Spectral Minimality of J-Positive Linear Systems of Finite Order

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Dedicated to Paul Fuhrmann on the occasion of his 60^{th} *birthday*

Abstract: It is shown that reachable completely J-positive linear systems of finite order are spectrally minimal.

1 Introduction

It is well-known (see e.g. [4],[7]) that a minimal finite dimensional linear system (A, B, C, D) is *spectrally minimal*, i.e.

$$\sigma(A) = \sigma(G),$$

where $\sigma(A)$ denotes the set of eigenvalues of A and $\sigma(G)$ denotes the set of singularities of the rational transfer function $G(z) = C(zI - A)^{-1}B + D$. This fact is one of the key relationships between the transfer function description and the state space description of system theory. Unfortunately, for infinite dimensional systems spectral minimality does not hold in general. The issue is very subtle and is, for example, related to the problem that the state space isomorphism theorem does not hold in general for infinite dimensional systems (see e.g. [4]), i.e. two reachable and observable infinite dimensional realization of a nonrational transfer function need not be equivalent.

The work by Fuhrmann and co-workers has been fundamental for the current understanding of the issue of spectral minimality for infinite dimensional systems (see [4] for an exposition). Through this work it is clear that spectral minimality results can still be obtained if certain restrictions are imposed on the problem. In ([4],[5]) it was shown that restricting the class of transfer functions to those which are strictly non-cyclic, spectral minimality can be established for reachable and observable realizations of such transfer functions. This work relies very heavily on the functional calculus for shifts restricted to invariant subspaces. Another mathematical tool, the spectral theorem and functional calculus for self-adjoint and, respectively, normal operators, provides the techniques underlying the second set of results. In these results ([1],[2]) spectral minimality is shown for infinite dimensional reachable and observable systems (A, B, C, D) such that A is selfadjoint and, respectively, normal. In a paper by Feintuch ([3]) further spectral minimality results were obtained for the case when A is compact or a spectral operator.

In this paper we will study systems with symmetry properties in an indefinite metric. We will examine a class of systems which we consider to be a prototype of a more general situation, more precisely, the class of completely J-positive systems of finite order. For these systems, our main result, cf. Theorem 5, shows that observable and reachable realizations are also spectrally minimal.

2 Preliminaries

Let $\mathcal{H}$ be a Hilbert space with the scalar product denoted by $\langle\cdot,\cdot\rangle$ and let J be a fixed symmetry on $\mathcal{H}$, that is $J^* = J = J^{-1}$. Then one can introduce an indefinite inner product on $\mathcal{H}$ denoted $[\cdot,\cdot]$

$$[x, y] = \langle Jx, y\rangle, \quad x, y \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}$ endowed with such an indefinite inner product $[\cdot,\cdot]$ is called a *Kreĭn space.* Most often one does not fix the positive definite inner product (there are infinitely many and all of them produce the same strong topology) of a Kreĭn space, but even though this point of view is the most natural, we will not follow this approach since we would have to introduce too much Kreĭn space terminology.

A bounded operator $A \in \mathcal{L}(\mathcal{H})$ is called *J-selfadjoint* if $JA = A^*J$. It is clear that the operator A is J-selfadjoint if and only if the operator JA is selfadjoint in the Hilbert space $\mathcal{H}$. A J-selfadjoint operator A on $\mathcal{H}$ is called *J-positive of order n* if $JA^n \geq 0$. Similarly one defines J-negative operators of order n. A J-positive operator of order 1 is called simply a J-positive operator.

We briefly review some of the main results on the spectral theory for J-positive operators which will be of importance in this paper. In the following we denote by $\mathcal{R}_0$ the Boole algebra generated by intervals Δ in $\mathbb{R}$ such that its boundary $\partial\Delta$ does not contain the point 0. We recall now a particular case of a celebrated theorem of H. Langer and some of its consequences, cf. [8], [9].

