ASYMPTOTIC STABILITY OF INFINITE DIMENSIONAL DISCRETE-TIME BALANCED REALIZATIONS

Yuanyin Wu* and Raimund Ober
Center of Engineering Mathematics
University of Texas at Dallas, TX 75083-0688, USA

Abstract

The question of power and asymptotic stability of infinite dimensional discrete-time state space systems is investigated. It is shown that every balanced realization is asymptotically stable. Conditions are given for balanced, input normal or output normal realization to be asymptotically and/or power stable.

1 Introduction

Balanced realizations for finite dimensional systems have received a great deal of attention. They were introduced as a means of performing model reduction in an easy fashion [6] and have subsequently been used in H^∞ control theory, for example, to evaluate the Hankel norm of a linear system [3], [4]. Recently, they have been used to study parametrization problems of certain sets of linear systems [8].

The elegant results obtained for finite dimensional balanced systems brought about some interest in the problem of the extension of the notion of a balanced realization to infinite dimensional systems. Glover, Curtain and Partington [4] derived continuous-time balanced realizations for a class of systems with nuclear Hankel operators. Young [13] developed a very general realization theory for infinite dimensional discrete-time systems. Similar results were obtained in the continuous time case by Ober and Montgomery-Smith [9].

One of the fundamental problems in systems theory is the question of the stability of the system. In this paper we will address this problem in the case of infinite dimensional balanced realizations and the closely related input and output normal realizations. By relating balanced realizations to restricted shift realizations, we are able to show that every balanced realization is asymptotically stable. In general, input normal and output normal realizations do not have the same stability properties as balanced realizations, but we can also give necessary and sufficient conditions for them to be asymptotically and/or power stable. It turns out that an input normal or output normal realization is power stable if only if its transfer function is rational whereas the power stability of par-balanced realization is more complicated to characterize in terms of the properties of the transfer function.

Let X, Y and U be separable Hilbert spaces. The linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \qquad k = 0, 1, \cdots \tag{1}$$

where A is a contraction on X, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$ and $D \in \mathcal{L}(U,Y)$, will be denoted by the quadruple of operators (A,B,C,D). The set of all such systems is denoted by $D_X^{U,Y}$. For (A,B,C,D) in $D_X^{U,Y}$, its observability operator is defined as

$$\mathcal{O} : X \to l_Y^2, \quad \mathcal{O}x = (CA^n x)_{n\geq 0},$$

for $x \in D(\mathcal{O}) = \{x \mid (CA^n x)_{n\geq 0} \in l_Y^2\}$. If $D(\mathcal{O}) = X$, $Ker(\mathcal{O}) = \{0\}$ and $\mathcal{O}$ is bounded then the system is said to be observable. The reachability of a system (A,B,C,D) is defined through its dual system, (A^*, C^*, B^*, D^*). If (A^*, C^*, B^*, D^*) is observable then (A,B,C,D) is said to be reachable. This is equivalent to that the operator

$$\begin{aligned} \mathcal{R} : l_U^2 &\to X \\ (u_n)_{n\geq 0} &\mapsto \textstyle\sum_{n\geq 0} A^n B u_n \end{aligned}$$

is bounded and has dense range in X. $\mathcal{R}$ is called the reachability operator of (A,B,C,D). If (A,B,C,D) is reachable and observable the observability gramian is defined to be $\mathcal{M} = \mathcal{O}^*\mathcal{O}$ and the reachability gramian is $\mathcal{W} = \mathcal{R}\mathcal{R}^*$.

For a system $(A,B,C,D) \in D_X^{U,Y}$, the $\mathcal{L}(U,Y)$ valued function $G(z) = C(zI-A)^{-1}B + D$, $z \in \mathbf{D}_e$, is called the transfer function of (A,B,C,D) while (A,B,C,D) is called a realization of G. In this case it is clear that G is analytic on $\mathbf{D}_e$ and at infinity. In this paper we restrict ourselves to the class of transfer functions that are bounded and analytic on $\mathbf{D}_e$, though most of the results apply to a larger class of transfer functions with some technical modifications. Notice that the boundedness and analyticity of a $\mathcal{L}(U,Y)$ valued function G on $\mathbf{D}_e$ means that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ where for $z \in \mathbf{D}$, $G^\perp(z) = z^{-1}[G(z^{-1}) - G(\infty)]$.

The symbols we use are defined as follows.

A^*	the adjoint of an operator A
Σ^*	the set $\{\overline{\sigma} : \sigma \in \Sigma\}$ if $\Sigma \subseteq \mathbf{C}$
$\mathbf{D}$	denotes the open unit disk
$\partial\mathbf{D}$	the unit circle
$\mathbf{D}_e$	the exterior of $(\partial\mathbf{D}) \cup \mathbf{D}$
$D_X^{U,Y}$	defined in this section
$G^\perp(z)$	$\frac{1}{z}[G(\frac{1}{z}) - G(\infty)]$, $z \in \mathbf{D}$, if G is a transfer function
$Z^\perp$	the orthogonal complement of the subspace Z of a Hilbert space
H_K	the Hankel operator with symbol K (see § 2)
$H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$	$\{F : \mathbf{D} \to \mathcal{L}(U,Y)$ analytic and bounded on $\mathbf{D}\}$
$H^2_Y(\mathbf{D})$	$\{f : \mathbf{D} \to Y$ analytic on $\mathbf{D}$ and $\sup_{0<r<1} \int_0^{2\pi} \Vert f(re^{it})\Vert^2 dt < \infty\}$
$\check{K}(z)$	$(K(\bar{z}))^*$ for $K \in L^\infty_{\mathcal{L}(U,Y)}(\partial\mathbf{D})$
$\mathcal{L}(U,Y)$	$\{A : U \to Y$ a bounded operator$\}$
$L^\infty_{\mathcal{L}(U,Y)}(\partial\mathbf{D})$	$\{F : \partial\mathbf{D} \to \mathcal{L}(U,Y)$ measurable and essentially bounded on $\partial\mathbf{D}\}$
$L^2_Y(\partial\mathbf{D})$	$\{f : \partial\mathbf{D} \to Y,\ \int_0^{2\pi} \Vert f(e^{it})\Vert^2 dt < \infty\}$
P_+	the orthogonal projection of $L^2_Y(\mathbf{D})$ onto $H^2_Y(\mathbf{D})$
P_X	the orthogonal projection of $H^2_Y(\mathbf{D})$ onto $X \subseteq H^\infty_Y(\mathbf{D})$
S	$(Sf)(z) = zf(z)$ for $f \in H^2_Y(\mathbf{D})$; the forward shift
S^*	$(S^*f)(z) = z^{-1}[f(z) - f(0)]$ for $f \in H^2_Y(\mathbf{D})$; the backward shift
$S(Q)$	$P_{(QH^\infty_Y(\mathbf{D}))^\perp} S\vert_{(QH^\infty_Y(\mathbf{D}))^\perp}$, $Q \in H^\infty_Y(\mathbf{D})$ inner
$S(Q)^*$	$S^*\vert_{(QH^\infty_Y(\mathbf{D}))^\perp}$, $Q \in H^\infty_Y(\mathbf{D})$ inner
$\sigma(A)$	the spectrum of an operator A
$\sigma_p(A)$	the point spectrum of an operator A
$\sigma(Q)$	the spectrum of an inner function Q, see § 3
$\sigma_s(G)$	the set of points in $\mathbf{C}$ where G has no analytic continuation, see Theorem 3.5

