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An introduction to the shift realization for finite dimensional continuous time systems

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A new approach is presented for the realization of continuous-time finite dimensional linear systems. Using standard results on Laplace transforms our results are also used to present a new derivation of Fuhrmann's shift realization for rational matrix functions.

1. Introduction

In the theory of linear dynamical systems the connection between internal state space description and external input–output description is of central importance. Here the realization problem, i.e. the determination of a state-space system from knowledge of the input–output map, is the difficult part. For finite dimensional systems this problem has been solved for a long time in the fundamental work by Kalman (see, e.g. Kalman *et al.* 1960). The field of contributions to realization theory is vast. In this brief inroduction it is therefore impossible to try to adequately review all the important contributions that were made. We therefore concentrate on the aspects of the theory that are of importance for this particular paper. Realization theory for infinite dimensional systems was studied by several authors in the 1970s (see e.g. Baras and Brockett 1973, Fuhrmann 1975, Dewilde 1976, Helton 1976 a, b) where the theory of shifts on invariant subspaces played a particularly important role. In all these contributions assumptions had to be made on the stability of the system that is being considered. Another important step was done in the work by Fuhrmann (1976). Motivated by the powerful results for infinite dimensional he examined finite dimensional systems from the point of view of shift realizations. Through the introduction of his polynomial and rational models he managed to bridge the gap between the approaches built on shift realizations and Kalman's module theoretic approach. At the same time he was able to remove the stability assumption that was necessary in the infinite dimensional context. In common with Kalman's work his work essentially concentrated on discrete-time systems. These results can of course be applied to the finite dimensional continuous-time realization problem. But it often appears artificial to determine a state space realization of a continuous-time system by what are essentially techniques motivated by and typically used for the study of discrete-time systems, e.g. the use of discrete-time shifts. In this paper an approach is developed that stays within the realm of continuous-time systems and does not rquire advanced mathematical methods such as, for example, the work by Yamamoto (1981, 1982), which is mainly devoted to infinite dimensional systems. In fact the approach to the realization problem presented in this paper was developed by the author as part of a graduate course on linear system theory which mainly focuses on finite dimensional continuous-time systems. Another objective of this paper is to try to clearly explain the role that Hankel type maps play in the derivation of realizations for finite dimensional continuous-time systems.

The paper has some tutorial character in that we try to present the complete development of the approach. In the appendix we will therefore also give the proofs for some known results formulated to fit our framework. Our approach is equally valid for systems with coefficients in the real or complex numbers. We denote by $\mathbb{K}$ the field of scalars which could either be the real field or the complex field.

A *(continuous-line) time-invariant linear n-dimensional state space system* is as usual given in the following way by a *state space X*, which is an n dimensional Euclidean space over the field $\mathbb{K}$, and by a quadruple of linear transformations $A: X \to X$, $B: \mathbb{K}^m \to X$, $C: X \to \mathbb{K}^p$, $D: \mathbb{K}^m \to \mathbb{K}^p$, such that

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0 \tag{1}$$

$$y(t) = Cx(t) + Du(t) \tag{2}$$

$t \geq t_0, t \in \mathbb{R}$. If a basis is given in the state space X we can identify X with $\mathbb{K}^n$ and we can think of the transformations A, B and C to be given in matrix form. We will also often refer to a system by referring to its quadruple of transformations (A, B, C, D).

If the input u is piecewise continuous, the solution to the set of differential equations is given by

$$x(t) = \int_{t_0}^{t} e^{(t-\tau)A} Bu(\tau)\, d\tau + e^{(t-t_0)A} x_0$$

and the output is given as a function of the input in the following way

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$$y(t) = \int_{t_0}^{t} C\,\mathrm{e}^{(t-\tau)A} Bu(\tau)\,\mathrm{d}\tau + C\,\mathrm{e}^{(t-t_0)A} x_0 + Du(t), \qquad t \geq t_0 \quad (3)$$

If we assume that the initial conditions are zero, then this convolution map (3) completely describes the input–output behaviour of the state space system. This leads us to the second definition of a system which we will consider, i.e. a definition through convolution maps.