Theorem 1 *Let $A \in \mathcal{L}(\mathcal{H})$ be a J-positive operator of order n. Then $\sigma(A) \subset \mathbb{R}$ and there exists a mapping $E\colon \mathcal{R}_0 \to \mathcal{L}(\mathcal{H})$, uniquely determined with the following properties:*

(1) *$E(\Delta)$ is J-selfadjoint for all $\Delta \in \mathcal{R}_0$.*

(2) *E is a Boole algebra morphism, that is, it is additive and multiplicative.*

(3) $E(\mathbb{R}) = I$.

(4) *For all $\Delta \in \mathcal{R}_0$ such that the polynomial t^n is positive (negative) on Δ, the operator $E(\Delta)$ is J-positive (J-negative).*

(5) *For all $\Delta \in \mathcal{R}_0$ the operator $E(\Delta)$ is in $\{A\}''$ (the bicommutant of the algebra generated by the operator A).*

(6) *For all $\Delta \in \mathcal{R}_0$ we have $\sigma(A|E(\Delta)\mathcal{H}) \subseteq \overline{\Delta}$.*

The mapping E which is uniquely associated to the J-positive operator A of some finite order n is called *the spectral function* of A. As a consequence of Theorem 1, the spectral function has also the following properties.

Corollary 1 *With the notation as in Theorem 1 let $\Delta \in \mathcal{R}_0$ be closed and such that $0 \notin \Delta$. Then:*

(a) *The function E_Δ defined by*

$$E_\Delta(\Lambda) \quad E(\Delta \cap \Lambda), \quad \Lambda \in \mathcal{R}_0,$$

can be extended uniquely to a bounded measure with $\operatorname{supp} E_\Delta \subseteq \Delta$.

(b) *The operator $AE(\Delta)$ is similar with a selfadjoint operator on a Hilbert space, in particular it has spectral measure.*

(c) *E_Δ is the spectral measure of the operator $AE(\Delta)$, in particular*

$$AE(\Delta) \quad \int_\Delta t \, \mathrm{d}\, E(t).$$

Corollary 1 shows that the spectral function E of a J-positive operator of some finite order n can be regarded as a, generally unbounded, spectral measure on $\mathbb{R} \setminus \{0\}$.

According to the general theory of a selfadjoint definitizable operator A in a Kreĭn space $\mathcal{K}$ [8], [9], the non-real points in the spectrum of A as well as the real points in the spectrum $\sigma(A)$ where the spectral function E is not defined, the so-called *critical points*, have some additional properties. For example, an isolated point in the spectrum of A is necessarily an eigenvalue of finite geometric multiplicity (that is, the maximal length of Jordan chains). Also, if a critical point is an eigenvalue then its geometric multiplicity is finite. In particular, in the case of a J-positive operator of order n, its only possible critical point is 0 and, if it is an eigenvalue, then its geometric multiplicity is less than or equal to $n+1$.

In this paper we consider *linear systems*, i.e. quadruples (A, B, C, D) where $A \in \mathcal{L}(\mathcal{H})$ is a contraction, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and $\mathcal{H}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces. Usually the spaces $\mathcal{U}$, $\mathcal{H}$, and $\mathcal{Y}$ are called, respectively, the *input space*, the *state space* and the *output space*. Also, the operators A, B, C, and D are called, respectively, the *main operator*, the *input operator*, the *output operator*, and the *feedthrough operator*.

With every linear system (A, B, C, D) there is associated its *transfer function* $G: \rho(A) \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ as follows

$$G(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \rho(A). \tag{1}$$

Since the main operator A is assumed contractive, the transfer function is defined and analytic for all $|\lambda| > 1$.

Let us assume that $\mathcal{U} = \mathcal{Y}$ and that on $\mathcal{H}$ there is fixed a symmetry J (and hence the associated Kreĭn space $(\mathcal{H}, [\cdot, \cdot])$). A linear system (A, B, C, D) is called *completely J-symmetric* if the operator A is J-selfadjoint, $C = JB^*$, and $D = D^*$. The completely J-symmetric system is called *completely J-positive of order n* if the operator A is J-positive of order n.