2 Hankel operators and shift realizations for discrete time systems

Our results will be mostly based on the investigation of restricted shift realizations whereby the shift realizations can be analyzed in terms of Hankel operators related to the transfer functions. Here we give a brief summary of some results on Hankel operators and the restricted shift realizations of discrete-time transfer functions. General references in this respect are e.g. Fuhrmann [2] and Helton [5].

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Definition 2.1 *If $K \in L^\infty_{\mathcal{L}(U,Y)}(\partial\mathbf{D})$, then the operator*

$$\begin{aligned} H_K: \ H^2_U(\mathbf{D}) &\to H^2_Y(\mathbf{D}) \\ u &\mapsto P_+ M_K J u \end{aligned}$$

where

$$\begin{aligned} Jf(z) &= f(\tfrac{1}{\bar z}) \\ M_K &\ \text{multiplication operator by } K \\ P_+ &\ \text{projection of } L^2_Y(\partial\mathbf{D}) \text{ onto } H^2_Y(\mathbf{D}) \end{aligned}$$

is called the Hankel operator with symbol K. □

Note that the Hankel operator is bounded since we assume that K is bounded. Hartmann's theorem shows that H_K is compact if K is a continuous function on $\partial\mathbf{D}$ with values in the set of compact operators (see [10]). It is well known and readily verified that the closure of the range of a Hankel operator H_K is the orthogonal complement of a right invariant subspace of $H^2_Y(\mathbf{D})$, i.e. the closure is a left invariant subspace.

Now we recall the restricted shift realization which was first introduced by Fuhrmann [1] and Helton [5] (see also [13]).

Theorem 2.1 *Let G be a $\mathcal{L}(U,Y)$ valued function such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$. Then G has a state space realization (A,B,C,D) with state space X, i.e. for $z \in \mathbf{D}_e$*

$$G(z) = C(zI - A)^{-1}B + D,$$

which is given in the following way:

The state space X is given by

$$X = \overline{range} H_{G^\perp} \subseteq H^2_Y(\mathbf{D}).$$

The state propagation operator $A: X \to X$, the input operator $B: U \to X$, the output operator $C: X \to Y$ and the feedthrough operator $D: U \to Y$ are given by the following, for $f \in X$,

$$\begin{aligned} (Af)(z) &:= (S^*f)(z) = \frac{f(z) - f(0)}{z}, \\ (Bu)(z) &:= G^\perp(z)u, \quad u \in U, \\ Cf &:= f(0), \\ Du &:= G(\infty)u, \quad u \in U; \end{aligned}$$

where S is the (forward) shift operator: $(Sf)(z) = zf(z)$, $f \in H^2_Y(\mathbf{D})$. The realization (A,B,C,D) is called the restricted shift realization *of the transfer function G.* □

The following proposition collects some results on the restricted shift realization of a discrete time transfer function concerning its observability and reachability. Note that by the canonical equivalence of l^2_Y and $H^2_Y(\mathbf{D})$ we can consider the observability operator $\mathcal{O}$ to map X into $H^2_Y(\mathbf{D})$. Similarly, the reachability operator $\mathcal{R}$ can be considered to be defined on $H^2_U(\mathbf{D})$.

Proposition 2.1 *Given the assumptions of Theorem 2.1, then the system (A,B,C,D) is observable and reachable. The observability operator $\mathcal{O}$ and reachability operator $\mathcal{R}$ of (A,B,C,D) are respectively given by*

$$\mathcal{O} = I_X : X \to H^2_Y(\mathbf{D}) \quad and \quad \mathcal{R} = H_{G^\perp} : H^2_U(\mathbf{D}) \to X.$$

Proof: See [2] or [13]. □

As a next step we are going to construct another realization which is the dual realization of the restricted shift realization. Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and let $(\tilde A, \tilde B, \tilde C, \tilde D)$ be the restricted shift realization of the transfer function $\tilde G(z) := (G(\bar z))^*$, $z \in \mathbf{D}_e$. Then the dual system $(\tilde A^*, \tilde C^*, \tilde B^*, \tilde D^*)$ is a realization of $\tilde G(z)$. It is called the **-restricted shift realization* of G.

Theorem 2.2 *The state space representation (A_*, B_*, C_*, D_*) of the *-restricted shift realization is given as follows:*

The state space X_* is $X_* = \overline{range} H_{\tilde G^\perp}$ with $\tilde G^\perp(z) = (G^\perp(\bar z))^*$. The operators A_*, B_*, C_* and D_* are defined as

$$\begin{aligned} A_* &= P_{X_*} S|_{X_*}, \\ B_* &: U \to X_*; u \mapsto P_{X_*} u, \\ C_* &: X_* \to Y; f \mapsto (H_{G^\perp} f)(0), \\ D_* &= G(\infty), \end{aligned}$$

where P_{X_*} is the orthogonal projection from $H^2_U(\mathbf{D})$ onto X_*.

