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Hankel norm approximation and control systems

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Abstract

Connections are established between problems of Hankel norm approximation, of finding approximating subspaces in Hilbert space $\mathcal{H}_2$ and stability and instability of control systems.

1 Introduction

Over the years it has become more and more evident that operator theory especially can be of great help in analyzing linear dynamical systems and in particular control systems (see e.g.[2]). This paper aims at establishing further connections between control theory and the theory of Hankel operators. We were motivated to do this work by results that interpreted robustness properties of control systems from the point of view of the geometry between the graph spaces of the plant and the controller (see e.g. [7],[1]). Let

$$\dot{x} = Ax + Bu, \quad x(0) = x_0,$$

$$y = Cx + Du$$

be a finite dimensional linear continous-time system, which we do not necessarily assume to be stable. By taking the Laplace transform and assuming $x_0 = 0$, we obtain the transfer function $G(s) = C(sI-A)^{-1}B+D$, which is a matrix-valued rational function (see e.g. [4]). In the transform domain the linear system can be seen as acting as a multplication operator on the Hardy space $\mathcal{H}_2$, which can be interpreted as the space of Laplace transforms of the space $L^2([0,\infty[)$. With the transfer function G we will associated the graph $\mathcal{G}(G)$ of the multiplication operator with symbol G, i.e. the operator $M_G : \mathcal{H}_2 \to \mathcal{H}_2; f \mapsto Gf$. Clearly, if the system is not stable and therefore G has poles in the closed right half plane then M_G will not be defined on the whole of $\mathcal{H}_2$.

In order to obtain a workable representation of the graph of M_G we need to introduce coprime factorizations (see e.g. [8]). The factorization $G = NM^{-1}$ ($G = \tilde{M}^{-1}\tilde{N}$) is a right (left) coprime factorization of G, if $N, M \in \mathcal{RH}_\infty$ ($\tilde{N}, \tilde{M} \in \mathcal{RH}_\infty$), the space of real rational functions with poles in the open left half-plane; M ($\tilde{M}$) is invertible as a proper rational function; and N, M ($\tilde{N}, \tilde{M}$) are right (left) coprime, i.e. there exist $\tilde{X}, \tilde{Y} \in \mathcal{RH}_\infty$ ($X, Y \in \mathcal{RH}_\infty$) such that $-\tilde{X}N + \tilde{Y}M = I$ ($-X\tilde{N} + \tilde{Y}M = I$). The factorization is called normalized if moreover $N^*N + M^*M = I$ ($\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$). With such a factorization the graph of M_G can be characterized as

$$\mathcal{G}(G) = \begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2.$$

In what follows we will be make much use of the following geometric notions in a Hilbert space H (see e.g. [3], [6], [9]). In our case the Hilbert space H will be the space $\mathcal{H}_2 \times \mathcal{H}_2$. Let A, $B \subseteq H$ be two closed subspaces, then it is possible to define the minimal angle and the gap between these two spaces as follows:

$$\cos\theta_{min}(A,B) = \sup_{u\in A, v\in B} \frac{|< u,v >|}{\|u\|\|v\|}$$

and

$$gap(A,B) = \|P_A - P_B\|,$$

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where P_C denotes the orthogonal projection on the closed subspace C. Alternatively the sine of the minimal angle can be defined by

$$\sin\theta_{min}(A,B) = \|P_{A\|B}\|^{-1},$$

where the skew projection $P_{A\|B}$ is defined by $P_{A\|B} : A + B \to A$, $u + v \mapsto u$, $u \in A$, $B \in B$. The skew projection is well defined on the Hilbert space H if $H = A + B$ and $A \cap B = \emptyset$. The skew-projection is bounded if and only if $\theta_{min}(A,B) > 0$. The following relationships hold,

$$\cos\theta_{min}(A,B) = \|P_A P_B\| = \|P_B P_A\|$$

$$= \sup_{u\in B,\|u\|=1} dist(u, A^{\perp}),$$

where $dist(u, A^{\perp}) = \inf_{v\in A^{\perp}} \|u - v\|$. The gap between two spaces can be characterized as follows,

$$gap(A,B) = max\{\|P_A P_{B^{\perp}}\|, \|P_{A^{\perp}} P_B\|\}$$

$$= max\{\cos\theta_{min}(A, B^{\perp}), \cos\theta_{min}(B, A^{\perp})\}$$

$$= max\{\sup_{u\in A,\|u\|=1} dist(u,B), \sup_{v\in B,\|v\|=1} dist(u,A)\}$$

If $gap(A,B) < 1$ then $\|P_A P_{B^{\perp}}\| = \|P_{A^{\perp}} P_B\|$.