In [6] an extensive analysis was carried out for J-positive systems of finite order, but the question of spectral minimality was only partially resolved. It is the topic of this paper to establish spectral minimality for reachable and observable J-positive systems of finite order.

The transfer function of a completely J-symmetric system is given in the following theorem.

Theorem 2 ([6]) *Let (A,B,C,D) be a linear system which is completely J-positive of order n, such that $\mathcal{U} = \mathcal{Y} = \mathbb{C}^m$, and consider its transfer function G as in (1). Then, there exist a J-positive operator $N \in \mathcal{L}(\mathcal{H})$, such that $N^2 = AN = 0$, and a symmetric matrix valued Borel measure $\mathrm{d}\,\nu$ on $[-1, 1] \setminus \{0\}$ such that*

$$G(\lambda) = D \quad \sum_{k=1}^{n} \frac{1}{\lambda^k} B^* J A^{k-1} B + \frac{1}{\lambda^{n+1}} B^* J N B + \frac{1}{\lambda^n} \int_{[-1,1]\setminus\{0\}} \frac{t^n}{(\lambda - t)} \mathrm{d} \quad). \tag{2}$$

The measure $\mathrm{d}\,\nu$ has also the following two properties.

(a) *$t^n \mathrm{d}\,\nu(t)$ is a positive matrix valued finite Borel measure on $[-1, 1]$;*

(b) *The function*

$$g(z) = \frac{1}{z}\left(G\left(\frac{1}{z}\right) - D\right) = \sum_{k \geq 0} a_k z^k \tag{3}$$

which is analytic for $|z| < 1$, has its Taylor coefficients

$$a_k = \begin{cases} B^*JA^kB, & 1 \le k \le n-1; \\ B^*JNB + \int\limits_{[-1,1]\setminus\{0\}} t^n \mathrm{d}\,\nu(t), & k = n; \\ \int\limits_{[-1,1]\setminus\{0\}} t^k \mathrm{d}\,\nu(t), & k \ge n+1. \end{cases} \tag{4}$$

The measure $\mathrm{d}\,\nu$ *is uniquely determined by these two properties, more precisely, if* E *denotes the spectral function of* A *we have* $\mathrm{d}\,\nu(t) = \mathrm{d}\,B^*JE(t)B$*, and the operator* N *can be chosen*

$$N = A^n - \int\limits_{\mathbb{R}\setminus\{0\}} t^n \mathrm{d}\,E(t). \tag{5}$$

If, in addition, $\pm 1 \notin \sigma_p(A)$ *then* $\mathrm{d}\,\nu(\{-1,1\}) = 0$ *and* $\lim_{k\to\infty} \|a_k\| = 0$.

The matrix valued measure $\mathrm{d}\,\nu$ as in Theorem 2 is called the *defining measure* of the system (A, B, C, D). Under the assumptions of Theorem 2 and as a consequence of the representation (2) it follows that the transfer function G has analytic continuation onto $\mathbb{C} \setminus \operatorname{supp}(\mathrm{d}\,\nu)$.

Let $\mathcal{U}$ be a Hilbert space and assume that $G\colon \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U})$ is an operator valued function which is analytic everywhere on its domain of definition and at infinity. One can define an operator valued analytic function $g\colon \mathbb{D} \to \mathcal{L}(\mathcal{U})$ by

$$g(z) = \frac{1}{z}\left(G\left(\frac{1}{z}\right) - G(\infty)\right), \quad |z| < 1.$$

