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OVERLAPPING BLOCK-BALANCED CANONICAL FORMS AND PARAMETRIZATIONS: THE STABLE SISO CASE

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Abstract: The balanced canonical form and parametrization of Ober for the case of SISO stable systems are extended to block- balanced canonical forms and related input-normal forms and parametrizations. They form an overlapping atlas of parametrizations of the manifold of stable SISO systems of given order. This extends the usefullness of these parametrizations, e.g. in gradient algorithms for system identification.

1. Introduction

In [5],[6] a canonical state space form was presented for the set of asymptotically stable linear systems, with the property that it is balanced, i.e. for each system represented in canonical form the corresponding observability and controllability Grammians are equal and diagonal (and positive definite). One motivation for studying balanced realizations and balanced canonical forms is their close relation to model reduction (see [6] and the references given there), Another motivation mentioned in [6] is the potential usefullness of balanced realizations for system identification. In many cases, in system identification as well as in related areas, one can reduce the problem at hand to an optimization problem in which some criterion function is optimized over a set of systems. Very often one cannot solve the optimization problem analytically and one has to use search algorithms (e.g. gradient algorithms), in which an initial point in the set of systems is adapted iteratively to give a hopefully good approximation of the optimal system. In such search algorithms one often uses a parametrization of the set of relevant systems. The balanced parametrization of [6] has the advantage that by construction, problems of identifiability are to a large extend avoided in such a search algorithm. The parametrization has the property that it contains structural indices (i.e. discrete-valued parameters), and with each possible choice of values for these indices corresponds a particular submanifold of systems, for which a parametrization in terms of real-valued parameters is given. To each system corresponds a unique set of structural indices. As the structural indices can take a large number of values, even for rather low-order systems (the number of possibilities increases fast with increasing order of the system), this means that in a search algorithm one has to either identify the structural indices by other means or one has to apply the search algorithm to a large number of parametrized submanifolds of systems. This is due to the fact that the parametrizations are disjoint.

Several authors (see e.g. [7,1] and the references given there) have investigated the possibility of using socalled overlapping parametrizations (in differential geometric terms: an atlas of coordinate charts). If one uses overlapping parametrizations, one does not have to search through each and every of the submanifolds, but instead one can search through the manifold as a whole, using the parametrizations to describe the manifold locally and changing from one parametrization to another when required. In case the search algorithm is of the gradient type, one can make sure that the decision rule for changing from one parametrization to another has little effect on the search algorithm by using a Riemannian gradient, with respect to some suitable Riemannian metric on the manifold (cf. [1] and the references given there).

In view of this it would be very desirable if the balanced parametrization of [6] could be extended to give a set of overlapping parametrizations. In this paper such an extension will be presented for the case of SISO stable systems. In the extension balancedness of the realization no longer holds for all realizations. Instead (what we will call) block-balanced realizations are used and the corresponding input-normal realizations. With a block-balanced canonical form we mean a canonical form for which the observability and controllability Grammian are equal and block-diagonal (and of course positive definite).

2. Canonical forms, balanced realizations and block-balanced realizations

Let us consider continuous time SISO systems of the form $\dot{x}_t = Ax_t + bu_t, y_t = cx_t$ with $t \in \mathbf{R}, u_t \in \mathbf{R}, x_t in \mathbf{R}^n, y_t \in \mathbf{R}, A \in \mathbf{R}^{n\times n}, b \in \mathbf{R}^{n\times 1}, c \in \mathbf{R}^{n\times 1}, (A,b,c)$ a minimal triple. Let for each $n \in \{1,2,3,\cdots\}$ the set C_n be given by $C_n = \{(A,b,c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n} | (A,b,c)$ minimal and the spectrum of A is contained in the open left half plane$\}$. As is well-known two minimal system representations (A_1,b_1,c_1) and (A_2,b_2,c_2) have the same transfer function $g(s) = c_1(sI - A_1)^{-1}b_1 = c_2(sI - A_2)^{-1}b_2$, and therefore describe the same input-output behaviour, iff there exists an $n \times n$ matrix $T \in Gl_n(\mathbf{R})$ such that $A_1 = TA_2T^{-1}, b_1 = Tb_2, c_1 = c_2T^{-1}$. In that case we say that (A_1,b_1,c_1) and (A_2,b_2,c_2) are i/o-equivalent. This is clearly an equivalence relation; write $(A_1,b_1,c_1) \sim (A_2,b_2,c_2)$. A unique representation of a linear system can be obtained by deriving a canonical form:

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Definition 2.1 *A canonical form for an equivalence relation '$\sim$' on a set X is a map $\Gamma : X \to X$ which satisfies for all $x, y \in X$: (i) $\Gamma(x) \sim x$ and (ii) $x \sim y \Longleftrightarrow \Gamma(x) = \Gamma(y)$*

Equivalently a canonical form can be given by the image set $\Gamma(X)$; a subset $B \subseteq X$ describes a canonical form if for each $x \in X$ there is precisely one element $b \in B$ such that $b \sim x$. The mapping $X \to B, x \mapsto b$ then describes a canonical form.

