Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

FA11 9:10

Another look at realization theory

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Abstract

A new approach is presented for the realization of continuous-time finite dimensional linear systems. Using standard results on Laplace transforms our results are also used to present a new derivation of Fuhrmann's shift realization for rational matrix functions.

1 Introduction

In the theory of linear dynamical systems the connection between internal state space description and external input-output description is of central importance. Here the realization problem, i.e the determination of a state-space system from knowledge of the input-ouput map, is the difficult part. For finite dimensional systems this problem has been solved for a long time in the fundamental work by Kalman (see e.g. [10]). The field of contributions to realization theory is vast. In this brief introduction it is therefore impossible to try to adequately review all the important contributions that were made. We therefore concentrate on the aspects of the theory that are of importance for this particular paper. Realization theory for infinite dimensional systems was studied by several authors in the seventies (see e.g. [1],[4], [8],[7],[2]) where the theory of shifts on invariant subspaces played a particularly important role. In all these contributions assumptions had to be made on the stability of the system that is being considered. Another important step was done in the work by Fuhrmann in ([5]). Motivated by the powerful results for infinite dimensional he examined finite dimensional systems from the point of view of shift realizations. Through the introduction of his polynomial and rational models he managed to bridge the gap between the approaches built on shift realizations and Kalman's module theoretic approach. At the same time he was able to remove the stability assumption that was necessary in the infinite dimensional context. In common with Kalman's work his work essentially concentrated on discrete-time systems. These results can of course be applied to the finite dimensional continuous-time realization problem. But it often appears artificial to determine a state space realization of a continuous-time system by what are essentially techniques motivated by and typically used for the study of discrete-time systems. We therefore tried to develop a setup that stays within the realm of continuous-time systems and does not require advanced mathematical methods such as for example the work by Yamamoto ([14],[15]) which is mainly devoted to infinite dimensional systems. In fact the approach to the realization problem presented in this paper was developed by the author as part of a graduate course on linear system theory which mainly focusses on finite dimensional continuous-time systems. Another objective of this paper is to try to clearly explain the role that Hankel type maps play in the derivation of realizations for finite dimensional continuous-time systems.

The paper has a tutorial character in that we try to present the complete development of the approach. Our approach is equally valid for systems with coefficients in the real or complex numbers. We denote by $\mathcal{K}$ the field of scalars which could either be the real field or the complex field.

A continuous-time time-invariant linear n-dimensional state space system is as usual given in the following way by a *state space* X, which is a n dimensional Euclidean space over the field $\mathcal{K}$, and by a quadruple of linear transformations $A : X \to X$, $B : \mathcal{K}^m \to X$, $C : X \to \mathcal{K}^p$, $D : \mathcal{K}^m \to \mathcal{K}^p$,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, && (1)\\ y(t) &= Cx(t) + Du(t), && (2) \end{aligned}$$

$t \geq t_0$, $t \in \Re$. If a basis is given in the state space X, then we can identify X with $\mathcal{K}^n$ and we can think of the transformations A, B and C to be given in matrix form. We will also often refer to a system by referring to its quadruple of transformations (A, B, C, D).

If the input u is piecewise continuous, the solution to the set of differential equations is given by

$$x(t) = \int_{t_0}^{t} e^{(t-\tau)A} Bu(\tau)d\tau + e^{(t-t_0)A}x_0$$

and the output is given as a function of the input in

[1]This research was supported in part by NSF grant: DMS 9501223

0-7803-3970-8/97 $10.00 © 1997 IEEE

the following way,

$$y(t) = \int_{t_0}^{t} Ce^{(t-\tau)A}Bu(\tau)d\tau + Ce^{(t-t_0)A}x_0 + Du(t), \quad (3)$$

$t \geq 0$. If we assume that the initial conditions are zero, then this convolution map (3) completely describes the input-output behavior of the state space system. This leads us to the second definition of a system which we will consider, i.e a definition through convolution maps.