The system (A_*, B_*, C_*, D_*) is observable and reachable. The reachability and observability operators $\mathcal{R}_*$ and $\mathcal{O}_*$ are respectively given by

$$\mathcal{R}_* = P_{X_*} : H^2_U(\mathbf{D}) \to X_* \quad and \quad \mathcal{O}_* = H^*_{\tilde G^\perp}|_{X_*} = H_{G^\perp}|_{X_*}.$$

Proof: Replacing G by $\tilde G$ in Theorem 2.1 we get the restricted shift realization of $\tilde G$ and the dual of this realization is the *-restricted shift realization stated in the theorem. □

From these results we see that the state space for the restricted shift realization is given as the closed range of the Hankel operator whose symbol is the transfer function mapped to the unit disk. The state propagation operator is just the backward shift restricted to the state space. For the *-restricted shift realization the state space is also the closed range of a Hankel operator while the state propagation operator is the forward shift compressed to this state space. As mentioned earlier, the closure of the range of a Hankel operator H_K is the orthogonal complement of a right invariant subspace of $H^2_Y(\mathbf{D})$. A vector valued version of Beurling's theorem (see e.g [2], p.186) asserts that a right invariant space in $H^2_Y(\mathbf{D})$ can only be either the trivial space $\{0\}$, or $QH^2_Y(\mathbf{D})$ where $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbf{D})$ is such that $\|Q\|_\infty \le 1$ and $Q(e^{it})$ is for a.e $t \in [0, 2\pi)$ a partial isometry with a fixed nonzero initial space. Such a function Q is called a rigid function. An important class of rigid functions is the so called class of inner functions. A rigid function Q is inner if $Q(e^{it})$ is for a.e $t \in [0, 2\pi)$ a unitary operator.

This discussion leads to the cyclicity of functions in $L^\infty_{\mathcal{L}(U,Y)}(\partial\mathbf{D})$ defined as follows.

Definition 2.2 *Let $K \in L^\infty_{\mathcal{L}(U,Y)}(\partial\mathbf{D})$. Then K is called*

1. cyclic, *if $(range H_K)^\perp = \{0\}$,*
2. noncyclic *if $(range H_K)^\perp = QH^2_Y(\mathbf{D})$ for some rigid function $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbf{D})$,*
3. strictly noncyclic *if $(range H_K)^\perp = QH^2_Y(\mathbf{D})$ for some inner function $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbf{D})$.* □

Note that if K is scalar then K is noncyclic if and only if it is strictly noncyclic. It is important to have characterizations for matrix-valued functions to be strictly noncyclic. To this end we introduce some definitions. Let K be in $H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$. The function $\hat K$ defined on $\mathbf{D}_e$ with values in $\mathcal{L}(U,Y)$ is called a *meromorphic pseudocontinuation of bounded type* of K if $\hat K$ is *of bounded type*, i.e.

$$\hat K(z) = \frac{F(z)}{h(z)}, \quad (z \in \mathbf{D}_e)$$

where F is a $\mathcal{L}(U,Y)$-valued function and h is a scalar-valued function, both bounded and analytic on $\mathbf{D}_e$; and K and $\hat K$ have the same strong radial limits on $\partial\mathbf{D}$.

Let $F_1 \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and $F_2 \in H^\infty_{\mathcal{L}(Z,Y)}(\mathbf{D})$. We say that F_1 and F_2 are *left weakly coprime* and write

$$(F_1, F_2)_L = I_Y$$

if $F_1 H^2_U(\mathbf{D}) \vee F_2 H^2_Z(\mathbf{D}) = H^2_Y(\mathbf{D})$, where $\vee$ stands for closed linear span.

Analogously we say that $K_1 \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and $K_2 \in H^\infty_{\mathcal{L}(U,Z)}(\mathbf{D})$ are *right weakly coprime* and write $(K_1, K_2)_R = I_U$ if $\tilde K_1$ and $\tilde K_2$ are weakly left coprime.

Using these notations, we have the following theorem ([2], Theorem 3.5, p. 254).

Theorem 2.3 *For $K \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbf{D})$ with U and Y finite dimensional, the following statements are equivalent.*

1. *K is strictly noncyclic.*
2. *On $\partial\mathbf{D}$ the function K can be factored as*
$$K = Q_1(zF_1)^* = (zF_2)^*Q_2.$$
Q_1 and Q_2 are inner functions in $H^{\infty}_{\mathcal{L}(Y)}(\mathbf{D})$ and $H^{\infty}_{\mathcal{L}(U)}(\mathbf{D})$ respectively. The functions F_1 and F_2 are in $H^{\infty}_{\mathcal{L}(Y,U)}(\mathbf{D})$ and in $H^{\infty}_{\mathcal{L}(U,Y)}(\mathbf{D})$ respectively, and the coprimeness conditions $(Q_1, zF_1)_R = I_Y, (Q_2, zF_2)_L = I_U$ hold. Here Q_1 (respectively Q_2) is unique up to right (left) multiplication by a constant unitary operator.
3. *K has a meromorphic pseudocontinuation of bounded type on $\mathbf{D}_e$.*

If 2.) holds then $Q_1H^2_Y(\mathbf{D}) = (rangeH_K)^{\perp}$ and $\tilde{Q}_2H^2_U(\mathbf{D}) = (rangeH_{\tilde{K}})^{\perp}$. □

Corollary 2.1 *In the notation of the theorem with U and Y finite dimensional, K is strictly noncyclic if and only if $\tilde{K}$ is strictly noncyclic.* □

From Theorem 2.1, Theorem 2.2, Theorem 2.3 and Definition 2.2 we see that the state space of a restricted shift realization of a transfer function G is the orthogonal complement of an invariant subspace which is characterized by a rigid function Q. The state propagation operator A is the backward shift S^* restricted to the state space $(QH^2_Y(\mathbf{D}))^{\perp}$, i.e. $A = S^*_{|(QH^2_Y(\mathbf{D}))^{\perp}}$, which we will denote by $S(Q)^*$. One of the important points in our context is that the function Q can be determined from the transfer function G, if $G^{\perp}$ is strictly noncyclic.