Then g has the Taylor expansion

$$g(z) = \sum_{k\ge 0} S_k z^k, \quad |z| < 1.$$

Associated with the function g one can consider the block-operator Hankel matrix

$$H = \begin{bmatrix} S_0 & S_1 & S_2 & \dots & S_k & \dots \\ S_1 & S_2 & S_3 & \dots & S_{k+1} & \dots \\ S_2 & S_3 & \dots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ \vdots & & & & & \end{bmatrix}. \tag{6}$$

We consider again a completely J-positive system (A, B, B^*J, D) of order n, where J is a fixed symmetry on the state space $\mathcal{H}$, and let $\mathcal{U}$ denote the input/output space. Following the general theory we consider $O\colon \mathcal{D}(O)(\subseteq \mathcal{H}) \to \ell^2_{\mathcal{U}}$, the *observability operator* defined by

$$\mathcal{D}(O) = \{h \in \mathcal{H} \mid \sum_{k\geq 0} \|B^* J A^k h\|^2 < \infty\},$$

$$Oh = (B^* J A^k h)_{k\geq 0}, \quad h \in \mathcal{D}(O).$$

By duality one introduces the *reachability operator* $R\colon \mathcal{D}(R)(\subseteq \ell^2_{\mathcal{U}}) \to \mathcal{H}$

$$\mathcal{D}(R) = \{(x_k)_{k\geq 0} \in \ell^2_{\mathcal{U}} \mid \sum_{k\geq 0} \|A^k B x_k\|^2 < \infty\},$$

$$R((x_k)_{k\geq 0}) = \sum_{k\geq 0} A^k B x_k, \quad (x_k)_{k\geq 0} \in \mathcal{D}(R).$$

Note that the domain of R is dense in $\ell^2_{\mathcal{U}}$ since it contains all the sequences with finite support.

In the following results the observability and reachability of J-positive systems of finite order are exmanined.

Theorem 3 ([6]) *If $\pm 1 \notin \sigma_p(A)$ then $\mathcal{D}(O)$ is dense in $\mathcal{H}$, $O = R^* J$ and $R = JO^*$, in particular both operators O and R are closed.*

Corollary 2 ([6]) *Assume that $\pm 1 \notin \sigma_p(A)$. Then the following assertions are equivalent:*

(i) *the observability operator O is bounded;*

(ii) *the reachability operator R is bounded.*

Recall that a system (A, B, C, D) is called *observable* if the observability operator O is bounded and injective. The system is called *reachable* if the reachability operator R is bounded and has dense range. Note that, as a consequence of Theorem 3, the completely J-positive system (A, B, B^*J, D) of order n, such that $\pm 1 \notin \sigma_p(A)$, is observable if and only if it is reachable. The kernel of the observability operator is characterized as follows.

Proposition 1 ([6]) *Let E be the spectral function and let N be the nilpotent operator associated with the main operator A of the completely J-positive system (A, B, B^*J, D) of order n. Then*

$$\ker(O) = \bigcap_{k=0}^{n} \ker(B^* J A^k) \cap \bigcap_{\Delta \in \mathcal{R}_0} \ker(B^* J E(\Delta)) \cap \ker(B^* J N).$$

Following N.J. Young [10], we say that a system (A, B, C, D) is *parbalanced* if the corresponding observability and reachability operators O and, respectively, R are bounded and the observability gramian O^*O coincide with the reachability gramian RR^*.

The following realization result was also established in [6]

Theorem 4 *Let $\mathcal{U}$ be a Hilbert space and let $G\colon \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U})$ be an operator valued function which is analytic on its domain and at infinity, such that G is symmetric, that is,*

$$G(\overline{z}) = G(z)^*, \quad |z| > 1.$$

If the Hankel block-operator matrix in (6) defines a bounded operator in $\ell^2_{\mathcal{U}}$ then there exists a Kreĭn state space $\mathcal{K}$ with a specified fundamental symmetry J on $\mathcal{K}$ such that G is realized by a completely J-symmetric linear system (A, B, B^*J, D) which is observable, reachable and parbalanced.

