ELSEVIER

Systems & Control Letters 35 (1998) 61–63

SYSTEMS & CONTROL LETTERS

Asymptotic stabilization of infinite-dimensional systems which cannot be exponentially stabilized

Raimund J. Ober *, 1

Center for Engineering Mathematics EC 35, University of Texas at Dallas, Richardson, TX 75083, USA

Received 30 September 1997; received in revised form 24 February 1998

Abstract

The question of the stabilization by state feedback of an infinite-dimensional continuous-time system is discussed. Systems are introduced for which no state feedback exists such that the closed-loop system is exponentially stable. But it is shown that a state feedback exists such that the closed-loop system is asymptotically stable. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Infinite dimensional linear system; Stabilization; State feedback; Asymptotic stability; Exponential stability; Blaschke product

1. Introduction

In this paper we will consider the stabilization of an infinite-dimensional continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad (1a)$$

$$y(t) = Cx(t) + Du(t), \qquad (1b)$$

where $x(t) \in X$ with X a separable Hilbert space, $u(t) \in U$, $y(t) \in Y$ and where we assume that U and Y are finite-dimensional spaces. For the purpose of the discussion of this paper it is possible to make the otherwise very restrictive assumption that the system operators A, B and C are bounded. We will refer to this system by referring to the quadruple of system operators (A, B, C, D). We call Eq. (1a) the input system and refer to it by the pair (A, B). The pair of operators (A, C) stands for the output system (1b).

Two stability notions are of importance in our context. The system (A, B, C, D) or the operator A is called *exponentially stable* if there exist constants $M > 0$ and $\omega_0 > 0$ such that

$$\|e^{tA}\| \leqslant M e^{-t\omega_0}$$

for $t \geqslant 0$. It is called *asymptotically stable* if for each $x \in X$

$$e^{tA}x \to 0$$

as $t \to \infty$. For a system which is not stable the classical state feedback stabilization problem is to find a state feedback $K : U \to X$ such that $A - BK$ is stable. The notion of stability which – to the best of my knowledge – is always used in the discussion of the stabilization of infinite-dimensional systems is that of exponential stability, i.e. a state feedback is sought such that the closed-loop system is exponentially stable (see e.g. [1]). The purpose of this paper is to present systems which cannot be exponentially stabilized, i.e. no bounded state feedback operator K exists such that the closed-loop system, i.e. $A - BK$ is

* E-mail: ober@utdallas.edu.

[1] This research was supported by NSF grant DMS-9501223.

PII: S0167-6911(98)00030-9

exponentially stable. But, we will show that it is nevertheless possible to find a bounded state feedback such that the closed-loop system is asymptotically stable. In fact, the systems that we consider are such that the spectrum $\sigma(A)$ of A is contained in the closed right half-plane. The "asymptotically stabilizing" state feedback is so constructed to shift the spectrum to the closed left half-plane. Based on the results presented here one could argue that it might be worthwhile to seek an "asymptotically stabilizing" state feedback when exponential stabilization is not possible.

We denote by $\sigma(T)$ the spectrum of the operator T. If T is an operator on a Hilbert space the symbol T^* stands for its adjoint.

2. The results

First, a result is presented which shows that if a system satisfies a number of properties then it cannot be exponentially stabilized but it can be asymptotically stabilized.

Theorem 2.1. *Let (A,B,C,D) be a continuous-time infinite-dimensional system with finite-dimensional input space, such that*
1. *A,B are bounded operators.*
2. *A^* is asymptotically anti-stable, i.e. for each $x\in X$, $\lim_{t\to\infty} e^{-tA^*}x=0$.*
3. *there exists a bounded and boundedly invertible P such that*

$$AP+PA^*=BB^*.$$

Then
1. *there exists no bounded operator $K:X\to U$ such that $A-BK$ is exponentially stable.*
2. *$A-BB^*P^{-1}$ is asymptotically stable.*

Proof. (1) Since A^* is asymptotically anti-stable, i.e. $-A^*$ is asymptotically stable, we have that $\sigma(A^*)\subseteq\{s\in\mathbb{C}\mid \mathrm{Real}(s)\geqslant 0\}$ (see e.g. ([5], Theorem 1.13, p. 109)). Since A is a bounded operator on an infinite-dimensional space the spectrum of A has an accumulation point in the closed right half-plane. By Weyl's theorem (see e.g. ([3], p. 96)) the accumulation point p of the spectrum cannot be shifted by a compact perturbation of A. Note that for any bounded $K:X\to U$ the operator BK is compact since U is finite dimensional. Therefore, there exists no bounded K such that $\sigma(A-BK)$ does not contain the accumulation point of $\sigma(A)$.