For the *-restricted shift realization the state space can be determined in a similar way to the derivation of the restricted shift realization. In this case the state propagation operator is the forward shift operator S compressed to the orthogonal complement of an invariant subspace that is determined by a rigid function Q_*, i.e. $P_{(Q_*H^2_U(\mathbf{D}))^{\perp}}S_{|(Q_*H^2_U(\mathbf{D}))^{\perp}}$, which we denote by $S(Q_*)$.

We summarize these results in the following proposition.

Proposition 2.2 *Let G be such that $G^{\perp} \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbf{D})$ with U and Y finite dimensional and let $(A,B,C,D) \in D^{U,Y}_X$ be its restricted shift realization and $(A_*,B_*,C_*,D_*) \in D^{U,Y}_{X_*}$ its *-restricted shift realization. Then*

1. *if $G^{\perp}$ is cyclic we have that*
   (a) *$A = S^*$ and $X = H^2_Y(\mathbf{D})$.*
   (b) *$A_* = S$ and $X_* = H^2_U(\mathbf{D})$.*
2. *if $G^{\perp}$ is noncyclic we have that*
   (a) *$A = S(Q)^*$, where $Q \in H^{\infty}_{\mathcal{L}(Y)}(\mathbf{D})$ is a rigid function such that*
   $$X = \overline{range}H_{G^{\perp}} = (QH^2_Y(\mathbf{D}))^{\perp}.$$
   If $G^{\perp}$ is strictly noncyclic with factorization $G^{\perp} = Q_1(zF_1)^*$, where $Q_1 \in H^{\infty}_{\mathcal{L}(Y)}(\mathbf{D})$ is inner and $F_1 \in H^{\infty}_{\mathcal{L}(Y,U)}(\mathbf{D})$ such that $(Q_1, zF_1)_R = I_Y$, then $Q = Q_1V_1$ for some unitary operator V_1 on Y.
   (b) *$A_* = P_{X_*}S_{|X_*} = S(Q_*)$, where $Q_* \in H^{\infty}_{\mathcal{L}(U)}(\mathbf{D})$ is a rigid function such that*
   $$X_* = \overline{range}H_{\tilde{G}^{\perp}} = (Q_*H^2_U(\mathbf{D}))^{\perp}.$$
   If $G^{\perp}$ is strictly noncyclic and $\tilde{G}^{\perp}$ has a factorization $\tilde{G}^{\perp} = Q_2(zF_2)^*$, where $Q_2 \in H^{\infty}_{\mathcal{L}(U)}(\mathbf{D})$ is inner and $F_2 \in H^{\infty}_{\mathcal{L}(U,Y)}(\mathbf{D})$ such that $(Q_2, F_2)_L = I_U$, then $Q_* = Q_2V_2$ for some unitary operator V_2 on U. □

3 Stability and spectral minimality of input normal and output normal realizations

In this section we discuss the stability and questions of spectral minimality of input normal and output normal realizations using the results on restricted and *-restricted shift realizations studied in Section 2.

The following definition recalls the notion of an input-normal and output-normal system as defined by Moore ([6]) for finite dimensional state space realizations. The definitions in infinite dimensional case are natural extensions of the finite dimensional notion (see e.g. [13]).

Definition 3.1 *Let $(A,B,C,D) \in D^{U,Y}_X$ be reachable and observable. Then the system is*

(i) output normal, *if $\mathcal{M} = I$,*

(ii) input normal, *if $\mathcal{W} = I$,*

(iii) par-balanced, *if $\mathcal{M} = \mathcal{W}$,*

(iv) balanced, *if $\mathcal{M} = \mathcal{W}$ and there is an orthonormal basis of the state space with respect to which $\mathcal{M}$ (and hence $\mathcal{W}$) has a diagonal matrix representation.* □

From our results on the restricted and the *-restricted shift realization we immediately have examples for input and output normal realizations.

Proposition 3.1 *The restricted shift realization is output normal whereas the *-restricted shift realization is input normal.*

Proof: The proof follows from Proposition 2.1 and Theorem 2.2. □

Next we are going to quote a result which establishes that an output-normal realization of a transfer function is unitarily equivalent to its restricted shift realization.

Two systems $(A_1,B_1,C_1,D_1) \in D^{U,Y}_{X_1}$ and $(A_2,B_2,C_2,D_2) \in D^{U,Y}_{X_2}$ are called *equivalent* (*unitarily equivalent*) if there exists a bounded and boundedly invertible operator (a unitary operator) V mapping the state space X_1 onto the state space X_2, such that
$$(A_1,B_1,C_1,D_1) = (V^{-1}A_2V, V^{-1}B_2, C_2V, D_2).$$
In this case V is called an equivalent (unitary) transformation.

Theorem 3.1 *([13]) If $(A_1,B_1,C_1,D_1) \in D^{U,Y}_{X_1}$ and $(A_2, B_2, C_2, D_2) \in D^{U,Y}_{X_2}$ are two output-normal realizations of a transfer function, then (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) are unitarily equivalent.* □

By a duality argument we have as a corollary that the same result holds for input-normal realizations, i.e. an input normal realization is unitarily equivalent to the *-restricted shift realization.

Corollary 3.1 *Let $(A_1,B_1,C_1,D_1) \in D^{U,Y}_{X_1}$ and $(A_2,B_2,C_2,D_2) \in D^{U,Y}_{X_2}$ be two input-normal realizations of a transfer function. Then (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) are unitarily equivalent.* □

We now turn to the study of stability. We introduce a classification of contractions according to their stability properties ([12]) which will simplify our notation.

