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Stability analysis of infinite dimensional discrete and continuous time linear systems

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Abstract. The question of power and asymptotic stability of infinite dimensional discrete-time state space systems is investigated. It is shown that every balanced realization is asymptotically stable. Conditions are given for balanced, input normal or output normal realization to be asymptotically and/or power stable.

1. Introduction. Balanced realizations for finite dimensional systems have received a great deal of attention. They were introduced as a means of performing model reduction in an easy fashion [7] and have subsequently been used in H^∞ control theory, for example, to evaluate the Hankel norm of a linear system [4], [5]. Recently, they have been used to study parametrization problems of certain sets of linear systems [8].

The elegant results obtained for finite dimensional balanced systems brought about some interest in the problem of the extension of the notion of a balanced realization to infinite dimensional systems. Glover, Curtain and Partington [5] derived continuous-time balanced realizations for a class of systems with nuclear Hankel operators. Young [12] developed a very general realization theory for infinite dimensional discrete-time systems. Similar results were obtained in the continuous time case by Ober and Montgomery-Smith [9].

One of the fundamental problems in systems theory is the question of the stability of the system. In this paper we will address this problem in the case of infinite dimensional balanced realizations and the closely related input and output normal realizations. In the case of discrete time systems, by relating balanced realizations to restricted shift realizations, we are able to show that every balanced realization is asymptotically stable. In general, input normal and output normal realizations do not have the same stability properties as balanced realizations, but we can also give necessary and sufficient conditions for them to be asymptotically and/or power stable.

Analogous results for continuous time systems can also be obtained. The main approach here is to use the generalization of a bilinear transformation that is routinely used in finite dimensional case to translate the results for discrete time systems to those for continuous time systems and vice versa.

Let X, Y and U be separable Hilbert spaces. The discrete time linear system

$$(1)\qquad \begin{array}{rcl} x_{k+1} &=& Ax_k + Bu_k \\ y_k &=& Cx_k + Du_k \end{array} \quad k = 0, 1, \cdots$$

where A is a contraction on X with $1 \notin \sigma_p(A)$, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$, will be denoted by the quadruple of operators (A, B, C, D). The set of all such systems is denoted by $D_X^{U,Y}$. For (A, B, C, D) in $D_X^{U,Y}$, its observability operator is defined as

$$\mathcal{O} : X \to l_Y^2, \quad \mathcal{O}x = (CA^n x)_{n \geq 0},$$

for $x \in D(\mathcal{O}) = \{x \mid (CA^n x)_{n\geq 0} \in l_Y^2\}$. If $D(\mathcal{O}) = X$, $Ker(\mathcal{O}) = \{0\}$ and $\mathcal{O}$ is bounded then the system is said to be observable. The reachability of a system (A, B, C, D) is defined through its dual system, (A^*, C^*, B^*, D^*). If (A^*, C^*, B^*, D^*) is observable then (A, B, C, D) is said to be reachable. This is equivalent to that the operator

$$\mathcal{R} : l_U^2 \to X, \quad (u_n)_{n\geq 0} \mapsto \sum_{n\geq 0} A^n B u_n$$

is bounded and has dense range in X. $\mathcal{R}$ is called the reachability operator of (A, B, C, D). If (A, B, C, D) is reachable and observable the observability gramian is defined to be $\mathcal{M} = \mathcal{O}^*\mathcal{O}$ and the reachability gramian is $\mathcal{W} = \mathcal{R}\mathcal{R}^*$.

For a discrete time system $(A, B, C, D) \in D_X^{U,Y}$, the $\mathcal{L}(U, Y)$ valued function $G(z) = C(zI - A)^{-1}B + D$, $z \in \mathbf{D}_e$, is called the (discrete time) transfer function of (A, B, C, D) while (A, B, C, D) is called a realization of G. In this case it is clear that G is analytic on $\mathbf{D}_e$ and at infinity. The discrete time transfer functions treated in this paper are supposed to be bounded and analytic on $\mathbf{D}_e$. Notice that the boundedness and analyticity of a $\mathcal{L}(U, Y)$ valued function G on $\mathbf{D}_e$ means that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ where for $z \in \mathbf{D}$, $G^\perp(z) = z^{-1}[G(z^{-1}) - G(\infty)]$.

The following symbols are used: $\mathbf{D}$ denotes the open unit disk; $\partial\mathbf{D}$ the unit circle; $\mathbf{D}_e$ the exterior of $(\partial\mathbf{D}) \cup \mathbf{D}$; $D(A) \subseteq X$ the domain of an operator A on X; RHP the open right half plane.

2. Stability of discrete time systems. Our results will be mostly based on the investigation of restricted shift realizations whereby the shift realizations can be analyzed in terms of Hankel operators related to the transfer functions. General references in realization theory are e.g. Fuhrmann [3] and Helton [6].

First we recall the restricted shift realization which was first introduced by Fuhrmann [2] and Helton [6] (see also [12]).