The central issue in the area of control theory is the stabilization of unstable systems by a control K. With a controller K we associate the transposed graph $\mathcal{G}^T(K)$ of the controller, i.e. if $K = UV^{-1}$ is a right coprime factorization of K, then

$$\mathcal{G}^T(K) := \begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2.$$

In [7] the following equivalent conditions were proved for a controller K to stabilize the plant G. For an alternative equivalent condition see [1].

Theorem 1.1 *Let $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be a right respectively left coprime factorization of a $p \times m$ plant G and $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ be a right respectively left coprime factorization of a controller K. Then the following statements are equivalent:*

S0) the control system (G, K) is internally stable,

S1) the function $-\tilde{N}U + \tilde{M}V$ is invertible in $\mathcal{RH}_\infty$,

S2) the function $-\tilde{U}N + \tilde{V}M$ is invertible in $\mathcal{RH}_\infty$,

S3) $\mathcal{G}(G) + \mathcal{G}^T(K) = \mathcal{H}_2^{p+m}$,

S4) $P_{[\mathcal{G}^T(K)]}\mathcal{G}(G) = [\mathcal{G}^T(K)]$,

S5) $\theta_{min}([\mathcal{G}(G)]^{\perp}, [\mathcal{G}^T(K)]^{\perp}) > 0$,

S6) $gap(\mathcal{G}(G), [\mathcal{G}^T(K)]^{\perp})) < 1$.

The conditions $S1$) and $S2$) are the classical conditions for internal stability of a control system (see e.g. [8]). A substantial part of this paper will be devoted to an extension of these results to the case when the control system has a certain number of unstable poles.

In [7] it was argued that $\theta_{min}([\mathcal{G}(G)]^{\perp}, [\mathcal{G}^T(K)]^{\perp})$ is a good indicator for how far a control system is away from instability. When designing a controller for a given plant G it therefore appears natural to try to find the controller that maximizes this angle, i.e. to find a controller K_0, such that

$$\theta_{min}([\mathcal{G}(G)]^{\perp}, [\mathcal{G}^T(K_0)]^{\perp})$$

$$= \sup_K([\mathcal{G}(G)]^{\perp}, [\mathcal{G}^T(K)]^{\perp}).$$

It was shown that such a controller does exist and that it in fact coincides with the optimally robust controller with respect to normalized coprime factor uncertainty as studied in [5]. This controller is characterized through the solution of the following Nehari extension problem, i.e. $K_0 = U_0 V_0^{-1}$, where

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V_0 \\ U_0 \end{bmatrix} \right\|_\infty$$

$$= \inf_{U,V\in\mathcal{RH}_\infty} \left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty$$

$$= \left\| H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}} \right\|,$$

where $G = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization of G and $H_{\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}}$ is the Hankel operator with symbol $\begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix}$.

Theorem 1.1 gives characterizations of stability in terms of the graph of the system and the transposed graph of the controller. It was shown that internal stability was equivalent to the minimal angle between the orthogonal complement of the graph associated with the system and the orthogonal complement of of the transposed graph associated with the controller being greater than zero. Therefore if the system is unstable, there exists an intersection between these subspaces. One of the aims of this paper is to characterize the intersection between these two subspaces. As this subspace is orthogonal to both the graph space associated with the system and the transposed graph space associated with the controller, the closed-loop system behaves as a stable system on the span of these graph spaces. This characterization enables most of the stability conditions in Theorem 1.1 to be generalized to unstable closed-loop systems with a finite number of poles in the open right half plane. Also the angles between these subspaces can be calculated from similar expressions to those of Theorem 1.1.

2 Graphs of Linear Systems and Instability

The first definition generalizes the usual definition of internal stability (see e.g. [8]) to include closed-loop systems with a finite number of poles in the open right half plane.

Definition 2.1 *Given a $p \times m$ system and a $m \times p$ controller with transfer functions G and K respectively, then the pair (G, K) is called* unstable to order k, $k = 0, 1, \ldots$ *if*

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I - GK)^{-1} & -(I - GK)^{-1}G \\ -K(I - GK)^{-1} & (I - KG)^{-1} \end{bmatrix}$$

$\in \mathcal{RH}_{\infty,k}^{(m+p)\times(m+p)}$.

It is evident that a pair (G, K) is unstable to order k only if it is also unstable to order $k - 1$, and that a pair (G, K) is unstable to order zero if and only if it is internally stable (see e.g [8]).

The following result is a generalization of well known stability criteria (see e.g. [8]).

Proposition 2.2 *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$ respectively and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$, then the following statements are equivalent,*

U0) *The pair (G, K) is unstable to order k.*

U1) *There exists a right inner-outer factorization,*

$$(\tilde{M}V - \tilde{N}U) = \tilde{\Theta}\tilde{S}_0$$

where $\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p}$ is an inner function of McMillan degree less than or equal to k and $\tilde{S}_0 \in \mathcal{RH}_\infty^{p\times p}$ is a unit. The factors are unique up to right respectively left multiplication by a constant unitary matrix.