Definition 3.2 *Let T be a contraction on the Hilbert space H. Then*

1. *$T \in C_{0\cdot}$ if $\lim_{n\to\infty} T^nh = 0$, for all $h \in H$,*
2. *$T \in C_{\cdot 0}$ if $\lim_{n\to\infty} (T^*)^nh = 0$, for all $h \in H$,*
3. *$T \in C_{1\cdot}$ if $\lim_{n\to\infty} T^nh \neq 0$, for all $h \in H$, $h \neq 0$,*
4. *$T \in C_{\cdot 1}$ if $\lim_{n\to\infty} (T^*)^nh \neq 0$, for all $h \in H$, $h \neq 0$.*

We further set $C_{ij} = C_{i\cdot} \cap C_{\cdot j}$, $i,j = 0,1$. □

Now we define the two notions of stability we will consider in the sequel.

Definition 3.3 *A discrete time system $(A,B,C,D) \in D^{U,Y}_X$ or the state propagation operator A is called*

1. asymptotically stable *if for every $x \in X$,*

$$A^k x \to 0, \qquad as\ k \to \infty,$$

i.e. if A is of class $C_{0\cdot}$,

2. power stable *if* $r < 1$ *where*

$$r := \inf\{\bar{r} \mid \text{there is } M_r > 0 \text{ such that } \|A^k\| \le M_r \bar{r}^k, k \ge 0\}.$$

The number r is called the degree of power stability. □

It is easy to see that stability, observability as well as reachability of discrete-time systems are preserved under equivalent transformations whereas input and output normality is preserved under unitary equivalence. Moreover, two equivalent power stable systems have the same degree of power stability.

Therefore by Theorem 3.1 and its corollary we can establish all stability and other important results concerning input- and output-normal realizations by restricting ourselves to *-restricted and restricted shift realizations. Furthermore, from Proposition 2.2 we can see that the study of stability and spectral properties of the restricted and *-restricted realizations reduces to the study of the operators $S(Q)^*$ and $S(Q_*)$ where Q and Q_* are rigid functions. We will need the following lemma (see Corollary, p.43, in [7]).

Lemma 3.1 *Let* $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbf{D})$ *be a rigid function. Denote by* P_X *the orthogonal projection of* $H^2_Y(\mathbf{D})^\perp$ *on* $X := (QH^2_Y(\mathbf{D}))^\perp$. *Then for* $f \in H^2_Y(\mathbf{D})$,

$$\lim_{n\to\infty} \|P_X S^n f\|^2 = \|f\|^2 - \|Q^* f\|^2. \qquad \square$$

The following theorem shows that an output normal realization of a transfer function is always asymptotically stable.

Theorem 3.2 *Let* G *be such that* $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ *and let* (A,B,C,D) *be an output normal realization of* G. *Then*

1. $A \in C_{0\cdot}$, *i.e.* A *is asymptotically stable.*
2. $A \in C_{00}$ *if and only if* $G^\perp$ *is strictly noncyclic.*
3. $A \in C_{01}$ *if* $G^\perp$ *is cyclic.*

Proof: By Proposition 3.1 we can assume without loss of generality that (A,B,C,D) is the restricted shift realization.

1.) The state propagation operator A of the restricted shift realization is the restriction of the backward shift S^* to a subspace of $H^2_Y(\mathbf{D})$. Since $(S^*)^k x_0 \to 0$ as $k \to \infty$, 1.) follows immediately.

2.) This follows from Proposition 2.2 and Lemma 3.1.

3.) If $G^\perp$ is cyclic then A is the backward shift S^* on the space $H^2_Y(\mathbf{D})$ and therefore $A \in C_{01}$. □

In the case of input normal realizations the situation is however such that we cannot in general expect that the realization is asymptotically stable since the state propagation operator of the *-restricted shift realization is the forward shift operator, compressed to a subspace of $H^2_U(\mathbf{D})$. The forward shift on $H^2_U(\mathbf{D})$ is not asymptotically stable. But the following corollary states that at least for an important class of transfer functions input normal realizations are asymptotically stable.

Corollary 3.2 *Let* (A,B,C,D) *be an input normal realization of* G. *Then*

1. $A \in C_{\cdot 0}$.
2. $A \in C_{00}$ *if and only if* $\tilde{G}^\perp$ *is strictly noncyclic.*
3. $A \in C_{10}$ *if* $\tilde{G}^\perp$ *is cyclic.*

Proof: Let (A,B,C,D) be the *-restricted realization of G. Recall that by definition (A,B,C,D) is the dual system of the restricted shift realization of $\tilde{G}$. Hence the result follows by duality from Theorem 3.2. □

We now proceed to power stability. The following result gives a characterization of power stability (see e.g. Przyluski [11]).

Proposition 3.2 *Let* T *be a contraction. Then the spectral radius* $r(T)$ *of* T, *i.e.*

$$r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\}$$

is given by

$$r(T) = \inf\{0 \le \bar{r} \le 1 \mid \text{there is } M_r \ge 0 \text{ such that } \|T^k\| \le M_r \bar{r}^k, k \ge 0\}.$$

Hence if T is power-stable, then the degree of power stability equals the spectral radius.

Proof: The proof follows from an application of the well-known formula

$$\sup\{|\lambda| \mid \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}. \qquad \square$$

In order to establish whether or not the output normal and input normal realizations are power stable it is therefore important to determine the spectral radius of its state propagation operator. To this end we appeal to C_0 contractions which play an important role in the theory of contractive operators. A detailed study of C_0 contractions can be found in [12]. We only point out that a C_0 contraction T on a Hilbert space is a *completely non-unitary* operator, for which the operator $u(T) := \lim_{r<1, r\to 1} u(rT)$ is a well defined bounded operator for any $u \in H^\infty$, and there exists an inner function m such that $m(T) = 0$. The least common divisor of all such inner functions is called the *minimal function* m_T of T. For the minimal function m_T we also have that $m_T(T) = 0$. Therefore the minimal function of a C_0 contraction can be seen to be a generalization of the minimal polynomial of a matrix.