THEOREM 2.1. *Let G be a $\mathcal{L}(U, Y)$ valued function such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$. Then G has a state space realization (A, B, C, D) with state space X, i.e. $G(z) = C(zI - A)^{-1}B + D$, for $z \in \mathbf{D}_e$ which is given in the following way:*

The state space X is given by $X = \overline{range} H_{G^\perp} \subseteq H_Y^2(\mathbf{D})$, where $H_{G^\perp} : H_U^2(\mathbf{D}) \to H_Y^2(\mathbf{D})$ is the Hankel operator with symbol $G^\perp$: $H_G^\perp u = P_+ G^\perp J u$, $(u \in H_U^2(\mathbf{D}))$, where $(Ju)(s) = u(-s)$.

The state propagation operator $A : X \to X$, the input operator $B : U \to X$, the output operator $C : X \to Y$ and the feedthrough operator $D : U \to Y$ are given by the following, for $f \in X$,

$$(Af)(z) := (S^* f)(z); \quad (Bu)(z) := G^\perp(z)u, \ (u \in U);$$
$$Cf := f(0); \quad Du := G(\infty)u, \ (u \in U);$$

where S is the (forward) shift operator: $(Sf)(z) = zf(z)$, $f \in H_Y^2(\mathbf{D})$. The realization (A, B, C, D) is called the restricted shift realization *of the transfer function G.*

This realization is observable and reachable. The observability operator $\mathcal{O}$ and reachability operator $\mathcal{R}$ of (A, B, C, D) are respectively given by $\mathcal{O} = I_X : X \to H_Y^2(\mathbf{D})$ and $\mathcal{R} = H_{G^\perp} : H_U^2(\mathbf{D}) \to X$. □

Another realization can be constructed as the dual realization of the restricted shift realization. Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be the restricted shift realization of the transfer function $\tilde{G}(z) := (G(\bar{z}))^*$, $z \in \mathbf{D}_e$. Then the dual system $(\tilde{A}^*, \tilde{C}^*, \tilde{B}^*, \tilde{D}^*)$ is a realization of $G(z)$. It is called the **-restricted shift realization* of G ([8]).

THEOREM 2.2. *The state space representation (A_*, B_*, C_*, D_*) of the *-restricted shift realization is given as follows:*

The state space X_* is $X_* = \overline{range} H_{\tilde{G}^\perp}$ with $\tilde{G}^\perp(z) = (G^\perp(\bar{z}))^*$. The operators A_*, B_*, C_* and D_* are defined as

$$A_* := P_{X_*} S_{|X_*}; \quad B_* u := P_{X_*} u, \ (u \in U);$$
$$C_* f := (H_{G^\perp} f)(0), \ (f \in X_*); \quad D_* = G(\infty),$$

where P_{X_*} is the orthogonal projection from $H_U^2(\mathbf{D})$ onto X_*.

This realization is observable and reachable. The reachability and observability operators $\mathcal{R}_*$ and $\mathcal{O}_*$ are respectively given by $\mathcal{R}_* = P_{X_*} : H_U^2(\mathbf{D}) \to X_*$ and $\mathcal{O}_* = H^*_{\tilde{G}^\perp}|X_* = H_{G^\perp}|X_*$. □

These two realizations are important because they represent two classes of systems: the input normal and output normal systems as defined in the following ([7]).

Let $(A, B, C, D) \in D_X^{U,Y}$ be a reachable and observable discrete-time system. Then the system is (i) *output normal*, if $\mathcal{M} = I$; (ii) *input normal*, if $\mathcal{W} = I$; (iii) *par-balanced*, if $\mathcal{M} = \mathcal{W}$; (iv) *balanced*, if $\mathcal{M} = \mathcal{W}$ and there is an orthonormal basis of the state space with respect to which $\mathcal{M}$ (and hence $\mathcal{W}$) has a diagonal matrix representation.

From our results on the restricted and the *-restricted shift realization we immediately have that the restricted shift realization is output normal whereas the *-restricted shift realization is input normal.

We now turn to the study of stability. The two notions of stability we will consider in the sequel are the follows:

A discrete time system $(A, B, C, D) \in D_X^{U,Y}$ or the state propagation operator A is called (i) *asymptotically stable* if for every $x \in X$ $A^k x \to 0$, as $k \to \infty$, (ii) *power stable* if $r := \inf\{\bar{r} | \exists M_{\bar{r}} > 0$ such that $\|A^k\| \leq M_{\bar{r}} \bar{r}^k$, $k = 0, 1, \ldots\} < 1$. The number r is called the *degree of power stability.*

It can be seen that stability, observability as well as reachability of discrete-time systems are preserved under equivalent transformations whereas input and output normality is preserved under unitary equivalence. Moreover, two equivalent power stable systems have the same degree of power stability. Note that two systems (A, B, C, D) and (A_1, B_1, C_1, D_1) are (unitarily) equivalent if $(A_1, B_1, C_1, D_1) = (VAV^{-1}, VB, CV^{-1}, D)$ where V is a bounded and boundedly invertible operator (a unitary operator).