Let $C(\mathbb{R})$ be the set of continuous $\mathbb{K}^p$-valued functions on $\mathbb{R}$ and let $PC(\mathbb{R})$ be the set of piecewise continuous functions on $\mathbb{R}$ and let $PC_{tr}(\mathbb{R})$ be the subset of $PC(\mathbb{R})$ of piecewise continuous functions on $\mathbb{R}$ that are zero for t 'negative enough', i.e. for each $f \in PC_{tr}(\mathbb{R})$ there exists t_0 such that $f(t) = 0$ for $t \leq t_0$.

Definition 1: Let $M(t) \in \mathbb{K}^{p\times m}$, $t \geq 0$, be a continuous function and let $D \in \mathbb{K}^{p\times m}$. Then the map

$$IO_{M,D} : PC_{tr}(\mathbb{R}) \to PC(\mathbb{R}); \qquad u \mapsto y$$

where $y(t) = \int_{-\infty}^{t} M(t-\tau)u(\tau)\,\mathrm{d}\tau + Du(t)$, $t \in \mathbb{R}$, is called the input–output map with symbol M and feedthrough term D.

To avoid convergence problems we have restricted the inputs to be zero for large enough negative time. If (A, B, C, D) is a state space system, then the input–output map $IO_{M,D}$ with symbol $M(t) = C\,\mathrm{e}^{tA}B$, $t \geq 0$, is called the *input–output map associated with the state space system*. It is easily seen that two state space systems that are related by a state space transformation have the same associated input–output map. A formulation of the realization problem is to determine conditions under which an input–output map is in fact the input–output map of a state space system.

In order to show that this input–output map is well defined with range in $PC(\mathbb{R})$, we show in the next proposition that if $D = 0$ then the range of $IO_{M,D}$ is in $C(\mathbb{R})$. A proof is given in the appendix.

Proposition 1: *Let M be a continuous function on $\mathbb{R}_+$. Then for the input–output map $IO_{M,D}$ with $D = 0$*

$$range(IO_{M,D}) \subseteq C(\mathbb{R})$$

The realization problem has been posed in a number of different, though equivalent, formulations. From our point of view there are essentially two classes of formulations that are of importance. One is a time domain formulation and the other is a frequency domain formulation. In the time domain formulation the question is asked whether a certain input–output description is the input–output description of a linear finite dimensional system. In the frequency domain formulation the question is asked whether given a proper rational function G, can it be written as the transfer function of a linear system (A, B, C, D), i.e. is $G(s) = C(sI - A)^{-1}B + D$, $s \in \mathbb{C}$?

In the following section we introduce the new definitions of observability, reachability and Hankel maps and discuss their connections. The observability and reachability maps are used to define the reachability and observability of a system. In the subsequent section the time domain realization problem is solved using a left shift realization. The construction heavily relies on properties of the range of the Hankel map since this range space serves as the state space of the realization. In the final section we will use the Laplace transform as a state space transformation to derive a solution for the frequency domain realization problem from the solution to the time domain problem. This in fact provides a novel approach to derive Fuhrmann's shift realization (Fuhrmann 1976) for rational transfer functions.