As in the case of matrices the spectrum of C_0 operators is given by the 'zeros' of the minimal function in the following sense. We define the *spectrum* $\sigma(Q)$ of an inner function $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbf{D})$ to be

$$\sigma(Q) = \{\lambda \in \bar{\mathbf{D}} \mid \liminf_{\xi\to\lambda, \xi\in\mathbf{D}} \inf_{\|y\|=1, y\in Y} \|Q(\xi)y\| = 0\}.$$

Then we have (see [7], p75),

Proposition 3.3 *If* T *is a* C_0 *operator, then*

$$\sigma(T) = \sigma(m_T) \ \text{ and } \ \sigma_p(T) = \sigma(m_T) \cap \mathbf{D}. \qquad \square$$

Now we use these results to analyze the spectrum of the operators $S(Q)$ and $S(Q)^*$. Firstly the following proposition ([7], p.73) shows when $S(Q)$ is a C_0 contraction.

Proposition 3.4 *If* $dim(U) < \infty$ *and* $Q \in H^\infty_{\mathcal{L}(U)}(\mathbf{D})$ *is a rigid function then the determinant* $d = det(Q)$ *is such that* $d(S(Q)) = 0$. *Therefore when* Q *is an inner function and* U *has finite dimension the operator* $S(Q)$ *is a* C_0 *contraction.* □

In fact $S(Q)$ and $S(Q)^*$ are both C_0 contractions when Q is inner and U is finite dimensional (see [2], p. 191 or [7], p.75).

One of the important results in the theory of the backward shift operator $S(Q)^*$ restricted to an invariant subspace is that its spectrum can be completely characterized by the associated inner function Q ([7], p.75).

Theorem 3.3 *1. (a) Let* S^* *be the backward shift on* $H^2_Y(\mathbf{D})$. *Then*

$$\sigma(S^*) = \bar{\mathbf{D}}, \qquad \sigma_p(S^*) = \mathbf{D}.$$

(b) Let S *be the forward shift on* $H^2_Y(\mathbf{D})$. *Then*

$$\sigma(S) = \bar{\mathbf{D}}, \qquad \sigma_p(S) = \emptyset.$$

2. Let $Q \in H^\infty_Y(\mathbf{D})$ *be an inner function with* Y *finite dimensional. Then*

(a)

$$\sigma(S(Q)^*) = \sigma(Q)^* = \sigma(m_{S(Q)^*}),$$

$$\sigma_p(S(Q)^*) = \sigma(S(Q)^*) \cap \mathbf{D} = \{\bar{\lambda} \in \mathbf{D} \mid KerQ(\lambda)^* \ne \{0\}\},$$

where $\sigma(Q)^* = \{\bar{\lambda} :\ \lambda \in \sigma(Q)\}$.

(b)

$$\sigma(S(Q)) = \sigma(Q) = \sigma(m_{S(Q)}),$$

$$\sigma_p(S(Q)) = \sigma(S(Q)) \cap \mathbb{D} = \{\lambda \in \mathbb{D} |\ Ker Q(\lambda) \neq \{0\}\}. \quad \square$$

The next result shows that we only need to be concerned with inner functions if we are interested in the case when the spectral radius of the restricted backward shift is less than one (see [2], p. 194).

Theorem 3.4 *Let U be finite dimensional and $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ a rigid function that is not inner. Then $\sigma_p(S(Q)^*) = \mathbb{D}$.* □

In terms of the restricted and *-restricted shift realizations, Theorems 3.3 and 3.4 can be translated into the following.

Proposition 3.5 *Let (A, B, C, D) and (A_*, B_*, C_*, D_*) be respectively the restricted and *-restricted shift realizations of a transfer function G with $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$, where U and Y have finite dimensions.*

1. *If $G^\perp$ is cyclic, then*

$$\sigma(A) = \bar{\mathbb{D}}, \qquad \sigma_p(A) = \mathbb{D},$$

$$\sigma(A_*) = \bar{\mathbb{D}}, \qquad \sigma_p(A) = \emptyset.$$

2. *If $G^\perp$ is noncyclic but not strictly noncyclic, then*

$$\sigma(A) = \bar{\mathbb{D}}, \qquad \sigma(A_*) = \bar{\mathbb{D}}.$$

3. *If $G^\perp$ is strictly noncyclic with factorization $G^\perp = Q_1(zF_1)^*$ and $\tilde{G}^\perp = Q_2(zF_2)^*$, where $Q_1 \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ and $Q_2 \in H^\infty_{\mathcal{L}(U)}(\mathbb{D})$ are inner, and where Q_1 and $F_1 \in H^\infty_{\mathcal{L}(Y,U)}(\mathbb{D})$ are weakly right coprime, and Q_2 and $F_2 \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ are also weakly right coprime, then*

$$\sigma(A) = \sigma(Q_1)^* = \sigma(m_A),$$

$$\sigma_p(A) = \sigma(Q_1)^* \cap \mathbb{D} = \{\bar{\lambda} \in \mathbb{D} |\ Ker Q_1(\lambda)^* \neq \{0\}\},$$

and

$$\sigma(A_*) = \sigma(Q_2) = \sigma(m_{A_*}),$$

$$\sigma_p(A_2) = \sigma(Q_2) \cap \mathbb{D} = \{\lambda \in \mathbb{D} |\ Ker Q_2(\lambda) \neq \{0\}\},$$

where $\sigma(Q_1)^* = \{\bar{\lambda} |\ \lambda \in \sigma(Q_1)\}$.

Proof: The proposition follows from Theorems 3.3 and 3.4 and Proposition 2.2. □

A very important property of finite dimensional systems is that the eigenvalues of the state propagation matrix correspond exactly to the poles of the transfer function. For infinite dimensional systems it is desirable to have the analogous property. This was shown to be true for strictly noncyclic transfer functions by Fuhrmann ([2], Chapter III). Notice that by Theorem 2.3 if $G^\perp$ is strictly noncyclic $G^\perp$ has a meromorphic pseudocontinuation of bounded type on $\mathbb{D}_e$. We take this as the definition of $G^\perp$ on $\mathbb{D}_e$ and hence define G on $\mathbb{D}$.