Let $C(\Re)$ be the set of continuous $\mathcal{K}^p$-valued functions on $\Re$ and let $PC(\Re)$ be the set of piecewise continuous functions on $\Re$ and let $PC_{tr}(\Re)$ be the subset of $PC(\Re)$ of piecewise continuous functions on $\Re$ that are zero for t small enough.

Definition 1.1 *Let $M(t) \in \mathcal{K}^{p\times m}$, $t \geq 0$, be a continuous function and let $D \in \mathcal{K}^{p\times m}$. Then the map*

$$IO_{M,D} : PC_{tr}(\Re) \rightarrow PC(\Re); \qquad u \mapsto y,$$

where $y(t) = \int_{-\infty}^{t} M(t-\tau)u(\tau)d\tau + Du(t)$, $t \in \Re$, is called the input-output map with symbol M and feedthrough term D.

To avoid convergence problems we have restricted the inputs to be zero for large enough negative time. If (A, B, C, D) is a state space system, then the input-output map $IO_{M,D}$ with symbol $M(t) = Ce^{tA}B$, $t \geq 0$, is called the *input-output map associated with the state space system.* It is easily seen that two state space systems that are related by a state space transformation have the same associated input-output map. A formulation of the realization problem is to determine conditions under which an input-output map is in fact the input-output map of a state space system.

In order to show that this input-output map is well defined with range in $PC(\Re)$, we show in the next Proposition that if $D = 0$ then the range of $IO_{M,D}$ is in $C(\Re)$.

Proposition 1.1 *Let M be a continuous function on $\Re_+$. Then for the input-output map $IO_{M,D}$ with $D = 0$,*

$$range(IO_{M,D}) \subseteq C(\Re).$$

In the following section we introduce the observability, reachability and Hankel maps and discuss their connections. In the subsequent section we derive the time domain realization result in terms of a shift realization where the state-space is the range of the Hankel map. Using the Laplace transform as a state space transformation in the final section Fuhrmann's shift realization of a rational function is rederived from the continuous-time realization result.

Due to the space constraints of these proceedings, no proofs can be given. They will be published elsewhere.

1.1 Notation

Denote by $\mathcal{K}$ the real field $\Re$ or the complex field $\mathcal{C}$. $\Re_+ = \{\lambda \in \Re | \lambda \geq 0\}$. $\Re_- = \{\lambda \in \Re | \lambda \leq 0\}$. By a Euclidean space we mean a finite dimensional vector space over $\mathcal{K} = \Re$ or $\mathcal{K} = \mathcal{C}$, with inner product $<\cdot,\cdot>$. The inner product is understood to be linear in the first component and anti-linear in the second component. $C(I)$ stands for the set of (vector/matrix-valued) continuous functions on the interval $I \subseteq \Re$, which can be unbounded. $PC(I)$ stands for the set of piecewise continuous (vector-valued) functions on the interval $I \subseteq \Re$. If I is a bounded interval, f is in $PC(I)$ if f has at most a finite number of discontinuities on I and at each point p in I the left and right limits of f exist and are finite. If I is infinite, f is in $PC(I)$ if for each bounded interval $I_b \subseteq I$, the restriction of f to I_b is in $PC(I_b)$. $PC_{tr}(I)$ denotes the subset of truncated functions of $PC_{tr}(I)$. A function $f \in PC(I)$ is called truncated if there exists $t_f \in \Re$, s.t. $f(t) = 0$ for $t < t_f$. The constant t_f depends on f. $\sigma(T)$ denotes the set of eigenvalues of the linear map T. $ker(T)$ stands for the kernel of the linear map T and $range(T)$ stands for the range of the linear map T. The rank of the linear map T is denoted by $rank(T)$. $T|_V$ denotes the restriction of the map T to the subspace V. $\|a\|$ stands for the norm of the map or vector a. $A \perp B$ denotes that the set A is orthogonal to the set B. The characteristic function χ_A is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

2 Reachability, observability and Hankel maps

Central notions in systems theory are those of the reachability and observability of a system. We give definitions of these notions which are suitable to our way of studying linear systems. The observability map $\mathcal{O}$ is defined as the map which maps the state of the system at time zero to the output of the system for positive time if no input is applied to the system. A system is then called observable if no non-zero state is mapped to the zero output. The reachability map maps a given input function in $PC_{tr}(\Re_-)$ applied from time $-\infty$ to time 0, to the state that is reached at time zero if this input is applied. A system is called reachable if all states can be reached in this way.