1.1. *Notation*

Denote by $\mathbb{K}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. $\mathbb{R}_+ = \{\lambda \in \mathbb{R} | \lambda \geq 0\}$. $\mathbb{R}_- = \{\lambda \in \mathbb{R} | \lambda \leq 0\}$. By a Euclidean space we mean a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, with inner product $\langle \cdot, \cdot \rangle$. The inner product is understood to be linear in the first component and anti-linear in the second component. $C(I)$ stands for the set of (vector/matrix-valued) continuous functions on the interval $I \subseteq \mathbb{R}$, which can be unbounded. $PC(I)$ stands for the set of piecewise continuous (vector-valued) functions on the interval $I \subseteq \mathbb{R}$. If I is a bounded interval, f is in $PC(I)$ if f has at most a finite number of discontinuities on I and at each point p in I the left and right limits of f exist and are finite. If I is infinite, f is in $PC(I)$ if for each bounded interval $I_b \subseteq I$, the restriction of f to I_b is in $PC(I_b)$. $PC_{tr}(I)$ denotes the subset of truncated functions of $PC_{tr}(I)$. A function $f \in PC(I)$ is called truncated if there exists $t_f \in \mathbb{R}$, s.t. $f(t) = 0$ for $t < t_f$. The constant t_f depends on f. $\sigma(T)$ denotes the set of eigenvalues of the linear map T. $ker(T)$ stands for the kernel of the linear map T and $range(T)$ stands for the range of the linear map T. The rank of the linear map T is denoted by $rank(T)$. $T|_V$ denotes the restriction of the map T to the subspace V. $\|a\|$ stands for the norm of the map or vector a. $A \perp B$ denotes that the set A is orthogonal to the set B. The characteristic function χ_A is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

2. Reachability, observability and Hankel maps

Central notions in systems theory are those of the reachability and observability of a system. We give definitions of these notions which are suitable to our way of studying linear systems. The observability map $\mathcal{O}$ is defined as the map which maps the state of the system at time zero to the output of the system for positive time if no input is applied to the system. A system is then called observable if no non-zero state is mapped to the zero output. The reachability map maps a given input function in $PC_{tr}(\mathbb{R}_-)$ to the state that is reached at time zero if this input is applied from time $-\infty$ to time 0. A system is called reachable if all states in the state space can be reached in this way.

Definition 2: Let (A, B, C, D) be a linear system with state space X, then

(1) the map

$$\mathcal{C}: X \to C(\mathbb{R}_+); \qquad x \mapsto (C\,\mathrm{e}^{tA}x)_{t\geq 0}$$

is called the *observability map* of the system. The system is called *observable* if $\mathcal{O}x \equiv 0$ implies that $x = 0$, i.e. if $ker(\mathcal{O}) = \{0\}$.

(2) the map

$$\mathcal{R}: PC_{tr}(\mathbb{R}_-) \to X; \qquad u \mapsto \int_{-\infty}^{0} \mathrm{e}^{-\tau A}Bu(\tau)\,\mathrm{d}\tau$$

is called the *reachability map* of the system. The system is called *reachable* if range $(\mathcal{R}) = X$.

A state space system is called *minimal* if it is both observable and reachable.

In order to avoid possible problems with the convergence of the integral in the definition of the reachability map for systems, the inputs have been restricted to the class of truncated piecewise continuous functions on $\mathbb{R}_-$, i.e. to functions in $PC_{tr}(\mathbb{R}_-)$.

Clearly $\mathcal{O}$ and $\mathcal{R}$ are linear maps. Since the domain of $\mathcal{O}$ and the range space of $\mathcal{R}$ is the n-dimensional state space X, the rank of $\mathcal{O}$ and $\mathcal{R}$ is at most n. An n-dimensional linear system is therefore observable if and only if the observability map $\mathcal{O}$ has rank n. It is reachable if and only if the reachability map has rank n.

The following proposition shows that our notion of reachability leads to the usual reachability criterion (see e.g. Kailath 1980). By duality the same also applies to the observability criterion.

Proposition 2: *An n-dimensional system (A, B, C, D) is reachable if and only if*

$$rank\,[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = n$$

Moreover, $range\,(\mathcal{R}) = span\{A^kB\mathbb{K}^m \mid k \geq 0\}$.