Theorem 3.5 *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ with U and Y finite dimensional. If $G^\perp$ is strictly noncyclic, then*

1. *every output normal realization (A, B, C, D) is* spectrally mini*mal, i.e.*

$$\sigma(A) = \sigma_s(G),$$

where $\sigma_s(G)$ denotes the set of points at which G is not analytical or has no analytical continuation.

2. *every input normal realization (A_*, B_*, C_*, D_*) is spectrally minimal.*

Proof: 1.) See Theorem 4.11, p. 267, in [2] for the case of the restricted shift realization.

2.) Without loss of generality we assume that (A, B, C, D) is the *-restricted shift realization of G.

Recall from Corollary 2.1 that $G^\perp$ is strictly noncyclic if and only if $\tilde{G}^\perp$ is strictly noncyclic. By 1.) and the construction of the *-restricted shift realization we have that $\sigma(A_*^*) = \sigma_s(\tilde{G})$. Since

$$\sigma_s(G) = (\sigma_s(\tilde{G}))^* = \sigma(A_*^*)^* = \sigma(A_*)$$

where $(\sigma_s(\tilde{G}))^* = \{\bar{\lambda} :\ \lambda \in \sigma_s(\tilde{G})\}$ and $\sigma(A_*^*)^* := \{\bar{\lambda} :\ \lambda \in \sigma(A_*^*)\}$, we have the spectral minimality of the *-restricted shift realization. □

We are now going to show that an input or output normal system can be power stable if and only if it is finite dimensional.

Theorem 3.6 *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$ and let U and Y be finite dimensional. Then an output normal or input normal realization of G is power-stable if and only if G is rational.*

Proof: The 'if' part is well known. To show the 'only if' part we assume, without loss of generality, that the restricted shift realization (A, B, C, D) of G is power stable. Thus $\sigma(A)$ is a closed set inside $\mathbb{D}$. Then it follows from Proposition 3.5 that $G^\perp$ is strictly noncyclic and $\sigma(A) = \sigma(Q)$ where $Q \in H^\infty_{\mathcal{L}(Y)}(\mathbb{D})$ is an inner function. This implies that $\sigma(A)$ is the set of points in $\mathbb{D}$ at which $det(Q(z)) = 0$. Hence $\sigma(A)$ must be finite as Q is inner. Now by Theorem 3.5 $\sigma(A) = \sigma_s(G)$. So $G^\perp$ has only a finite number of singular points on $\mathbb{D}_e$. It then follows from Theorem 2.3 that $G^\perp$ must be rational and therefore G is also rational. □

4 Balanced realizations

This section is devoted to the study of the stability properties of balanced realizations with infinite dimensional state space.

Balanced realizations of finite dimensional systems have played an important role in model reduction and Hankel norm approximations of linear systems [6], [3]. In finite dimensions it is straightforward to construct a balanced realization from input-normal or output-normal realizations. In infinite dimensions it is not trivial to guarantee that this can be done since the state space transformation that is involved in general has an unbounded inverse. That this is nevertheless possible was shown by Young [13]. Note that the subscripts o and i in the following theorem signify output and input normal realizations respectively.

Theorem 4.1 *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbb{D})$. Let (A_o, B_o, C_o, D_o) be the restricted shift realization of G with state space $X_o = \overline{range} H_{G^\perp}$ and let (A_i, B_i, C_i, D_i) be the *-restricted realization with state space $X_i = \overline{range} H_{\tilde{G}^\perp}$. Set*

$$\mathcal{W}_o = H_{G^\perp} H^*_{G^\perp}|_{X_o}, \qquad \mathcal{M}_i = H_{\tilde{G}^\perp} H^*_{\tilde{G}^\perp}|_{X_i}.$$

1. *There exist par-balanced realizations $(A_{b1}, B_{b1}, C_{b1}, D_{b1}) \in D^{U,Y}_{X_o}$ and $(A_{b2}, B_{b2}, C_{b2}, D_{b2}) \in D^{U,Y}_{X_i}$ of G that satisfy*

$$\mathcal{W}_o^{1/4} A_{b1} = A_o \mathcal{W}_o^{1/4}, \qquad A_{b2} \mathcal{M}_i^{1/4} = \mathcal{M}_i^{1/4} A_i.$$

2. *All par-balanced realizations of G are unitarily equivalent.*
3. *If G is continuous on $\partial\mathbb{D}$ with values in the set of compact operators, then there exists a balanced realization whose state space is equal to the closure of the range of the Hankel operator with symbol $G^\perp$. The diagonal entries of the gramian are the singular values of the Hankel operator with symbol $G^\perp$.*

Proof: 2.), 3.) and the existence of the first par-balanced realization of 1.) can be found in [13]. The second realization of 1.) can be obtained by taking the dual of the par-balanced realization of $\tilde{G}$ constructed by the method of the first realization. □

Now we study further how the state propagation operator of a balanced realization is related to the state space propagation operators of input- and output-normal realizations. The following lemma is very helpful.

Lemma 4.1 *([2], p. 124) Let $A : H_1 \to H$ and $B : H_2 \to H$ be two linear operators from Hilbert spaces H_1 and H_2, respectively, into a Hilbert space H. Then $AA^* \leq BB^*$ if and only if there exists a contraction $V : H_1 \to H_2$ such that $A = BV$. Moreover, $AA^* = BB^*$ if and only if V is a partial isometry with final space equal to $\overline{range}(B^*)$.*

Proposition 4.1 *In Theorem 4.1, both $\mathcal{W}_o^{1/2}$ and $\mathcal{W}_o^{1/4}$ are bounded injective operators on X_o and have dense ranges. The same also applies to $\mathcal{M}_i^{1/2}$ and $\mathcal{M}_i^{1/4}$ on X_i.*

In the terminology of [12], we say that A_{b1} is a quasi-affine trans-form *of A_o and A_i a quasi-affine transform of A_{b2}.*

Proof: Notice that by Lemma 4.1 there exists a partial isometry V with final space equal to the closure of $H^*_{G^\perp}X_o$ such that $\mathcal{W}_o^{1/2} = H_{G^\perp}V$. From this it will follow that $\mathcal{W}_o^{1/2}$ has dense range and hence $\mathcal{W}_o^{1/4}$ also has dense range. The statement regarding $\mathcal{M}_i^{1/2}$ and $\mathcal{M}_o^{1/4}$ follows from a duality argument. □

The main result in this section is that all par-balanced realizations are asymptotically stable. Notice that (A, B, C, D) is a par-balanced realization if and only if (A^*, C^*, B^*, C^*) is par-balanced.