Definition 2.1 *Let (A, B, C, D) be a linear system with state space X, then*

1. *the map*

$$\mathcal{O} : X \rightarrow C(\Re_+); \quad x \mapsto (Ce^{tA}x)_{t\geq 0},$$

is called the observability map *of the system. The system is called* observable *if* $\mathcal{O}x \equiv 0$ *implies that* $x = 0$, *i.e. if* $ker(\mathcal{O}) = \{0\}$.

2. *the map*

$$\mathcal{R} : PC_{tr}(\Re_-) \to X; \quad u \mapsto \int_{-\infty}^{0} e^{-\tau A} B u(\tau) d\tau,$$

is called the reachability map *of the system. Here* $PC_{tr}(\Re_-)$ *is the set of truncated piecewise continuous* $\mathcal{K}^m$*-valued functions on the negative real line, i.e. those functions which are zero for* t *small enough. The system is called* reachable *if* $range(\mathcal{R}) = X$.

A state space system is called minimal *if it is both observable and reachable.*

In order to avoid possible problems with the convergence of the integral in the definition of the reachability map for systems, the inputs have been restricted to the class of truncated piecewise continuous functions on $\Re_-$, i.e. to functions in $PC_{tr}(\Re_-)$.

Clearly $\mathcal{O}$ and $\mathcal{R}$ are linear maps. Since the domain of $\mathcal{O}$ and the range space of $\mathcal{R}$ is the n-dimensional state space X, the rank of $\mathcal{O}$ and $\mathcal{R}$ is at most n. An n-dimensional linear system is therefore observable if and only if the observability map $\mathcal{O}$ has rank n. It is reachable if and only the reachability map has rank n.

The following proposition shows that our notion of reachability leads to the usual reachability criterion (see e.g. [9]). By duality the same also applies to the observability criterion.

Proposition 2.1
An n*-dimensional system* (A, B, C, D) *is reachable if and only if*

$$rank \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n,$$

Moreover, $range(\mathcal{R}) = span\{A^k B \mathcal{K}^m \mid k \geq 0\}$.

An important map in our study of linear systems is the so-called *Hankel map.* The Hankel map that is associated with a system maps 'past' inputs of the system into 'future' outputs. Before clarifying this point further, we will define what we mean by a Hankel map.

Definition 2.2 *Let* $M(t) \in \mathcal{K}^{p \times m}, t \geq 0$, *be a continuous function. Then the map*

$$H_M : PC_{tr}(\Re_-) \to C(\Re_+);$$

$$u(\cdot) \mapsto y(\cdot) = \int_{-\infty}^{0} M(\cdot - \tau) u(\tau) d\tau,$$

is called the Hankel map with symbol M.

The Hankel map H_M is related to the input-output map $IO_{M,D}$ as follows. If $D = 0$ and $u \in PC_{tr}(\Re)$ is such that $u(t) = 0$ for $t > 0$, then

$$(H_M(u))(t) = (IO_{M,D}(u))(t),$$

for $t \geq 0$. Hence the Hankel map maps those inputs of an input-output map with $D = 0$, that are zero for positive time, to the outputs of the system for positive time. That the range of the Hankel map is contained in $C(\Re_+)$ is a consequence of Proposition 1.1. The definition of the Hankel map therefore follows the same philosophy that underlies the use of Hankel operators in the realization theory of stable continuous-time (infinite-dimensional) systems (see e.g. [6]). What is important from our point of view is that with this definition of the Hankel map no stability assumptions have to be made on the systems that are being studied.

The particular application of Hankel maps that we have in mind is when the symbol M of the Hankel map is in fact the impulse response of a linear system (A, B, C, D), i.e. $M(t) = Ce^{tA}B$, $t \geq 0$. This Hankel map is called the Hankel map *associated* with the linear system.

