$

For the Q_0 that achieves the infimum $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] Q_0$ is a coprime factorization of K by the above arguments, implying that every maximally stabilizing controller has a coprime factorization satisfying the bound (7). □

The theorem shows that the maximally stabilizing controller can be constructed from the Nehari extension of the transfer function $[{}^{\tilde{M}^*}_{-\tilde{N}^*}]$, and the minimum angle between the graph space of the system G and the orthogonal complement of the transposed graph of the controller K is $\cos\theta_{\min}^{\text{opt}} = \sigma_1$.

If Theorem 6.4 is compared to Theorem 4.2 and 4.3 of McFarlane and Glover [8], it is clear that the maximally stabilizing controller is also an optimal controller with respect to the normalized left coprime factor perturbations problem. The coprime factor perturbation models were introduced in [15]. That the connection between the uncertainty description in the coprime factor framework and in the gap framework is not accidental follows from [6]. See also [11] for further discussion and results on this connection.

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