In [12], it is shown that any two input normal systems are unitarily equivalent; similarly two output normal or two par-balanced systems are equivalent. Therefore we can establish all stability and other important results concerning input and output normal realizations by restricting ourselves to restricted and *-restricted shift realizations. Notice that the state propagation operators in the restricted and *-restricted realizations are respectively restricted left and right shift operators. Hence the stability study of input/output normal realizations amounts to the study of the restricted shift operators. The following is the summary of our main results. In formulating the results we will use the notion of cyclicity of analytic transfer functions. A transfer function G is said to be strictly noncyclic if $G = QF^*$ where Q is inner, Q, $F \in H^\infty_{\mathcal{L}(U,Y)}$ and Q and F are *weakly left coprime.*

THEOREM 2.3. *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and let (A, B, C, D) be an output normal and (A_1, B_1, C_1, D_1) an input normal realization of G. Then*

1. *(A, B, C, D) is asymptotically stable.*
2. *If $\tilde{G}^\perp$ is strictly noncyclic, then (A_1, B_1, C_1, D_1) is asymptotically stable.*
3. *Every par-balanced realization is asymptotically stable.*

THEOREM 2.4. *Let G be such that $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and let U and Y be finite dimensional. Then an output normal or input normal realization of G is power-stable if and only if G is rational.*

THEOREM 2.5. *Let $G^\perp \in H^\infty_{\mathcal{L}(U,Y)}(\mathbf{D})$ and U and Y be finite dimensional. Assume that $G^\perp$ is strictly noncyclic. Then a par-balanced realization of G is power-stable if and only if G is rational.*

3. Stability of continuous time systems. We restrict ourselves to the so called *admissible* continuous time systems defined below. More details concerning infinite dimensional continuous time state space systems and realizations of nonrational transfer functions can be found in [3], [9], [11] and [1].

DEFINITION 3.1. *A quadruple of operators (A_c, B_c, C_c, D_c) is called an* admissible continuous time system *with state space X, input space U and output space Y, where X, U, Y are separable Hilbert spaces, if*

1. *$(A_c, D(A_c))$ is the generator of a strongly continuous semigroup of contractions on X.*
2. *$B_c : U \to (D(A_c^*)^{(\prime)}, \|\cdot\|')$ is a bounded linear operator.*
3. *$C_c : D(C_c) \to Y$ is linear with $D(C_c) = D(A_c) + (I - A_c)^{-1} B_c U$ and $C_{c|D(A_c)} : (D(A_c), \|\cdot\|_{A_c}) \to Y$ is bounded.*
4. *$C_c(I - A_c)^{-1} B_c \in L(U, Y)$*
5. *A_c, B_c, C_c are such that $\lim_{\substack{s \in \mathbf{R} \\ s \to +\infty}} C_c(sI - A_c)^{-1} B_c = 0$ in the norm topology.*
6. *$D_c \in L(U, Y)$.*

We write $C_X^{U,Y}$ for the set of admissible continuous time systems with input space U, output space Y and state space X. □

By the resolvant identity, 4 of the definition implies that $G_c(s) := C_c(sI - A_c)^{-1} B_c \in L(U, Y)$ for all $s \in RHP$ and G_c is analytic on the RHP. The function G_c is called the transfer function of the system and (A_c, B_c, C_c, D_c) is called a realization of G_c.

Because of lack of space, we refer the reader to [9] [10] [11] for the concept of equivalence, reachability, observability, input/output normality, par-balancing and balancing, asymptotic and exponential stability of infinite dimensional continuous time systems.

We can carry most of the stability results in the previous section to continuous time linear systems using a bilinear transformation that maps a discrete time linear system to a continuous time linear system, though the exponential stability properties of continuous time systems have to be obtained independently. Here are the asymptotic stability results. The symbol $TLC^{U,Y}$ will stands for the class of transfer functions that have reachable and observable realizations.

THEOREM 3.1. *Let $G_c \in TLC^{U,Y}$.*

1. *Every output normal realization of G_c is asymptotically stable.*
2. *If G_c is strictly noncyclic, then every input normal realization is asymptotically stable.*
3. *Every par-balanced realization of G_c is asymptotically stable.*

For exponential stability, we have the following:

THEOREM 3.2. *Let $G_c \in TLC^{U,Y}$, with U, Y finite dimensional.*

1. *An input or output normal realization is exponentially stable if and only if G_c is strictly noncyclic and there is $\epsilon > 0$ such that G can be analytically continued to $Re(s) > -\epsilon$.*
2. *If G_c is strictly non-cyclic, then a par-balanced realization is exponentially stable if and only if there is $\epsilon > 0$ such that G can be analytically continued to $Re(s) > -\epsilon$.* □

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