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A functional approach to LQG balancing

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LQG balanced realizations have become important in various areas of control and systems theory mainly in the context of model reduction and robust control. LQG balanced realizations were introduced by Jonckheere and Silverman [4] to be able to perform model reduction on a closed loop system where the controller is an LQG controller. They defined LQG balanced realizations as realizations such that the solutions to the controller and filter Riccati equation are identical and diagonal. They have also become important in robust control since they are suitable realizations to calculate optimally robust controllers for coprime factor uncertainties ([5]).

In [6] a parametrization of the class of all linear systems of fixed McMillan degree was given in terms of LQG balanced realizations. The approach that was taken there was to relate a state space realization of a transfer function to the state space realization of a coinner function made up by the normalized coprime factors of the transfer function. In this way a canonical form for stable coinner functions could be used to produce a canonical form for transfer functions of minimal systems.

Because of the strong relationship between balanced realizations and Hankel operators this approach pointed to the possibility of studying LQG balanced realizations from an operator theoretic point of view by relating LQG balanced realizations to Hankel operators with coinner symbols. This is the topic of the present paper. The techniques which are used are polynomial and functional analytic methods as they were used in [2] to study Lyapunov balanced realizations. In particular it is shown how LQG balanced realizations can be defined in terms of rational models. It becomes clear that the point of view taken here leads to substantial insight into the properties of LQG balanced realizations.

Balanced realizations for aysmptotically stable systems and minimal systems are defined by the requirement that the positive definite solution to a Lyapunov equation respectively algebraic Riccati equation and the solution to its dual equations are identical and diagonal. The diagonal entries to this solution are invariants of the system.

For asymptotically stable balanced systems, i.e. Lyapunov balanced systems it is well known that these invariants are the singular values of the Hankel operator that corresponds to the system ([3],[2]). The significance of these invariants was not well known for the case of LQG balanced realizations and we are going to show that for for LQG balanced realizations these quantities have an interpretation as singular values of a Hankel operator that can be associated with the system. The methods that we will be using are those of polynomial and rational models. This study also leads to an interpretation of the sign parameters that appear in the parametrization of the various classes of systems if restricted to scalar systems ([6]).

For a detailed analysis of this problem see [1].

Let $G = NM^{-1} = \overline{M}^{-1}\overline{N}$ be the normalized right and left coprime factorizations of the transfer function G.

Lemma 0.1 *Consider the Bezout equation* $\overline{M}V - \overline{N}U = I$.

1. *There exists a unique solution* $[U_L, V_L]$ *of the*

Bezout equation such that

$$R^* = M^*U_L + N^*V_L \in H^\infty_-$$

and R^* is strictly proper.

Our aim is to study the Hankel operator $H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$, given a NLCF $g = \overline{M}^{-1}\overline{N}$ of the transfer function $g = \frac{e}{d}$, where e and d are coprime polynomials, with d monic of degree n. Note that since g is assumed to be a scalar transfer function that the NLCF and the NRCF coincide. Our study will be based on the relationship between $H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$ and H_{R^*}. We recall the basic setup and at the same time introduce the notation which we will be using in the sequel. To this end, let us take a polynomial spectral factorization

$$ee^* + dd^* = tt^*$$

or

$$(\frac{e}{t})(\frac{e}{t})^* + (\frac{d}{t})(\frac{d}{t})^* = 1.$$

Here t is stable and normalized to be monic. In this case

$$\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} d/t \\ e/t \end{pmatrix}$$

and

$$\begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix} = \begin{pmatrix} -\frac{e}{t} & \frac{d}{t} \end{pmatrix}.$$

For the associated function R^* we have $R^* =: r^*/t^*$. Note that the polynomials r and t are coprime, since the McMillan degrees of R and g are the same. Then $KerH_{R^*} = \frac{t^*}{t}H^2_+$ and so

$$\{KerH_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}\}^\perp = \{KerH_{R^*}\}^\perp$$

$$= \{S_K H^2_+\}^\perp = X^t,$$

with $S_K = \frac{t^*}{t}$. The function $\frac{p_i}{t}$ is defined to be a (non-normalized) Schmidt vector of H_{R^*}, then $\frac{p_i}{t}$ is also a (non-normalized) Schmidt vector of $H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$. The functions in $ImH_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$ are of the form $\begin{pmatrix} a/t^* \\ b/t^* \end{pmatrix}$ for some polynomials a and b. Therefore the i^{th} Schmidt pair of $H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$ with singular value σ_i can be written as

$$\left(\begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix}, \frac{p_i}{t}\right),$$

for some polynomials $\hat{p}_1^{(i)}$, $\hat{p}_2^{(i)}$ whose degrees are less than the degree of t.