If, in addition, for some $n \geq 0$ and all $|\lambda| > 1$ the function G has the representation

$$G(\lambda) = D + \sum_{k=1}^{n} \frac{1}{\lambda^k} S_{k-1} + \frac{1}{\lambda^{n+1}} \Gamma + \frac{1}{\lambda^n} \int\limits_{[-1,1]\setminus\{0\}} \frac{t^n}{(\lambda - t)} \mathrm{d}\,\nu(t),$$

where $\{S_k\}_{k=0}^{n-1}$ is a family of bounded selfadjoint operators on $\mathcal{U}$, $D \in \mathcal{L}(\mathcal{U})$, $D = D^*$, $\Gamma \in \mathcal{L}(\mathcal{U})$, $\Gamma \geq 0$, and ν is a hermitian $\mathcal{L}(\mathcal{U})$-valued measure on $[-1, 1] \setminus \{0\}$ such that $t^n \mathrm{d}\,\nu(t)$ is a finite and positive measure, then the realization (A, B, C, D) constructed above is completely J-positive of order n.

3 Spectral Minimality

So far we have considered the transfer function of a discrete-time system (A,B,C,D) as a function defined outside the closed disk in the complex plane with center 0 and radius larger than $\|A\|$. In order to study the question of spectral minimality one needs to extend this definition. It is clear that the transfer function can be defined as an analytic function on the resolvent set of A. Let now G be the maximal analytic continuation of this function. The set of singularities $\sigma(G)$ of G are then defined to be the complement of the points of analyticity of G. Clearly $\sigma(G)$ is a closed set. The system (A, B, C, D) is called *spectrally minimal* if $\sigma(G) = \sigma(A)$.

Proposition 2 *Let (A_i, B_i, C_i) be a discrete-time system with bounded system operators and state space $\mathcal{H}_i$ such that the corresponding observability (reachability) operator O_i (R_i) is bounded $i = 1, 2$. Assume that $\sigma(A_1) \cap \sigma(A_2) = \emptyset$. Also assume* *that the resolvent sets of A_1 and A_2 have only one component. Then the system (A, B, C) given by*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

with state space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ is observable (reachable) if and only if the systems (A_1, B_1, C_1) and (A_2, B_2, C_2) are obervable (reachable).

Proof: Let $\mathcal{H}_i$ be the state space of the system (A_i, B_i, C_i), $i = 1, 2$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is the state space of (A, B, C) and $O : \mathcal{H} \to l^2_{\mathcal{Y}}$, $x = (x_1, x_2) \mapsto O_1x_1 + O_2x_2$ is the observability operator of (A, B, C). Clearly O is bounded if and only if O_1 and O_2 are bounded. Moreover the injectivity of O implies injectivity of O_1 and O_2. Therefore the observability of (A, B, C) implies the observability of (A_1, B_1, C_1) and (A_2, B_2, C_2).

Now assume that (A_1, B_1, C_1) and (A_2, B_2, C_2) are observable but that (A, B, C) is not observable. Then there exists $x = (x_1, x_2) \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $0 = Ox = O_1x_1 + O_2x_2$. Let G_i be the transfer function of the discrete-time system (A_i, x_i, C_i). Note that for $|z| > \|A_i\|$, $i = 1, 2$,

$$G_i(z) = C_i(zI - A_i)^{-1}x_i = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} C_i A_i^n x_i.$$

Since $(C_1A_1^n x_1)_{n \geq 1} = O_1x_1 = -O_2x_2 = -(C_2A_2^n x_2)_{n \geq 0}$, we have that $G_1(z) = -G_2(z)$, $|z| > \max\{\|A_1\|, \|A_2\|\}$. Since by assumption the resolvent sets of A_1 and A_2 only have one component, this implies that $G_1 = -G_2$ and $\sigma(G_1) = \sigma(-G_2) = \sigma(G_2)$.

