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GRAPHS OF LINEAR SYSTEMS AND STABILIZATION

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Abstract

It is discussed how in a natural way geometric ideas can give new insight into problems of robust stabilization. Stability criteria for control systems are phrased using geometric notions involving the graph of the plant and the graph of the controller. In this framework the connection between coprime factor uncertainty and uncertainty in the gap metric is examined. Necessary and sufficient conditions for robust stabilization are given. In the geometric framework the notion of a maximally robust controller is defined and it is shown that this controller is identical to the optimally robust controller by McFarlane-Glover.

Keywords: Graphs of linear systems, Stability criteria, Geometric interpretation, Coprime factor uncertainty, Gap Metric, Robust Control.

1 Introduction

The purpose of this paper is to show how geometric ideas can be applied in control theory and in particular in robust control to give further insight and understanding of fundamental issues. If G is the $p \times m$ rational transfer function of a plant and $G = NM^{-1}$ a normalized right coprime factorization, i.e. N, $M \in \mathcal{RH}_\infty$, M invertible, N, M right coprime over $\mathcal{RH}_\infty$ and $N^*N + M^*M = I$, then the graph of the multiplication operator acting between the input space $\mathcal{H}_2^m$ and the output space $\mathcal{H}_2^p$ with symbol G is given by $\mathcal{G}(M_G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m$. In the space $\begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$ which contains $\mathcal{G}(M_G)$ as a closed shift invariant subspace, the usual geometric notions can be defined such as the minimal angle $\theta_{min}(A,B) \in [0,\pi[$ between two closed subspaces A and B which is given by $\cos\theta_{min}(A,B) = sup_{u\in A, v\in B}\frac{|<u,v>|}{\|u\|\,\|v\|}$. A standard measure of the distance between two closed subspaces A, B the gap, defined by $gap(A,B) := \|P_A - P_B\|$, where P_A is the orthogonal projection onto the subspace A. Of importance are the following relationships:

$$\begin{aligned} gap(A,B) &= max\{\|P_A P_{B^\perp}\|, \|P_{A^\perp}P_B\|\} \\ &= max\{\cos\theta_{min}(A,B^\perp), \cos\theta_{min}(B,A^\perp)\}. \end{aligned}$$

In the following sections we apply these notions to analyze stability and robustness properties of control systems.

Due to space constraints no proofs are given, they can be found in [6],[7],[8].

2 Characterizations of Closed-Loop Stability

We are now going to show how stability criteria for control systems can be stated in terms of geometric notions in the Hilbert space $\begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$.

Theorem 2.1 *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$ respectively and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$, then the following statements are equivalent,*

S0) *The pair (G, K) is internally stable, i.e.*

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-GK)^{-1} & -(I-GK)^{-1}G \\ -K(I-GK)^{-1} & (I-KG)^{-1} \end{bmatrix} \in \mathcal{H}_\infty^{(m+p)\times(m+p)}.$$

S1) $\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m + \begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p = \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$.

S2) $P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp}\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m\right) = \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p\right)^\perp$.

S3) $\left\| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp} \right\| < 1.$

S4) $\theta_{min}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m\right)^\perp, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p\right)^\perp\right) > 0.$

S5) $gap\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p\right)^\perp\right) < 1.$

Moreover

$$\cos\theta_{min}\left(\left[\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m\right]^\perp, \left[\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p\right]^\perp\right)$$

$$= gap\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2^p\right)^\perp\right) = \| P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp} \|.$$

A further equivalent condition appeared in [1]. The quantities which are used to characterize internal stability can be easily calculated. This is of particular importance for the application of these results to robust control where it is necessary to have the precise quantaties available. In the following proposition it is shown that the quantities can in fact be calculated as standard $\mathcal{H}_\infty$-optimization problems.

Proposition 2.2 *Suppose the $p \times m$ transfer function G has a* normalized *r.c.f. (N, M) and a* normalized *l.c.f. $(\tilde{N}, \tilde{M})$. Assume that the $m \times p$ transfer function K has a* normalized *r.c.f. (U, V) and a* normalized *l.c.f. $(\tilde{U}, \tilde{V})$ then,*

$$1)\quad \left\| P_{\left(\left[\begin{smallmatrix} V\\ U\end{smallmatrix}\right]\mathcal{H}_2^p\right)^\perp} P_{\left(\left[\begin{smallmatrix} N\\ M\end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} \right\| = \inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty = \inf_{Q\in\mathcal{RH}_\infty^{m\times m}} \left\| \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right\|_\infty$$