Theorem 4.2 *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and let (A_b, B_b, C_b, D_b) be a par-balanced realization of G. Then $A_b \in C_{00}$.*

Proof: We only point out a few key steps here. We use the notation in Theorem 4.1. First it can be verified that

$$H^*_{G^\perp}A^*_o = H^*_{G^\perp}S|_{X_o} = S^*|_{X_o}H^*_{G^\perp}|_{X_o} = A_oH^*_{G^\perp}|_{X_o}.$$

This will lead to $A_o\mathcal{W}_oA^*_o \leq \mathcal{W}_o$. Thus by Lemma 4.1 there exists a contraction V on X_o such that $A_o\mathcal{W}_o^{1/2} = \mathcal{W}_o^{1/2}V$, and hence for any positive integer n $A^n_o\mathcal{W}_o^{1/2} = \mathcal{W}_o^{1/2}V^n$. Since $\mathcal{W}_o^{1/4}A_{b1} = A_o\mathcal{W}_o^{1/4}$, it is possible to show that for $x \in \mathcal{W}_o^{1/2}X_o$

$$\|A^n_{b1}x\|^2 = < \mathcal{W}_o^{-\frac{1}{4}}A^n_o\mathcal{W}_o^{\frac{1}{4}}x, \mathcal{W}_o^{-\frac{1}{4}}A^n_o\mathcal{W}_o^{\frac{1}{4}}x > = < A^n_o\mathcal{W}_o^{\frac{1}{2}}x, V^ny > \rightarrow 0,$$

where $y = \mathcal{W}_o^{-1/4}x$. Now using the deneness of $\mathcal{W}_o^{1/2}X_o$ in X_o we can show that $\|A^n_{b1}x\|^2 \rightarrow 0$ for any $x \in X_o$. Hence $A_{b1} \in C_0$.. Then by Theorem 4.1 and the comments preceding the theorem we have $A_b \in C_{00}$ for any par-balanced realization (A_b, B_b, C_b, D_b) of G. □

We are going to discuss the spectral properties of a par-balanced realization and relate these properties to the characterization of power stability of par-balanced realizations.

Proposition 4.2 *Let (A_b, B_b, C_b, D_b), (A_i, B_i, C_i, D_i) and (A_o, B_o, C_o, D_o) be respectively a balanced, an input-normal and an output-normal realization of G with $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}$ and U and Y finite dimensional. If $G^\perp$ is strictly noncyclic, then*

1. *A_b is a C_0 operator with minimal function m.*
2. *A_i is a C_0 operator with minimal function m.*
3. *A_o is a C_0 operator with minimal function m.*

Proof: Note that the assumption implies that A_i, A_o and A_b are all completely non-unitary. The result now follows from Proposition 4.1 and from Proposition 4.6, p. 125, in [12]. □

For the spectrum of the state propagation operators we obtain the following result.

Corollary 4.1 *Under the assumption of Proposition 4.2, we have*

$$\sigma(A_b) = \sigma(A_i) = \sigma(A_o) \ \ \textit{and} \ \ \sigma_p(A_b) = \sigma_p(A_i) = \sigma_p(A_o).$$

Proof: The proof is an immediate consequence of Proposition 4.2 and Proposition 3.3. □

For the question of the spectral minimality we have the same result as for input- and output-normal realizations in the case of finite dimensional U and Y.

Corollary 4.2 *Under the assumption of Proposition 4.2, the systems (A_b, B_b, C_b, D_b), (A_i, B_i, C_i, D_i) and (A_o, B_o, C_o, D_o) are spectrally minimal, i.e.*

$$\sigma_s(G) = \sigma(A_b) = \sigma(A_i) = \sigma(A_o).$$

Proof: Combining Theorem 3.5 with Theorem 3.1 we have that

$$\sigma_s(G) = \sigma(A_i) = \sigma(A_o).$$

Corollary 4.1 now implies the result. □

The criteria for power-stability are also identical to those in the input- and output-normal case if $G^\perp$ is strictly noncyclic.

Corollary 4.3 *Let (A_b, B_b, C_b, D_b) be a par-balanced realization of G with $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and U and Y finite dimensional. Assume that $G^\perp$ is strictly noncyclic. Then A_b is power-stable if and only if G is rational.*

Proof: The proof follows from Corollary 4.1 and Theorem 3.6. □

This corollary shows that a par-balanced realization of G, with $G^\perp$ non-rational and strictly noncyclic, cannot be power stable. When $G^\perp$ is not strictly non-cyclic the situation is complicated. Here we give an example of a power stable par-balanced realization of a cyclic function with l_2 as its state space.

Example: Let S and S^* be the right and left shifts on the space l_2. Let $A = \frac{1}{5}(I + S + S^*)$. Clearly $\|A\| \leq 3/5$. Define $B : \mathbf{C} \rightarrow l_2$ as

$$B(\lambda) = (\lambda, 0, 0, \cdots)^T, \ \ \lambda \in \mathbf{C}$$

and $C : l_2 \rightarrow \mathbf{C}$ as $C(x_k)_{k\geq 1} = x_1$, $(x_k)_{k\geq 1} \in l_2$. We take D to be zero. It can be verified that this is an observable and reachable realization. It is power stable and par-balanced. Also for the transfer function $g(z) = C(zI - A)^{-1}B$ we have $g^\perp \in H^\infty$ due to the fact that $\|A\| < 1$. By Corollary 4.3 $g^\perp$ must be cyclic.

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