Proof: First recall that there exist continuous functions $\alpha_0(t), \ldots, \alpha_{n-1}(t)$, $t \geq 0$, such that

$$\mathrm{e}^{tA} = \sum_{k=0}^{\infty}(1/k!)t^kA^k = \sum_{k=0}^{n-1}\alpha_k(t)A^k, t \geq 0$$

Let $x \perp range\,([B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B])$. Since then for $u \in PC_{tr}(\mathbb{R}_-)$

$$x \perp \sum_{k=0}^{n-1} A^kB\int_{-\infty}^{0} \alpha_k(-\tau)u(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{0} \mathrm{e}^{-\tau A}Bu(\tau)\,\mathrm{d}\tau$$
$$= \mathcal{R}u$$

we have that $x \perp range(\mathcal{R})$ and hence $range\,(\mathcal{R}) \subseteq range\,([B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B])$.

Let now $x \perp range\,(\mathcal{R})$. Hence $\langle x, \int_{-\infty}^{0} \mathrm{e}^{-\tau A}B \times u(\tau)\,\mathrm{d}\tau\rangle = 0$ for all $u \in PC_{tr}(\mathbb{R}_-)$. Set $u(\tau) := B^*\mathrm{e}^{-\tau A^*}x \times \chi_{[-1,0]}(\tau)$, for $\tau \leq 0$. Then $u \in PC_{tr}(\mathbb{R}_-)$ and

$$0 = \left\langle x, \int_{-\infty}^{0} \mathrm{e}^{\tau A}BB^*\,e^{-\tau A^*}x\chi_{[-1,0]}(\tau)\,\mathrm{d}\tau \right\rangle$$
$$= \int_{-1}^{0} \|B^*\,\mathrm{e}^{-\tau A^*}x\|^2\,\mathrm{d}\tau$$

Hence $B^*\,\mathrm{e}^{-\tau A^*}x = 0$, for $-1 \leq \tau \leq 0$. Taking the kth derivative of this function and evaluating it at 0 we have for $k \geq 0$, that $B^*(A^*)^kx = 0$, and therefore for $u_0 \in \mathbb{K}^m$, $0 = \langle x, A^kBu_0\rangle$, which implies that $x \perp range\,([B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B])$. Hence

$$range\,([B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]) \subseteq range\,(\mathcal{R})$$

Together with the previous inclusion this implies equality of the sets. Therefore

$$n = rank\,([B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B])$$

if and only if $rank\,(\mathcal{R}) = n$ which is the case if and only if the system is reachable.

By a standard application of the Cayley Hamilton theorem we have moreover that

$$range\,(\mathcal{R}) = span\,\{A^kB\mathbb{K}^m \mid k \geq 0\} \qquad \square$$

An important map in our study of linear systems is the so-called *Hankel map*. The Hankel map that is associated with a system maps 'past' inputs of the system into 'future' outputs. Before clarifying this point further, we will define what we mean by a Hankel map.

Definition 3: Let $M(t) \in \mathbb{K}^{p\times m}$, $t \geq 0$, be a continuous function. Then the map

$$H_M : PC_{tr}(\mathbb{R}_-) \to C(\mathbb{R}_+);$$
$$u(\cdot) \mapsto y(\cdot) = \int_{-\infty}^{0} M(\cdot - \tau)u(\tau)\,\mathrm{d}\tau$$

is called the *Hankel map with symbol M*.

The Hankel map H_M is related to the input–output map $IO_{M,D}$ as follows. Let $D = 0$ and let $u \in PC_{tr}(\mathbb{R})$ be an input function defined for negative time. We can extend this function to a function u_e defined on the whole time axis by requiring it to be zero for positive time, i.e. $u_e(t) = u(t)$ for $t \leq 0$ and $u_e(t) = 0$ for $t > 0$. Then

$$[H_M(u)](t) = [IO_{M,D}(u_e)](t)$$

for $t \geq 0$. Hence the Hankel map maps those inputs of an input–output map with $D = 0$, that are zero for positive time, to the outputs of the system for positive time. That the range of the Hankel map is contained in $C(\mathbb{R}_+)$ is therefore a consequence of Proposition 1. The definition of the Hankel map follows the same philosophy that underlies the use of Hankel operators in the realization theory of stable continuous-time (infinite-dimensional) systems (see e.g. Fuhrmann 1981). What is important from our point of view is that with this definition of the Hankel map no stability assumptions have to be made on the systems that are being studied.