$$2)\quad \left\| P_{\left(\left[\begin{smallmatrix} V\\ U\end{smallmatrix}\right]\mathcal{H}_2^p\right)} P_{\left(\left[\begin{smallmatrix} N\\ M\end{smallmatrix}\right]\mathcal{H}_2^m\right)} \right\| = \inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} Q \right\|_\infty = \inf_{Q\in\mathcal{RH}_\infty^{m\times m}} \left\| \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{bmatrix} Q \right\|_\infty$$

$$3)\quad \left\| P_{\left(\left[\begin{smallmatrix} V\\ U\end{smallmatrix}\right]\mathcal{H}_2^p\right)} - P_{\left(\left[\begin{smallmatrix} N\\ M\end{smallmatrix}\right]\mathcal{H}_2^m\right)^\perp} \right\| = \inf_{Q\in\mathcal{RH}_\infty^{p\times p}} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|_\infty$$

If (G,K) is internally stable then

$$gap\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right)^\perp\right) = \|N^*V + M^*U\|_\infty = \sqrt{1-(\tau(\tilde{V}M-\tilde{U}N))^2},$$

where for $F \in \mathcal{L}_\infty$ we set $\tau(F) = ess\ \inf\{\sigma_{min}(F(s)) \mid Re(s) = 0\}$.

3 Uncertainty Description

Two ways of modelling uncertainty in robust control have received a considerable amount of attention: uncertainty in the gap metric and coprime factor perturbations. In this section the connection between these two uncertainty descriptions will be discussed.

The gap between the graph of the plant G and the perturbed plant G_Δ was introduced in [2] as a measure of distance between two plants, i.e. $\delta(G_1,G_2) := gap(\mathcal{G}(M_G),\mathcal{G}(M_{G_\Delta}))$. The directed gap is defined as $\vec{\delta}(G_1,G_2) := \|P_{\mathcal{G}(M_{G_2})^\perp}P_{\mathcal{G}(M_{G_2})}\|$, so that $\delta(G_1,G_2) = \max\{\vec{\delta}(G_1,G_2),\vec{\delta}(G_2,G_1)\}$.

In [3] it was shown how the directed gap can be calculated by an $\mathcal{H}_\infty$-optimization problem. For a more elementary proof see [8].

Proposition 3.1 *Given two $p\times m$ systems G_1, G_2 with normalized right coprime factors (N_1,M_1) and (N_2,M_2) respectively, then*

$$\vec{\delta}(G_1,G_2) = \left\|P_{\mathcal{G}(M_{G_2})^\perp}P_{\mathcal{G}(M_{G_1})}\right\| = \inf_{Q\in\mathcal{RH}_\infty}\left\|\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q\right\|_\infty .$$

Since $\vec{\delta}(G_1,G_2) = \vec{\delta}(G_2,G_1)$ if $\delta(G_1,G_2) = 1$ it is of importance to have a characterization for when $\delta(G_1,G_2) < 1$. This is given in the following theorem.

Theorem 3.2 *Given two $p\times m$ systems G_1, G_2 with normalized right coprime factors (N_1,M_1) and (N_2,M_2) respectively, then the following statements are equivalent,*

1) $\delta(G_1,G_2) < 1$.

2) *The Toeplitz operator $T_{(N_1^*N_2+M_1^*M_2)}$ is invertible.*

3) *There exists a* $Q \in \mathcal{RH}_\infty$ *such that* $\left\| \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} - \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} Q \right\|_\infty < 1$ *and* $Q^{-1} \in \mathcal{RH}_\infty$.

4) *There exists a* $Q \in \mathcal{RH}_\infty$ *such that* $\left\| \begin{bmatrix} N_2 \\ M_2 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\|_\infty < 1$ *and* $Q^{-1} \in \mathcal{RH}_\infty$.

Another way to describe uncertainty in the plant description is to allows for uncertainty in the coprime factors of the plant ([9]). Given a system $G = NM^{-1}$ then any other system of the same input/output dimensions can be written in the form $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}$ for a $\Delta_N, \Delta_M \in \mathcal{H}_\infty$. It is shown here that if the correct restrictions are placed on the class of allowable perturbations, the class of systems generated by $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}$ is equivalent to an open ball in the gap metric. The gap ball and the directed gap ball are now defined.