In the following proposition we show that if the symbol of the Hankel map is the impulse response of a system then the reachability and observability maps of the system introduce a factorization of the Hankel map.

Proposition 2.2 *Let* (A, B, C, D) *be a linear system and let* $M(t) := Ce^{tA}B$, $t \geq 0$. *Let* $\mathcal{O}$ *be the observability map and let* $\mathcal{R}$ *be the reachability map of the system* (A, B, C, D), *then the Hankel map* H_M *with symbol* M *can be factored as*

$$H_M = \mathcal{O}\mathcal{R}.$$

In the following proposition we are going to show that the Hankel map of a minimal n-dimensional system has rank n.

Proposition 2.3 *Let* (A, B, C, D) *be a* n*-dimensional system. Let* $M(t) := Ce^{tA}B$, $t \geq 0$. *Then*

1. *the Hankel map with symbol* M *has at most rank* n.

2. *the Hankel map has rank* n *if and only if* (A, B, C, D) *is minimal.*

The range of the Hankel map associated with a system can be described easily in terms of the system maps. This is of importance in particular in connection with our approach to realization theory.

Corollary 2.1 *Assume that (A, B, C, D) is a minimal system. With the notation of the previous proposition we have*

$$range(H_M) = \{(Ce^{tA}x)_{t\geq 0} \mid x \in X\}$$

where X is the state-space of the system. Moreover each $f \in range(H_M)$ is infinitely often continuously differentiable.

3 Realization theory

In the introductory section we discussed two representations for linear systems. One representation was in terms of state space models, the other representation was in terms of input-output maps. We immediately saw that a state space model gave rise in a natural way to an input-output map. In this section we are going to further clarify the connection between these two types of system representations. More precisely we are going to study the realization problem, i.e. we are going to establish necessary and sufficient conditions for an input-output map to be the input-output map of a state space system. The conditions will be in terms of a finite rank assumption on the Hankel map associated with the input-output map. If a realization exists we are going to construct the left shift realization.

Before proving this realization theorem we will need to establish that the Hankel map intertwines the left shift and prove a number of other preliminary results. The left shift is defined as follows.

Definition 3.1 *For $t \geq 0$, the map*

$$L_t : C(\Re_+) \to C(\Re_+);$$

$$(f(\tau))_{\tau\geq 0} \mapsto (L_t f) = (f(\tau + t))_{\tau\geq 0},$$

is called the left shift (with increment t) *on $C(\Re_+)$. Similarly, for $t \geq 0$, the map*

$$L_t : PC_{tr}(\Re_-) \to PC_{tr}(\Re_-);$$

$$(f(\tau))_{\tau\leq 0} \mapsto L_t f = (L_t f(\tau))_{\tau\leq 0}$$

$$= \begin{cases} f(\tau + t), & \tau + t \leq 0, \\ 0, & \tau + t > 0, \end{cases} \quad \tau \leq 0,$$

is called the left shift (with increment t) *on $PC_{tr}(\Re_-)$.*

Note that there is a slight abuse of notation in that we use the same letter to denote the left shift both on the space $C(\Re_+)$ and on the space $PC_{tr}(\Re_-)$.

In the following proposition we are going to show that the Hankel map intertwines the left shift.

Proposition 3.1 *Let H_M be a Hankel map. Then for each $t \geq 0$,*

$$H_M L_t = L_t H_M.$$

The following corollary shows that the range and kernel of a Hankel map are invariant under the left shift. This is an important property and will be crucial for the realization theory presented later.

Corollary 3.1 *Under the assumptions of the previous proposition we have,*

1. *$range(H_M)$ is invariant under L_t, i.e. for each $t \geq 0$, $L_t(range(H_M)) \subseteq range(H_M)$.*
2. *$ker(H_M)$ is invariant under L_t, i.e. for each $t \geq 0$, $L_t(ker(H_M)) \subseteq ker(H_M)$.*

The following well-known proposition states that a continuous semigroup of operators on a finite dimensional Euclidean space can be written as the exponential function of a linear map.