For an operator $\tilde{A}$ to be exponentially stable it is necessary that $\sup\{\mathrm{Real}(\lambda)\mid\lambda\in\sigma(\tilde{A})\}<0$ (see e.g. ([5], Corollary 1.5, p. 105)). This implies that no state feedback exists such that $A-BK$ is exponentially stable.

(2) As

$$AP+PA^*=BB^*$$

we have that

$$A-BB^*P^{-1}=-PA^*P^{-1}.$$

Since A^* is asymptotically anti-stable, $-A^*$ is asymptotically stable. As P is bounded with bounded inverse P^{-1} we have that $-PA^*P^{-1}$ is asymptotically stable and therefore, $A-BB^*P^{-1}$ is asymptotically stable. □

Having proved the general result we now need to show the existence of a system with the required properties. We will need the following well-known lemma. For the sake of completeness we will give the short proof. Let (A,C) be an output system with bounded operators A and C. Then the operator

$$\mathcal{O}:X\to L^2([0,\infty[),$$
$$x\mapsto(Ce^{tA}x)_{t\geqslant 0}$$

is called the *observability operator* of (A,C) (see e.g. [2]).

Lemma 2.1. *Let (A,B,C,D) be a continuous-time system with bounded operators A and C. Moreover, assume that the observability operator $\mathcal{O}$ is bounded and that A is asymptotically stable, i.e. $\lim_{t\to\infty}e^{tA}x=0$, for each $x\in X$. Then with $P:=\mathcal{O}^*\mathcal{O}$,*

$$A^*P+PA=-C^*C.$$

Proof. For $x,y\in X$ we have with $P=\mathcal{O}^*\mathcal{O}$ that

$$\begin{aligned}
&\langle (A^*P+PA)x,y\rangle\\
&\quad=\langle \mathcal{O}x,\mathcal{O}Ax\rangle+\langle \mathcal{O}Ax,\mathcal{O}y\rangle\\
&\quad=\int_0^\infty \left\langle Ce^{tA}x,Ce^{tA}Ay\right\rangle+\left\langle Ce^{tA}Ax,Ce^{tA}y\right\rangle\mathrm{d}t\\
&\quad=\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left\langle Ce^{tA}x,Ce^{tA}y\right\rangle\mathrm{d}t\\
&\quad=\lim_{\tau\to\infty}\left\langle Ce^{tA}x,Ce^{tA}y\right\rangle\big|_0^\tau=-\langle Cx,Cy\rangle\\
&\quad=-\langle C^*Cx,y\rangle,
\end{aligned}$$

which implies the result. □

Theorem 2.2. *There exists a continuous-time input system (A,B) with one-dimensional input space U and bounded operators A and B such that*
1. A^* *is asymptotically anti-stable, i.e.* $\lim_{\to\infty} e^{-tA^*}x = 0$ *for each* $x \in X$.
2. *there exists a bounded operator P with bounded inverse such that*

$$AP + PA^* = BB^*.$$

Proof. Let $\mathrm{Bl_s}$ be an infinite Blaschke product which is analytic in the open right half-plane and whose zeros are contained in a bounded set. Such a Blaschke product can easily be constructed, take e.g.

$$\mathrm{Bl}(s) := \prod_{n=1}^{\infty} \frac{|1-\beta_n^2|}{1-\beta_n^2}\frac{s-\beta_n}{s+\overline{\beta_n}},$$