Let $(A, b, c) \in C_n$. As is well-known the controllability Grammian W_c can be obtained as the unique, positive definite symmetric solution of the Lyapunov equation $AW_c + W_c A^T = -bb^T$. In a dual fashion, the observability Grammian W_o is the unique, positive definite symmetric solution of the Lyapunov equation $A^T W_o + W_o A = -c^T c$.

Definition 2.2 *Let $(A, b, c) \in C_n$, then (A, b, c) is called* balanced *if the corresponding observability and controllability Grammians are equal and diagonal, i.e. there exist positive numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ such that*

$$W_o = W_c = diag(\sigma_1, \dots, \sigma_n) =: \Sigma \tag{1}$$

The numbers $\sigma_1, \dots, \sigma_n$ are called the (Hankel) singular values of the system.

The singular values are known to be uniquely determined by the input-output behaviour of the system.

Theorem 2.3 (Moore 1981) *Let $(A, b, c) \in C_n$ with*

$$\Sigma = diag(\sigma_1 I_{n(1)}, \dots, \sigma_k I_{n(k)}), \sigma_1 > \sigma_2 > \dots \sigma_k > 0 \text{ and } \sum_{j=1}^{k} n(j) = n.$$

Then (A, b, c) is unique up to an orthogonal state-space transformation of the form

$$Q = diag(Q_1, Q_2, \dots, Q_k)$$

with orthogonal $Q_i \in \mathbf{R}^{n(i) \times n(i)}, i = 1, \dots, k$.

Definition 2.4 *Let $(A, b, c) \in C_n$, then (A, b, c) is called* input-normal *if $W_c = I_n$ and will be called σ-*input-normal *if $W_c = \sigma I_n$.*

Similarly (A, b, c) is called output-normal *if $W_o = I_n$ and σ-*output-normal *if $W_o = \sigma I_n$.*

It is not difficult to show that an input-normal realization is unique up to an arbitrary orthogonal state-space transformation.

The following definition is new and basic to our considerations in this paper.

Definition 2.5 *Let $(A, b, c) \in C_n$, then (A, b, c) will be called* block-balanced, *with indices $n(i) \in \mathbf{N}, i = 1, \dots, k$, adding up to n, if the observability Grammian and the controllability Grammian are equal and block-diagonal, i.e. there exist $n(i) \times n(i)$ positive definite matrices $\Sigma_i, i = 1, \dots, k$, such that*

$$W_o = W_c = diag(\Sigma_1, \dots, \Sigma_k)$$

It will be convenient to call an arbitrary system representation $(A, b, c) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times 1} \times \mathbf{R}^{1 \times n}$ block-balanced if the pair of Lyapunov equations $A\Sigma + \Sigma A^T = -bb^T, A^T\Sigma + \Sigma A = -c^T c$ has a positive definite solution of the form $\Sigma = diag(\Sigma_1, \dots, \Sigma_k)$ (assuming neither asymptotic stability nor minimality).

Remark. The matrices $\Sigma_i, i = 1, \dots, k$ are in general *not* uniquely determined by the input-output behaviour of the system. However the eigenvalues $\lambda_1(\Sigma_i) \geq \lambda_2(\Sigma_i) \geq \dots \geq \lambda_{n(i)}(\Sigma_i)$ of the matrices $\Sigma_i, i = 1, \dots, k$ together form the set of Hankel singular values of the system, which are uniquely determined by the input- output behaviour of the system, as remarked before.

Theorem 2.6 *Suppose $(A, b, c) \in C_n$ is block-balanced with indices $n(j) \in \mathbf{N}, j = 1, \dots, k, \sum_{j=1}^{k} n(j) = n$ and with the additional property $\lambda_1(\Sigma_1) \geq \lambda_{n(1)}(\Sigma_1) > \lambda_1(\Sigma_2) \geq \lambda_{n(2)}(\Sigma_2) > \dots > \lambda_1(\Sigma_k) \geq \lambda_{n(k)}(\Sigma_k) > 0$. This uniquely determines (A, b, c) up to an orthogonal state-space transformation of the form $Q = diag(Q_1, \dots, Q_k)$ with orthogonal $Q_i \in \mathbf{R}^{n(i) \times n(i)}, i = 1, \dots, k$*

Proof. See [2].