The particular application of Hankel maps that we have in mind is when the symbol M of the Hankel map is in fact the impulse response of a linear system (A, B, C, D), i.e. $M(t) = C\mathrm{e}^{tA}B$, $t \geq 0$. This Hankel map is called the Hankel map *associated* with the linear system.

In the following proposition we show that if the symbol of the Hankel map is the impulse response of a system then the reachability and observability maps of the system introduce a factorization of the Hankel map.

Proposition 3: *Let (A, B, C, D) be a linear system and let $M(t) := C\mathrm{e}^{tA}B$, $t \geq 0$. Let $\mathcal{O}$ be the observability map and let $\mathcal{R}$ be the reachability map of the system (A, B, C, D), then the Hankel map H_M with symbol M can be factored as*

$$H_M = \mathcal{O}\mathcal{R}$$

Proof: The proof is a simple verification. □

In the following proposition we are going to show that the Hankel map of a minimal n-dimensional system has rank n.

Proposition 4: *Let (A, B, C, D) be an n-dimensional system. Let $M(t) := C\mathrm{e}^{tA}B$, $t \geq 0$. Then*

(1) *the Hankel map with symbol M has at most rank n.*

(2) *the Hankel map has rank n if and only if (A, B, C, D) is minimal.*

Proof:

(1) By Proposition 3 we have that

$$H_M = \mathcal{O}\mathcal{R}$$

Since $\mathcal{O}$ has as its domain the n-dimensional state space X and $\mathcal{R}$ maps into X we have that

$$rank\,(H_M) \leq min\,(rank\,(\mathcal{O}), rank\,(\mathcal{R})) \leq n$$

(2) Now assume that (A, B, C, D) is minimal. Since $range\,(\mathcal{R}) = X$ which is an n-dimensional space we have that $rank\,(\mathcal{R}) = n$. Since the domain of $\mathcal{O}$ is X we therefore have that $rank\,(H_M) = n$ if $rank\,(\mathcal{O}) = n$. But this is the case since the system is observable.

Now assume that $rank\,(H_M) = n$, but that (A, B, C, D) is not minimal. Then either $rank\,(\mathcal{O})$ or $rank\,(\mathcal{R})$ is strictly less than n. But this shows that

$$rank\,(H_M) \leq min\,(rank\,(\mathcal{O}), rank\,(\mathcal{R})) < n$$

which is a contradiction. □

The range of the Hankel map associated with a system can be described easily in terms of the system maps. As a consequence we have that all functions in the range of such a Hankel map are infinitely often continuously differentiable. This is of importance in particular in connection with our approach to realization theory.

Corollary 1: *Assume that (A, B, C, D) is a reachable system. With the notation of the previous proposition we have*

$$range\,(H_M) = \{(C\mathrm{e}^{tA}x)_{t \geq 0} \mid x \in X\}$$

where X is the state-space of the system. Moreover each $f \in range\,(H_M)$ is infinitely often continuously differentiable.

Proof: The result follows since

$$range\,(H_M) = range\,(\mathcal{O}\mathcal{R}) = \{\mathcal{O}x \mid x \in X\}$$
$$= \{(C\mathrm{e}^{tA}x)_{t \geq 0} \mid x \in X\}$$

This representation implies that each function in $range\,(H_M)$ is infinitely often continuously differentiable. □

3. The time domain realization problem

In the introductory section we discussed two representations for linear systems. One representation was in terms of state space models, the other representation was in terms of input–output maps. We immediately saw that a state space model gave rise in a natural way to an input–output map. In this section we are going to study the time domain realization problem, i.e. we are going to establish necessary and sufficient conditions for an input–output map to be the input–output map of a state space system. This problem is of course the same as asking under which conditions can a function M in $C(\mathbb{R}_+)$ be written as the impulse response of a linear system (A, B, C, D), i.e. $M(t) = C\mathrm{e}^{tA}B$ for $t \geq 0$. It follows from the results in the previous section that a necessary condition is that the Hankel map H_M is of finite rank. We will show that this condition is also sufficient.