with $\beta_n := 1/(n+1)^2$, $n = 1,2,\ldots,$ is a well defined Blaschke product since $\sum_{n=1}^{\infty} \mathrm{Real}(\beta_n)/(1+|\beta_n|^2) < \infty$ (see e.g. ([4], p. 132)). Note that a Blaschke product $\mathrm{Bl_s}$ with the above properties is analytic at ∞ and hence $\lim_{\lambda\to\infty,\lambda\in\mathbb{R}} \mathrm{Bl_s}(\lambda)$ exists, which shows that the general assumption on transfer functions of [6] is satisfied. Let now (A_s, B_s, C_s, D_s) be the restricted shift realization of $\mathrm{Bl_s}$ ([2,6]). This realization is spectrally minimal, i.e. $\sigma(A_s) = \sigma(\mathrm{Bl_s})$, since $\mathrm{Bl_s}$ is non-cyclic [2, 6]. Here $\sigma(\mathrm{Bl_s})$ is the set of singularities of $\mathrm{Bl_s}$, i.e. the complement of the maximal set of analyticity of $\mathrm{Bl_s}$. As $\mathrm{Bl_s}$ is analytic at infinity we have that A_s is bounded and hence also B_s and C_s are bounded (Lemma 8.1 in [6]). Since (A_s, B_s, C_s, D_s) is the left shift realization it is output normal, i.e. the observability operator $\mathcal{O}$ is bounded and $P := \mathcal{O}^*\mathcal{O} = I$ [6]. By the Lemma, we therefore, have that

$$A_s^*P + PA_s = -C_s^*C_s.$$

As A_s is the generator of the left shift semigroup we have that A_s is asymptotically stable, i.e. $\lim_{t\to\infty} e^{tA_s}x = 0$ for each $x \in X$.

Consider now the "anti-stable" transfer function $\mathrm{Bl_a}$, i.e. the transfer function which is analytic in the open left half-plane, given by

$$\mathrm{Bl_a}(s) = (\mathrm{Bl_s}(-\bar{s}))^*, \quad s \in \mathbb{C}, \ \mathrm{Real}(s) < 0.$$

As for $s \in \mathbb{C}$, $\mathrm{Real}(s) < 0$,

$$\begin{aligned}\mathrm{Bl_a}(s) &= (\mathrm{Bl_s}(-\bar{s}))^* = (C_s(-\bar{s}I - A_s)^{-1}B_s + D_s)^* \\ &= -B_s^*(sI + A_s^*)^{-1}C_s^* + D_s^*\end{aligned}$$

we have that

$$(A,B,C,D) := (-A_s^*, C_s^*, -B_s^*, D_s^*)$$

is a realization of $\mathrm{Bl_a}$ such that A and B are bounded, A is asymptotically anti-stable, i.e. $e^{-tA^*}x = e^{tA_s}x \to 0$ as $t \to \infty$ for each $x \in X$. We also have that

$$AP + PA^* - BB^* = (-A_s^*)P + P(-A_s^*)^* - C_s^*C_s = 0.$$

Hence the input system (A,B) has the required properties. □

In the proof of the previous theorem an example was constructed of a system that satisfies the conditions of Theorem 2.1 using known results of the realization theory of Blaschke products [6]. In fact, we have constructed a realization of an "anti-stable" Blaschke product $\mathrm{Bl_a}$ which has all its roots in the open right half-plane. The input system (A,B) is such that the spectrum $\sigma(A)$ of A equals the complement of the maximal set of analyticity of $\mathrm{Bl_a}$. As shown, this system cannot be exponentially stabilized. But, we find a state feedback such that the closed loop system $A - BK$ is asymptotically stable. In fact, the state feedback shifts the infinite spectrum of A to the closed left half-plane.

As has been pointed out correctly by a reviewer of this paper state feedback for infinite-dimensional systems has practical problems. The results presented here should therefore not be taken to necessarily imply a prescription for performing an actual control design. But, it is hoped that the results are of interest to those who use state-space methods for the analysis and design of infinite-dimensional control systems where state feedback does play an important role as part of an overall strategy which may also include the design of an observer.

References

[1] R. Curtain, H. Zwart, An Introduction to Infinite-Dimensional System Theory, Springer, Berlin, 1995.

[2] P. Fuhrmann, Linear Systems and Operators in Hilbert Space, McGraw-Hill, New York, 1981.

[3] P. Halmos, A Hilbert Space Problem Book, 2nd ed., Springer, Berlin, 1974.

[4] K. Hoffmann, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[5] R. Nagel, One Parameter Semigroups of Positive Operators, Lecture Notes in Mathematics, vol. 1184, Springer, Berlin, 1986.

[6] R.J. Ober, Y. Wu, Infinite dimensional continuous time linear systems: stability and structure analysis, SIAM J. Control Optim. (1996) 757–812.