The operator $\hat{H}_{[-\overline{N}\ \overline{M}]}$ restricted to $\{Ker\hat{H}_{(-\overline{N}\ \overline{M})}\}^\perp$ acts by multiplication. Therefore the *singular value/singular vector equations* for $H_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}$ can be written as

$$H_{\begin{pmatrix} -\overline{N}^*\overline{M}^* \end{pmatrix}}\frac{p_i}{t} = P_-\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}\frac{p_i}{t}$$

$$= \sigma_i\begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix}$$

$$H^*_{\begin{pmatrix} -\overline{N}^* \\ \overline{M}^* \end{pmatrix}}\begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix}$$

$$= \begin{pmatrix} -\overline{N} & \overline{M} \end{pmatrix}\begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix} = \sigma_i\frac{p_i}{t}.$$

By partial fraction decomposition there exist polynomials $\pi_1^{(i)}, \pi_2^{(i)}$ of degree at most $n-1$, such that

$$\begin{pmatrix} -e^*/t^* \\ d^*/t^* \end{pmatrix}\frac{p_i}{t} = \sigma_i\begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix} + \begin{pmatrix} \pi_1^{(i)}/t \\ \pi_2^{(i)}/t \end{pmatrix}$$

$$\begin{pmatrix} -\frac{e}{t} & \frac{d}{t} \end{pmatrix}\begin{pmatrix} p_1^{(i)}/t^* \\ p_2^{(i)}/t^* \end{pmatrix} = \sigma_i\frac{p_i}{t},$$

or, rewritten polynomially,

$$\begin{pmatrix} -e^* \\ d^* \end{pmatrix} p_i$$

$$= \sigma_i t \begin{pmatrix} \hat{p}_1^{(i)} \\ \hat{p}_2^{(i)} \end{pmatrix} + t^* \begin{pmatrix} \pi_1^{(i)} \\ \pi_2^{(i)} \end{pmatrix}$$

$$-e\hat{p}_1^{(i)} + d\hat{p}_2^{(i)} = \sigma_i t^* p_i$$

Another way to rewrite these equations is

$$\begin{pmatrix} -e^*/t^* \\ d^*/t^* \end{pmatrix}$$

$$= \sigma_i \frac{t}{t^*} \begin{pmatrix} \hat{p}_1^{(i)}/p_i \\ \hat{p}_2^{(i)}/p_i \end{pmatrix} + \begin{pmatrix} \pi_1^{(i)}/p_i \\ \pi_2^{(i)}/p_i \end{pmatrix}.$$

A LQG balanced realization is defined as follows.

Definition 0.1 *A minimal system (A, B, C) is called* LQG balanced *if there exists a diagonal matrix $\Sigma = diag(\mu_1, \dots, \mu_n) > 0$, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$, such that*

$$\begin{cases} A\Sigma + \Sigma\tilde{A} + B\tilde{B} - \Sigma\tilde{C}C\Sigma = 0 \\ \tilde{A}\Sigma + \Sigma A + \tilde{C}C - \Sigma B\tilde{B}\Sigma = 0. \end{cases}$$

The diagonal entries of the matrix Σ are called the LQG singular values *of the system.*

We now come to derive a LQG balanced realization of the transfer function g. Again this realization will be shown to be the matrix representation of the shift realization with respect to a basis that is constructed from the Schmidt vectors of H_{R^*}.

Theorem 0.1 *Let $g = \frac{e}{d}$ and let $\frac{e}{t}$, $\frac{d}{t}$ be the normalized coprime factors of g. Let $R^* = \frac{r^*}{t^*}$ be the function associated with the LQG controller. Assume the singular values of H_{R^*} are $\mu_1 > \dots > \mu_n > 0$. Let $\{\frac{p_i}{t}, \epsilon_i \frac{p_i^*}{t^*}\}$ be the μ_i-Schmidt pairs of H_{R^*}. Then*

1. *$\{\frac{p_i}{d}\}_{i=1}^n$ is a basis for X^d.*
2. *If we normalize the basis so that*

$$\left\|\frac{p_i}{t}\right\|^2 = \sigma_i(1 - \sigma_i^2)^{\frac{1}{2}}$$

then the matrix representation of the shift realization of g with respect to this basis is LQG balanced. Specifically we have

$$\begin{cases} a_{ji} &= -\epsilon_j p_{i,n-1} p_{j,n-1} \left(\dfrac{1 - \lambda_i\lambda_j}{\lambda_i + \lambda_j}\right) \\ b_i &= \epsilon_i p_{i,n-1} \\ c_i &= p_{i,n-1} = \epsilon_i b_i, \end{cases}$$

where $\lambda_i = \epsilon_i \mu_i$.

3. *The previous LQG balanced realization is signature symmetric. Specifically, with $J = diag(\epsilon_1, \dots, \epsilon_n)$ we have*

$$JA = \tilde{A}J$$

$$\tilde{C} = JB.$$

4. *With respect to the constructed LQG balanced realization we have*

$$\frac{p_i}{d} = C(zI - A)^{-1}e_i.$$

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