The following theorem will be fundamental for our results.

Theorem 2.7 (Pernebo and Silverman, [8], Kabamba, [3]) *Let $(A, b, c) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times 1} \times \mathbf{R}^{1 \times n}$ be conformally partitioned as follows:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, c = \begin{pmatrix} c_1 & c_2 \end{pmatrix},$$

with $A_{ii} \in \mathbf{R}^{n(i) \times n(i)}, i = 1, 2$ and let (A, b, c) be block-balanced with indices $n(1), n(2)$ such that $\Sigma_1, \Sigma_2 > 0$ have no eigenvalues in common.

Then $(A, b, c) \in C_n \Leftrightarrow (A_{ii}, b_i, c_i) \in C_{n(i)}, i = 1, 2$.

3. The case k=1: a Schwarz-like canonical form for stable SISO systems in continuous time

Theorem 3.1 *Consider the set B_n of all $(A, b, c) \in C_n$ of the following form:*

$$A = \begin{pmatrix} a_{11} & \alpha_1 & & 0 \\ -\alpha_1 & 0 & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ 0 & & -\alpha_{n-1} & 0 \end{pmatrix}, a_{11} = -\frac{b_1^2}{2} < 0,$$

$$\alpha_i > 0, i = 1, \dots, n-1,$$

$$b = \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_1 > 0,$$

$$c = \begin{pmatrix} c_1 & \gamma_1 & \dots & \gamma_{n-1} \end{pmatrix}, c_1 \in \mathbf{R}, \gamma_j \in \mathbf{R}, j = 1, \dots, n-1.$$

Each triple $(A, b, c) \in B_n$ is input-normal.

Let S_n be the set of values of the vector of parameters $(b_1, \alpha_1, \dots, \alpha_{n-1}, c_1, \gamma_1, \dots, \gamma_{n-1})$ such that the corresponding triple $(A, b, c) \in B_n$, i.e. such that $b_1 > 0, \alpha_i > 0, i = 1, \dots, n$ and $c_1, \gamma_1, \dots, \gamma_{n-1}$ such that the pair (c, A) is observable.

The set B_n describes a continuous canonical form and the parametrization mapping $S_n \longrightarrow B_n$, which maps each parameter vector to the corresponding triple (A, b, c), is a homeomorphism.

If $(\gamma_1, \dots, \gamma_{n-1}) \neq 0 \in \mathbf{R}^{n-1}, n \geq 2$, then the system has several different singular values.

Proof See [2].

Remarks (i) If $c_1 \neq 0$ we define $\sigma := \left|\frac{c_1}{b_1}\right| > 0$, which we will call a pseudo-singular value. If the vector $\gamma = (\gamma_1, \dots, \gamma_{n-1})$ is close enough to zero the pseudo-singular value will be close to the true singular values of the system, because of continuity of the singular values as a function of γ and the fact that if $\gamma = 0$, the system has only one singular value and its value is σ. If $c_1 \neq 0$ the system can be brought simply into $\sigma-$input-normal form by multiplying c by $\sigma^{-\frac{1}{2}}$ and b by $\sigma^{\frac{1}{2}}$. The resulting $\sigma-$input -normal form is a *canonical* form locally around $\gamma = 0$, but not globally because the systems which have $c_1 = 0$ in the previous canonical form cannot be represented in this way. (ii)

Clearly the canonical forms presented are controllable (because they are input-normal, resp. σ-input-normal), but observability will fail for certain choices of c; the observability Grammian will be singular for such a choice of c. If $\gamma = 0, c_1 \neq 0$, the system is observable, because the observability Grammian will be $\sigma^2 I$, resp. σI. Therefore also in some open neighbourhood around such a system, observability will still hold. (iii) This canonical form is closely related to the so-called Schwarz canonical form, cf. e.g. [4], [9].