Clearly $\sigma(G_i) \subseteq \sigma(A_i)$, $i = 1, 2$. Since by assumption $\sigma(A_1) \cap \sigma(A_2) = \emptyset$ we have that $\sigma(G_1) \cap \sigma(G_2) = \emptyset$. As $\sigma(G_1) = \sigma(G_2)$, this implies that $\sigma(G_1) = \sigma(G_2) = \emptyset$. Hence G_1 and G_2 are analytic on $\mathbb{C}$. But $\lim_{|s| \to \infty} \|G_i(s)\| = 0$, $i = 1, 2$. Hence $G_1 = G_2 = 0$ on $\mathbb{C}$. Therefore $O_1x_1 = -O_2x_2 = 0$. By the injectivity of O_1 and O_2 we have that $x_1 = 0$, $x_2 = 0$ and hence $x = (x_1, x_2) = 0$. Therefore O is injective and hence (A, B, C) is observable.

The statements concerning reachability follow by duality. ∎

We can now determine the spectral minimality of a reachable and observable completely J-positive realization of finite order. The part of the proof which deals with the non-zero spectrum is inspired by the proof of the spectral minimality for continuous-time symmetric systems with bounded operators in ([1]).

Theorem 5 *Let (A, B, B^*J, D) be a completely J-positive realization of finite order of the transfer function G such that the reachability operator R has dense range. Then the system is spectrally minimal, i.e. $\sigma(A) = \sigma(G)$.*

Proof: Note that the regions of analyticity of G and the resolvent of A have only one connected component. By the earlier remarks we have that $\sigma(G) \subseteq \sigma(A)$. We therefore need to consider the reverse set inclusion.

Let Δ be a compact real interval which does not contain 0. Then either $\Delta \subset (-\infty, 0)$ or $\Delta \subset (0, +\infty)$. To make a choice, assume that $\Delta \subset (0, +\infty)$.

From the construction of the spectral function A of a J-positive operator of finite order as in [9] we have

$$E(\Delta) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \frac{1}{2\pi \mathrm{i}} \int_{C^{\delta}_{\Delta_\varepsilon}} (zI - A)^{-1} \mathrm{d}\, z, \tag{7}$$

where $\Delta_\varepsilon = [a - \varepsilon, b + \varepsilon]$, assuming that $\Delta = [a, b]$, $C^{\delta}_{\Delta_\varepsilon}$ is a rectangle symmetric with respect to the real axis constructed around the interval Δ_ε from which we remove two segments of length 2δ around the points of coordinates $(-\varepsilon + a, 0)$ and $(b + \varepsilon, 0)$.

Let us assume now that G has analytic continuation in a neighbourhood of Δ. Then, from (7) and taking into account that the system (A, B, B^*J, D) is a realization of G, applying the Cauchy formula we get

$$B^*JE(\Delta)B = \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \frac{1}{2\pi \mathrm{i}} \int_{C^{\delta}_{\Delta_\varepsilon}} (G(z) - D)\mathrm{d}\, z = 0.$$

Taking into account of Theorem 1 it follows that $E(\Delta)\mathcal{H}$ is a uniformly positive subspace of $\mathcal{H}$ and hence, for arbitrary u in the input/output space $\mathcal{U}$ we have

$$0 = \langle B^*JE(\Delta)Bu, u\rangle = [E(\Delta)Bu, E(\Delta)Bu] \geq \alpha \|E(\Delta)Bu\|^2,$$

for some $\alpha > 0$. Then $E(\Delta)B = 0$ follows and since the spectral function E commutes with the main operator A we obtain

$$0 = A^k E(\Delta)B = E(\Delta)A^k B, \quad k \geq 0.$$

From here we obtain that $E(\Delta)|\mathcal{R}(R) = 0$ and since it is assumed that the reachability operator R has dense range, it follows that $E(\Delta) = 0$ and hence $\Delta \cap \sigma(A) = \emptyset$.