Proposition 3.2 *Let X be a finite-dimensional Euclidean space and let $T(t) : X \to X$, $t \geq 0$ be a semigroup of linear maps on X, i.e. $T(0) = I$, $T(t + s) = T(t)T(s)$, for $t, s \geq 0$, and $\|T(t) - I\| \to 0$ as $t \to 0+$. Then there exists a unique linear map $A : X \to X$ such that*

$$T(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k,$$

for $t \geq 0$.

Let $IO_{M,D}$ be an input-output map with symbol M. We need to show that if the Hankel map H_M has finite rank then Mu_0 is in the range of H_M for each $u_0 \in \mathcal{K}^m$.

Lemma 3.1 *Let $g_1, g_2, \ldots, g_n$ be a basis of an n-dimensional subspace of the space of vector-valued functions $C(\Re_+)$. Then there exist distinct points $t_1, t_2, \ldots, t_n \in \Re_+$ such that the matrix*

$$\mathcal{G} := \begin{pmatrix} g_1(t_1) & g_2(t_1) & \ldots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \ldots & g_n(t_2) \\ \vdots & \vdots & & \vdots \\ g_1(t_n) & g_2(t_n) & \ldots & g_n(t_n) \end{pmatrix}$$

has zero kernel. The left inverse $\mathcal{F}$ of $\mathcal{G}$, i.e. $\mathcal{F}\mathcal{G} = I$, is such that if $f \in C(\Re_+)$ is a linear combination of $g_1, g_2, \ldots, g_n$, i.e.

$$f = \alpha^{(1)} g_1 + \alpha^{(2)} g_2 + \cdots + \alpha^{(n)} g_n$$

for some $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)} \in \mathcal{K}$ *then*

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix} = \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix}.$$

Moreover the subspace spanned by $g_1, g_2, \ldots, g_n$ *is closed under pointwise convergence.*

Lemma 3.2 *Let* H_M *be a Hankel map of finite rank* n. *Then* $Mu_0 \in range(H_M)$ *for all* $u_0 \in \mathcal{K}^m$.

We are now in a position to prove the realization theorem.

Theorem 3.1 *Let* H_M *be a Hankel map of finite rank* n. *Then there exists a minimal* n*-dimensional state space system* (A, B, C, D) *such that*

$$M(t) = Ce^{tA}B, \quad t \geq 0.$$

Such a system is given by

$$X := range(H_M) \subseteq C(\Re_+),$$
$$B : \mathcal{K}^m \to X; \quad u \mapsto Bu := Mu,$$
$$C : X \to \mathcal{K}^p; \quad x \mapsto Cx := x(0),$$
$$A : X \to X,$$

is such that for $t \geq 0$,

$$e^{tA} : X \to X; \quad x \mapsto L_t x.$$

The map $D : \mathcal{K}^m \to \mathcal{K}^p$ *is linear and arbitrary. The state space* X *has an arbitrary Euclidean structure.*

The realization (A, B, C, D) introduced in the previous theorem is called the *(left) shift realization.* For different approaches to obtain the left-shift realization for infinite-dimensional continuous-time systems see e.g. ([8],[13]).

In Section 1 it was shown that each state space system gives rise to an input-output system. The following Corollary states that an input-output system is the input-output system of a state space system if and only if the associated Hankel map has finite rank.

Corollary 3.2 *Let* M *be a* $\mathcal{K}^{p\times m}$*-valued function on* $[0, \infty[$. *Then, the following two statements are equivalent,*

1. *the input-output map with kernel function* M *is the input-output map of a state space system.*
2. *the Hankel map* H_M *has finite rank.*

In the following corollary we show that for the left shift realization the reachability and observability maps can be derived. It turns out that the observability map is the identity map on the state space, whereas the reachability map is the Hankel map with symbol M whose range is restricted to the state space.