U2) *There exists a right inner-outer factorization,*

$$(\tilde{V}M - \tilde{U}N) = \Theta S_0$$

where $\Theta \in \mathcal{RH}_\infty^{m\times m}$ is an inner function of McMillan degree less than or equal to k and $S_0 \in \mathcal{RH}_\infty^{m\times m}$ is a unit. The factors are unique up to right respectively left multiplication by a constant unitary matrix.

One of the aims of this section is to interprete the previous result from a geometric point of view by considering the graph of the plant and the graph of the controller.

The next proposition connects the unique right inner-outer factorizations of $\tilde{S} = (\tilde{M}V - \tilde{N}U)$ and $S = (\tilde{V}M - \tilde{U}N)$ directly to a decomposition of the space $\mathcal{H}_2^p \times \mathcal{H}_2^m$ in terms of the graph space of the system G and the transposed graph of the controller K.

Proposition 2.3 *Given the assumptions of Proposition 2.2,*

1. *Let $\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p}$ be an inner function, then*

$$\left(\mathcal{G}(G) + \mathcal{G}^T(K)\right) \oplus T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}(\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p)$$

$$= \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$$

if and only if

$$(\tilde{M}V - \tilde{N}U) = \tilde{\Theta}\tilde{S}_0$$

for some unit $\tilde{S}_0 \in \mathcal{RH}_\infty^{p\times p}$.

2. *Let $\Theta \in \mathcal{RH}_\infty^{m\times m}$ be an inner function such that,*

$$\left(\mathcal{G}(G) + \mathcal{G}^T(K)\right) \oplus T_{\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]}(\mathcal{H}_2^p \ominus \Theta\mathcal{H}_2^p)$$

$$= \begin{bmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{bmatrix}$$

if and only if

$$(\tilde{V}M - \tilde{U}N) = \Theta S_0$$

for some unit $S_0 \in \mathcal{RH}_\infty^{m\times m}$.

The following theorem summarizes the previous results. It gives further necessary and sufficient conditions for a control system to be a unstable to order k. The importance of the result in our context is that the stability properties of the control system are characterized in terms of the graph of the plant and the transposed graph of the controller. This result generalizes the result on stable control systems given in [7],[1].

Theorem 2.4 *Given the assumptions of Proposition 2.2 , the following statements are equivalent,*

U0) *the pair (G, K) is unstable to order k,*

U3) *there exists an inner transfer function $\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p}$ of McMillan degree less than or equal to k such that the space $\mathcal{H}_2^p \times \mathcal{H}_2^m$ can be decomposed as*

$$\mathcal{H}_2^{p+m} = \left(\mathcal{G}(G) + \mathcal{G}^T(K)\right)$$

$$\oplus T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}(\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p),$$

The inner function $\tilde{\Theta}$ is unique up to right multiplication by a constant unitary matrix.

U4) *there exists a unique inner transfer function $\Theta \in \mathcal{RH}_\infty^{m\times m}$ of McMillan degree less than or equal to k such that the space $\mathcal{H}_2^p \times \mathcal{H}_2^m$ can be decomposed as*

$$\mathcal{H}_2^{p+m} = \left(\mathcal{G}(G) + \mathcal{G}^T(K)\right) \oplus$$

$$T_{\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]}(\mathcal{H}_2^p \ominus \Theta\mathcal{H}_2^p).$$

The inner function Θ is unique up to right multiplication by a constant unitary matrix.

Furthermore,

$$T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]}(\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p)$$

$$= T_{\left[\begin{smallmatrix} -\tilde{U}^* \\ \tilde{V}^* \end{smallmatrix}\right]}(\mathcal{H}_2^m \ominus \Theta\mathcal{H}_2^m).$$

3 Gap between graph spaces and control systems

In the previous section it was shown how order-k stability of a closed-loop system can be characterized in terms of spanning conditions of the graph of the plant and the transposed graph of the controller. In this section we are going to give further interpretations of these results in terms of the gap and minimum angle between certain graph spaces. These results generalize the results in [7] which were derived for stable closed-loop systems.

For ease of notation we define the following class of inner transfer functions, $\mathcal{B}_k^p := \{\tilde{\Theta} \in \mathcal{RH}_\infty^{p\times p} : \tilde{\Theta}^*\tilde{\Theta} = I_p;$ McMillan degree of $\tilde{\Theta} \leq k\}$ Therefore the class $\mathcal{B}_0^p = \{U \in \mathcal{C}^{p\times p} : U^*U = I_p\}$, and define the class $\mathcal{B}_{-1}^p = \{0\}$.