Definition 3.3 *Given a* $p \times m$ *system* G_1 *with normalized r.c.f.* (N_1, M_1) *then the following classes of transfer functions are defined for* $\epsilon > 0$,

$$\begin{aligned} \mathcal{B}^\epsilon_{G_1} &:= \{G_2 : \delta(G_1, G_2) < \epsilon\} \\ \vec{\mathcal{B}}^\epsilon_{G_1} &:= \{G_2 : \vec{\delta}(G_1, G_2) < \epsilon\} \end{aligned}$$

which are called the gap ball and directed gap ball of G_1 *respectively. Also define the following classes*

$$\begin{aligned} \mathcal{G}^\epsilon_{G_1} &:= \{(N_1 + \Delta_N)(M_1 + \Delta_M)^{-1} : \left[\begin{smallmatrix} \Delta_N \\ \Delta_M \end{smallmatrix}\right] \in \mathcal{H}_\infty^{(p+m)\times m}; \\ &\qquad ((N_1 + \Delta_N), (M_1 + \Delta_M)) \text{ right coprime}; \left\|\left[\begin{smallmatrix} \Delta_N \\ \Delta_M \end{smallmatrix}\right]\right\|_\infty < \epsilon\} \\ \vec{\mathcal{G}}^\epsilon_{G_1} &:= \{(N_1 + \Delta_N)(M_1 + \Delta_M)^{-1} : \left[\begin{smallmatrix} \Delta_N \\ \Delta_M \end{smallmatrix}\right] \in \mathcal{H}_\infty^{(p+m)\times m}; \left\|\left[\begin{smallmatrix} \Delta_N \\ \Delta_M \end{smallmatrix}\right]\right\|_\infty < \epsilon\} \end{aligned}$$

In [4] it was shown that for given ϵ the directed gap ball $\vec{\mathcal{B}}^\epsilon_{G_1}$ and the coprime factor ball $\vec{\mathcal{G}}^\epsilon_{G_1}$ are identical. In the same paper it was also shown that for small enough ϵ the gap ball $\mathcal{B}^\epsilon_{G_1}$ and the coprime factor ball $\mathcal{G}^\epsilon_{G_1}$ are also identical. The following theorem states that gap balls $\mathcal{B}^\epsilon_{G_1}$ can be fully characterized in terms of in terms of coprime factor uncertainty balls $\mathcal{G}^\epsilon_{G_1}$.

Theorem 3.4 *Given a* $p \times m$ *system* G_1 *with normalized r.c.f.* (N_1, M_1) *then for* $\epsilon > 0$,

$$\mathcal{B}^\epsilon_{G_1} = \mathcal{G}^\epsilon_{G_1}.$$

4 Robust Stabilization

We can now give a result that gives a full characterization of the maximal ball in the gap metric that can be stabilized by a controller.

Theorem 4.1 *Given a* $p \times m$ *system* G, *a* $m \times p$ *stabilizing controller* K *then for all perturbed systems* $G_\Delta \in \vec{\mathcal{B}}^\epsilon_G$,

$$(G_\Delta, K) \text{ is internally stable}$$

if and only if

$$\epsilon \leq sin\theta_{min}\left(\left[\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}_2^m\right]^\perp, \left[\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}_2^p\right]^\perp\right)$$

In the previous theorem it was shown that a control design is the more robust the smaller the gap is between the orthogonal complement of the graph of the plant and the orthogonal complement of the transposed graph of the controller. Therefore a natural question is whether for a given plant there is a controller that maximizes the robustness of the control system, as measured in this sense. Given a $p \times m$ system G, the optimal minimal angle θ_{min}^{opt} is defined by,

$$\cos\theta_{min}^{opt} := \inf_{K\ stabilizing} gap(\mathcal{G}(M_G), \left[\mathcal{G}^T(M_K)\right]^\perp).$$

Further a controller, K, achieving this infinum is called a maximally stable controller.

Theorem 4.2 *Suppose the* $p \times m$ *transfer function* G *has a* normalized *r.c.f.* (N, M) *and a* normalized *l.c.f.* $(\tilde{N}, \tilde{M})$ *respectively then,*

$$\cos\theta_{min}^{opt} := \inf_{K\ stabilizing} gap(\mathcal{G}(M_G), \left[\mathcal{G}^T(M_K)\right]^\perp) = \sigma_1,$$

where $\sigma_1 = \|H_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}\|$. *A maximally stable controller exists and each maximally stable controller has a right coprime factorization* (U, V) *satisfying the extension,*

$$\left\|\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}\right\|_\infty = \sigma_1.$$

Notice that the maximally stable controller is precisely the optimally robust controller derived in [5]. That the approach taken here arrives at the same controller as the approach via coprime factor uncertainty is no surprise if the results on the connection between corime factor uncertainty and uncertainty in the gap are taken into account.

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