4. An input-normal and a block-balanced canonical form

Let $n(1), \ldots, n(k) \in \{1, 2, \ldots, n\}, \sum_{j=1}^{k} n(j) = n$, denote a partition of n as before. Let $C_{n(1),n(2),\ldots,n(k)}$ denote the subset of all systems in C_n with the property that their n Hankel singular values (multiplicities *included*) $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) > 0$ can be partitioned into k disjoint sets of singular values (again with multiplicities included) in the following way:

$$\begin{aligned} \sigma(1) &\geq \ldots \geq \sigma(n(1)) > \sigma(n(1)+1) \geq \\ &\geq \ldots \geq \sigma(n(1)+n(2)) > \sigma(n(1)+n(2)+1) \geq \\ &\geq \ldots \geq \sigma(\sum_{j=1}^{l} n(j)) > \sigma((\sum_{j=1}^{l} n(j)) + 1) \geq \\ &\geq \ldots > 0 \end{aligned} \tag{2}$$

So we require that $\sigma(\sum_{j=1}^{l} n(j)) > \sigma((\sum_{j=1}^{l} n(j)) + 1)$ for $l = 1, 2, \ldots, k-1$ and $\sigma(n) > 0$ of course. Note that the notation is consistent with the fact that C_n denotes the set of stable systems which have as their only "restriction" that there are n positive singular values (multiplicities included), i.e. that the order of the system is n.

The other extreme is $C_{1,1,\ldots,1}$, which denotes the set of n-th order stable systems with n *distinct* singular values. For this set of systems a balanced canonical form was derived in [3].

Next we will present a canonical form on $C_{n(1),\ldots,n(k)}$.

Theorem 4.1 *Consider the set $B_{n(1),\ldots,n(k)}$ of triples (A, b, c) of the following form:*

$$\begin{aligned} A &= (A(i,j))_{1 \leq i,j \leq k}, \\ A(i,j) &\in \mathbf{R}^{n(i) \times n(j)}, i,j \in \{1, \ldots, k\} \\ b &= \begin{pmatrix} b(1) \\ b(2) \\ \vdots \\ b(k) \end{pmatrix}, b(i) \in \mathbf{R}^{n(i)}, i = 1, \ldots, k, \\ c &= (c(1), \ldots, c(k)), c(j)^T \in \mathbf{R}^{n(j)}, j = 1, \ldots, k, \\ A(i,i) &= \begin{pmatrix} a(i,i)_{11} & \alpha(i)_1 & 0 & \ldots & 0 \\ -\alpha(i)_1 & 0 & \alpha(i)_2 & \ddots & \vdots \\ 0 & -\alpha(i)_2 & & \ddots & 0 \\ \vdots & \ddots & \ddots & & \alpha(i)_{n(i)-1} \\ 0 & \ldots & 0 & -\alpha(i)_{n(i)-1} & 0 \end{pmatrix}, \\ a(i,i)_{11} &= -\frac{b_i^2}{2}, \\ \alpha(i)_j &> 0, j = 1, \ldots, n(i) - 1, \\ b(i) &= \begin{pmatrix} b_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_i > 0, \\ c(i) &= \left(c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}\right), i = 1, \ldots, k, \end{aligned}$$

where the parameters are to be taken such that the corresponding observability Grammians $\Sigma_i^2, i = 1, \ldots, k$, which satisfy the observability Lyapunov equations

$$\Sigma_i^2 A(i,i) + A(i,i)^T \Sigma_i^2 = -c(i)^T c(i) \tag{3}$$

are fulfilling the following matrix inequalities

$$\Sigma_1^2 > \Sigma_2^2 > \ldots > \Sigma_k^2 > 0; \tag{4}$$

for each pair $(i,j), i \neq j$, the matrices $A(i,j), A(j,i)$ are determined (uniquely!) from the following pair of linear matrix equations:

$$\begin{aligned} A(i,j) + A(j,i)^T &= -b(i)b(j)^T \\ \Sigma_i^2 A(i,j) + A(j,i)^T \Sigma_j^2 &= -c(i)^T c(j) \end{aligned} \tag{5}$$

The set $B_{n(1),\ldots,n(k)}$ describes a continuous canonical form on $C_{n(1),\ldots,n(k)}$. The $2n$ "free" parameters of the canonical form are

$$b_i, \alpha(i)_1, \ldots, \alpha(i)_{n(i)-1}, c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}, i = 1, \ldots, k.$$

Let $S_{n(1),\ldots,n(k)} \subset \mathbf{R}^{2n}$ be the set of all values of the parameter vector for which the corresponding triple $(A, b, c) \in B_{n(1),\ldots,n(k)}$, i.e. for all $i \in \{1, \ldots, k\}$: $b_i > 0, \alpha(i)_j > 0, j = 1, \ldots, n(i) - 1$, and $c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}$ such that the matrices $\Sigma_i, i = 1, \ldots, k$, found in (3) satisfy the inequalities (4). The mapping $S_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$ which maps a parameter vector to the corresponding triple (A, b, c) is a homeomorphism.