We have therefore shown that $\sigma(A) \setminus \{0\} \subseteq \sigma(G)$ and hence that $\sigma(A) \setminus \{0\} = \sigma(G) \setminus \{0\}$.

Now we need to clarify the role of the point 0. If $0 \in \sigma(G)$ then the proof is completed. Now assume that $0 \notin \sigma(G)$. Since $\sigma(G)$ is closed this implies that there exists a $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \cap \sigma(G) = \emptyset$. If $0 \notin \sigma(A)$ the proof is also completed. It therefore remains to exclude the case that $0 \in \sigma(A)$. To do this assume that $0 \in \sigma(A)$. Note that since $\sigma(A) \setminus \{0\} = \sigma(G) \setminus \{0\}$, we have that 0 is an isolated spectral point of A as by assumption $[-\varepsilon, \varepsilon] \cap \sigma(G) = \emptyset$. But by [9] this implies that 0 is an eigenvalue of A.

Now consider $\Delta_r :=]-\infty, -\varepsilon[\cup]\varepsilon, \infty[$ and $\Delta_0 = [-\varepsilon, \varepsilon]$. Let

$$(A_r, B_r, C_r, D) := (E(\Delta_r)AE(\Delta_r), E(\Delta_r)B, CE(\Delta_r), D),$$

$$(A_0, B_0, C_0, 0) := (E(\Delta_0)AE(\Delta_0), E(\Delta_0)B, CE(\Delta_0), 0)$$

and define the corresponding state spaces $H_r = E(\Delta_r)H$ and $H_0 = E(\Delta_0)H$. Let G_r and G_0 be the corresponding transfer functions. Note that since $0 \in \sigma_p(A)$ the system $(A_0, B_0, C_0, 0)$ is not the zero system.

We have that the system (A, B, C, D) is the sum of these two systems. Since (A, B, C, D) is reachable and observable and the spectra of A_r and A_0 are disjoint these two systems are also reachable and observable by the previous Proposition. Also note that $\sigma(A_r) = \sigma(G) \cap \Delta_r$ and $\sigma(A_0) = \sigma_p(A_0) = \{0\}$, where $\sigma_p(A_0)$ denotes the point spectrum of A_0.

Since H_r is a uniformly positive subspace the system (A_r, B_r, C_r, D) is completely J-symmetric with $J = I$. By the spectral minimality result of the first part of the proof (or the spectral minimality result for self-adjoint systems in [1]) it follows that

$$\sigma(G_r) = \sigma(A_r) = \sigma(G) \cap \Delta_r.$$

Since $G = G_r + G_0$ and the sets of singularities of the two functions G_r and G_0 are disjoint, we have that

$$\sigma(G) = \sigma(G_r) \cup \sigma(G_0) = (\sigma(G) \cap \Delta_r) \cup \sigma(G_0) = \sigma(G) \cup \sigma(G_0).$$

Since $\sigma(G) \cap \{0\} = \emptyset$ and $\sigma(G_0) = \{0\}$, this implies that $\sigma(G_0) = \emptyset$.

On the other hand A_0 is a nilpotent operator of finite order, more precisely, $A_0^{n+1} = 0$, cf. [8], [9]. But then for $z \neq 0$,

$$G_0(z) = C_0(zI - A_0)^{-1}B_0 = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} C_0 A_0^n B_0 = \sum_{n=0}^{N} \frac{1}{z^{n+1}} C_0 A_0^n B_0.$$

Therefore G_0 is a rational function. Since the system $(A_0, B_0, C_0, 0)$ is minimal this implies that this system is finite dimensional. By the spectral minimality of minimal finite dimensional systems we have that

$$\{0\} = \sigma(A_0) = \sigma(G_0).$$

This is a contradiction to $\sigma(G_0) = \emptyset$. Hence $0 \notin \sigma(A)$ if $0 \notin \sigma(G)$ and therefore $\sigma(A) = \sigma(G)$. ■

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