The main (in-) stability results are now stated.

Theorem 3.1 *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f. $(\tilde{N}, \tilde{M})$ respectively and the $m \times p$ transfer function K has a r.c.f. (U, V) and a l.c.f. $(\tilde{U}, \tilde{V})$, then the following statements are equivalent,*

U0) *The pair (G, K) is unstable to order k.*

U5) *There exists an inner transfer function $\tilde{\Theta} \in \mathcal{B}_k^p$ such that*

$$\| P_{[\mathcal{G}^T(K)]^\perp} P_{[\mathcal{G}(M)]^\perp} \Big| \mathcal{H}_2^{p+m}$$

$$\ominus T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} (\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p) \| < 1$$

U6) *There exists an inner transfer function $\tilde{\Theta} \in \mathcal{B}_k^p$ such that*

$$gap\left(\mathcal{G}^T(K), T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} Z_G^{-1}(\tilde{\Theta}\mathcal{H}_2^p)\right) < 1$$

U7) *There exists an inner transfer function $\tilde{\Theta} \in \mathcal{B}_k^p$ such that*

$$\| P_{\mathcal{G}^T(K)} - P_{[G(G)]^\perp} \Big| \begin{pmatrix} \mathcal{H}_2^p \\ \mathcal{H}_2^m \end{pmatrix}$$

$$\ominus T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} (\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p) \| < 1$$

In case the system (G, K) is stable to order k we have the following result.

Theorem 3.2 *Given the assumptions of Theorem 3.1, and further assume that the pair (G, K) is stable to order k then,*

$$\inf_{\tilde{\Theta}\in\mathcal{B}_k^p} gap\left(K, T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} Z_G^{-1}(\tilde{\Theta}\mathcal{H}_2^p)\right)$$

$$= \inf_{\tilde{\Theta}\in\mathcal{B}_k^p} \| P_{\mathcal{G}^T(K)}$$

$$- P_{[\mathcal{G}(G)]^\perp} \mid \mathcal{H}_2^{p+m} \ominus T_{\left[\begin{smallmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{smallmatrix}\right]} (\mathcal{H}_2^p \ominus \tilde{\Theta}\mathcal{H}_2^p) \|$$

$$= \left\| P_{\mathcal{G}^T(K)} P_{\mathcal{G}(G)} \right\| = \| M^*U + N^*V \|_\infty$$

4 Optimally unstable controllers and Hankel norm approximations

The optimal unstable controller to order k of a system G could be defined as the controller which maximizes the minimum angle between the graph of the system and the transposed graph of the controller, subject to the constraint that the closed-loop system is unstable to order k. Though this controller has little significance in terms of design, the result is interesting in that it gives a geometrical interpretation of Hankel norm approximation of non-square inner functions. For this reason the analysis is pursued in this section.

Definition 4.1 *Given a $p\times m$ system G, the optimal minimal angle to order k, $(\theta_{min}^{opt})_k$ is defined by*

$$\cos(\theta_{min}^{opt})_k$$

$$:= \inf_{K\in\mathcal{K}_G} \cos\theta_{min}(\mathcal{G}(M_G), \mathcal{G}^T(M_K))$$

where $\mathcal{K}_G := \{m \times p$ functions K s.t. (G, K) unstable to order $k\}$. Further a controller achieving this infinum is called an optimal unstable controller to order k.

Theorem 4.2 *Suppose the $p \times m$ transfer function G has a r.c.f. (N, M) and a l.c.f.*

$(\tilde{N}, \tilde{M})$ *respectively and let* σ_j *with multiplicity* r_j *be the singular values of the Hankel operator with symbol* $[\tilde{M} \quad - \tilde{N}]^*$ *where* $\sigma_1 > \sigma_2 > \ldots > \sigma_j > \ldots$ *then,*

$$\cos(\theta_{min}^{opt})_k$$

$$:= \inf_{K \in \mathcal{K}_G} \cos \theta_{min}(\mathcal{G}(M_G), \mathcal{G}^T(M_K)) = \sigma_i$$

where $\sum_{j=1}^{i-1} r_j \leq k < \sum_{j=1}^{i}$. *An optimal unstable controller to order k exists and every optimal unstable controller has a right coprime factorization* (U, V) *such that*

$$\left\| \begin{bmatrix} \tilde{M}^* \\ -\tilde{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} \tilde{\Theta}^* \right\|_\infty = \sigma_i$$

where $\tilde{\Theta} \in \mathcal{RH}_\infty^{p \times p}$ *is the unique inner function such that* $(\tilde{M}V - \tilde{N}U)\tilde{\Theta}^*$ *is a unit.*

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