The form is input-normal, i.e. $A + A^T = -bb^T$ and has block-diagonal observability Grammian $\Sigma^2 := diag(\Sigma_1^2, \ldots, \Sigma_k^2) > 0$.

Let $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) > 0$ denote the n positive Hankel singular values of the system (with their multiplicities).

If for some $i \in \{1, \ldots, k\}$ the vector $\gamma(i) = 0$, then Σ_i^2 is a scalar matrix $\Sigma_i^2 = \sigma^2\left(1 + \sum_{j=1}^{i-1} n(j)\right) . I_{n(i)}$, and

$\sigma\left(\sum_{j=1}^{i-1} n(j)\right) > \sigma\left(1 + \sum_{j=1}^{i-1} n(j)\right) = \ldots = \sigma\left(\sum_{j=1}^{i} n(j)\right) > \sigma\left(1 + \sum_{j=1}^{i} n(j)\right)$

If for all $i \in \{1, \ldots, k\}, \gamma(i) = 0$, then the observability Grammian is consequently diagonal.

Remark. A block-balanced realization can be obtained from the presented canonical form by applying a state-space transformation

$$T := \Sigma^{\frac{1}{2}} = \operatorname{diag}\left(\Sigma_1^{\frac{1}{2}}, \ldots, \Sigma_k^{\frac{1}{2}}\right) > 0 \tag{6}$$

The corresponding controllability and observability Grammians will both be equal to

$$\Sigma = \operatorname{diag}(\Sigma_1, \ldots, \Sigma_k) > 0$$

Proof. See [2].

5. An atlas of overlapping block-balanced canonical forms

Theorem 5.1 *Let the state space dimension n be fixed. The continuous canonical forms $C_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$, $n(j) \in \{1, \ldots, n\}; j = 1, \ldots, k$; $\sum_{j=1}^{k} n(j) = n$; $k \in \{1, \ldots, n\}$, form an overlapping set of continuous canonical forms covering C_n. Each of the sets $C_{n(1),\ldots,n(k)}$, $\sum_{j=1}^{k} n(j) = n$, is an open subset of C_n and together they cover C_n.*

Proof. See [2].

Corollary 5.2 *The set of mappings $C_{n(1),\dots,n(k)}/\sim \longrightarrow S_{n(1),\dots,n(k)} \subset \mathbf{R}^{2n}$, $(n(1),\dots,n(k)) \in P(n;k), k = 1,\dots,n$, which map each equivalence class of triples to the corresponding parameter vector in the canonical form, forms an atlas for the manifold of stable SISO i/o-systems of order n.*

Proof. See [2].

Remark. A motivation for using this atlas rather than e.g. just the Schwarz-like canonical form B_n is the following. Suppose one wants to use *balanced realizations*. Then one can use the balanced canonical form of [6]. However this form is discontinuous at all points of $C_{n(1),\dots,n(k)} \setminus C_{1,\dots,1}$, i.e. in all triples $(\tilde{A},\tilde{b},\tilde{c})$ which have two or more coinciding singular values. And the complement $C_{1,\dots,1}$, of the set of discontinuity points consists of 2^n topological components, one component for each sign pattern; this should be compared to C_n which has only $n+1$ topological components (the Brockett components). It appears that this is a serious disadvantage if one wants to use balanced realizations and canonical forms in e.g. search algorithms for system identification.

In order to overcome these difficulties one could use the overlapping block-balanced canonical forms as follows. If $(\tilde{A},\tilde{b},\tilde{c})$ has k distinct Hankel singular values $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$ with multiplicities resp. $n(1),\dots,n(k)$, then one can use the block-balanced continuous canonical form on $C_{n(1),\dots,n(k)}$ *locally around* $(\tilde{A},\tilde{b},\tilde{c})$. If one is moving away from $(\tilde{A},\tilde{b},\tilde{c})$ in a search algorithm for example, one has to decide whether the canonical form corresponding to a different partition should be used: if the largest $n(1)$ singular values differ sufficiently from each other one could use e.g. $C_{1,\dots,1,n(2),\dots,n(k)}$ (where there are $n(1)$ ones in the subindex before $n(2)$) etc. In this way one would use balanced realizations and "almost- balanced" realizations while moving around in the set of n-th order systems, without encountering discontinuity points.

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