Corollary 3.3 *We use the notation of the previous theorem. Then the observability map* $\mathcal{O}$ *of the system* (A, B, C, D) *is given by*

$$\mathcal{O} : X \to C(\Re_+); \quad x \mapsto x,$$

and the reachability map $\mathcal{R}$ *of the system is*

$$\mathcal{R} : PC_{tr}(\mathcal{R}_-) \to X; \quad u \mapsto H_M u.$$

As an immediate consequence of the above realization result we have that the functions in the range of the Hankel map H_M are infinitely often differentiable.

Corollary 3.4 *Let* $H_M : PC_{tr}(\Re_-) \to C(\Re_+)$ *be a finite rank Hankel map. Then*

1. *each* $f \in range(H_M)$ *is infinitely often continuously differentiable.*
2. M *is infinitely often continuously differentiable.*
3. *if* (A, B, C, D) *is the left shift realization of* M *we have for* $f \in range(H_M)$,

$$Af = f'.$$

4 Transfer functions

We can now consider the realization problem for rational functions. Using the realization result of the previous section we are going to derive Fuhrmann's shift realization for rational models ([5]). The main tool to achieve this will be a state space transformation which is obtained by applying the Laplace transform to the elements in the state space of the left shift realization. Following the tutorial style of this paper we will recall without proof some facts on Laplace transforms which are important for our development. As general references to these results on Laplace transforms see e.g. ([11],[12],[3]).

Let $\mathcal{E}$ be the set of piecewise continuous functions $f : \mathcal{R}_+ \to \mathcal{K}^{r\times l}$ such that $f(t)e^{-at} \to 0$ as $t \to \infty$ for some $a \in \mathcal{R}$. A function $f \in \mathcal{E}$ is said to be in the set $\mathcal{E}_a$ if

$f(t)e^{-bt} \to 0$ as $t \to \infty$ for all $b > a$. For $f \in \mathcal{E}_a$ the *(one-sided) Laplace transform* of f is given by

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}dt,$$

for all $s \in \mathcal{C}$ with $Re(s) > a$. Clearly the Laplace transform could be defined on a larger class of functions, but the presently chosen setup will be sufficient for our purposes.

In the following Proposition some standard results are being stated concerning the Laplace transform of the impulse response of a system.

Proposition 4.1 *Let (A, B, C, D) be a minimal system and let $M(t) := Ce^{tA}B$, $t \geq 0$. Then*

1. *$M \in \mathcal{E}_a$, where $a = \max\{Re(\lambda) \mid \lambda \in \sigma(A)\}$.*
2. *$(\mathcal{L}M)(s) = C(sI - A)^{-1}B$, for all $s \in \mathcal{C}$, $Re(s) > a$.*
3. *$\mathcal{L}M$ is a rational matrix function which has no poles in the half plane $\{s \in \mathcal{C} \mid Re(s) > a\}$.*

We can now prove the realization result for rational functions, i.e that each proper rational function can be written as the transfer function of a linear system. An important aspect of the result is that the particular realization, which is derived, is the transfer domain equivalent of the left shift realization. The following result is nothing else but Fuhrmann's shift realization for rational models ([5]) stated and proved in our framework.

Theorem 4.1 *Let G be a proper rational matrix function. Then there exists a state space X and a minimal system (A, B, C, D) such that*

$$G(s) = C(sI - A)^{-1}B + D, \quad s \in \mathcal{C} \setminus \sigma(A),$$

which is defined as follows: for a strictly proper rational function f let $(F(f))(s) := sf(s) - \lim_{s\to\infty} sf(s)$, $s \in \mathcal{C}$. Let $G_p(s) := G(s) - G(\infty)$, $s \in \mathcal{C}$. The state space X is defined to be the linear span of the spaces $F^k(G_p\mathcal{K}^m)$, $k = 0, 1, 2, \ldots$. The system maps are defined as,

$$B : \mathcal{K}^m \to X; u \mapsto G_p u,$$

$$C : X \to \mathcal{K}^p; x \mapsto \lim_{s\to\infty} sx(s),$$

$$A : X \to X; x \mapsto Ax,$$

where

$$(Ax)(s) = sx(s) - \lim_{s\to\infty} sx(s), \quad s \in \mathcal{C}.$$

$$D = G(\infty).$$

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