Before discussing the approach to the construction of the realization we need to introduce the left shift since it plays a crucial role in what follows.

Definition 4: For $t \geq 0$, the map

$$L_t : C(\mathbb{R}_+) \to C(\mathbb{R}_+); \quad [f(\tau)]_{\tau \geq 0} \mapsto (L_t f) = [f(\tau + t)]_{\tau \geq 0}$$

is called the *left shift* (*with increment* t) on $C(\mathbb{R}_+)$. Similarly, for $t \geq 0$, the map

$$L_t : PC_{tr}(\mathbb{R}_-) \to PC_{tr}(\mathbb{R}_-)$$

$$[f(\tau)]_{\tau \leq 0} \mapsto L_t f = [L_t f(\tau)]_{\tau \leq 0}$$

$$= \begin{cases} f(\tau + t), & \tau + t \leq 0, \\ 0, & \tau + t > 0, \end{cases} \quad \tau \leq 0$$

is called the *left shift* (*with increment* t) on $PC_{tr}(\mathbb{R}_-)$.

Note that there is a slight abuse of notation in that we use the same letter to denote the left shift both on the space $C(\mathbb{R}_+)$ and on the space $PC_{tr}(\mathbb{R}_-)$.

Our approach to the time domain realization problem of a function $M \in C(\mathbb{R}_+)$ is as follows. Analogously to the derivation of the left shift realization in the discrete-time and infinite-dimensional case (Fuhrman 1981) the state space X is chosen to be the range of the Hankel map H_M, i.e.

$$X = range\,(H_M)$$

The input map B: $\mathbb{K}^m \to X$ is then defined as the map multiplying a constant vector $u \in \mathbb{K}^M$ by the symbol M, i.e. for $u \in \mathbb{K}^m$ we set

$$Bu := Mu$$

Clearly Bu is in $C(\mathbb{R}_+)$ for each $u \in \mathbb{K}^m$ since M is assumed to be continuous. But it is not immediately clear that B as defined here maps into the state space, i.e. into the range of H_M. That this is indeed the case is established in Lemma 1. The readout map C is defined by the point evaluation

$$Cx = x(0)$$

for $x \in X$. Since X is a finite dimensional subspace of the space of continuous function $C(\mathbb{R}_+)$ no questions of boundedness of this map arise. The other key element in the construction of the realization is the left-shift semigroup $(L_t)_{t \geq 0}$. We have for $t \geq 0$ and $u \in \mathbb{K}^m$ that

$$CL_t Bu = CL_t Mu = (L_t Mu)(0) = M(t)u$$

This simple computation shows that the realization problem is solved if we can find a linear map A acting on the state space X such that $e^{tA} = (L_t)_{|X}$ for $t \geq 0$. For this to be true the left shift L_t has to leave the state space invariant, i.e. the range of the Hankel map H_M. This is established in Corollary 2. The remaining question whether a semigroup of linear maps on a finite dimensional space can be the representation as the exponential of a linear map is addressed in Proposition 6.

The key step in showing that the left shift leaves the range of the Hankel map invariant, is given in the following result. There it is shown that the Hankel map intertwines the left shift.

Proposition 5: *Let H_M be a Hankel map. Then for each $t \geq 0$*

$$H_M L_t = L_t H_M$$

Proof: In order to prove the statement, let $u \in PC_{tr}(\mathbb{R}_-)$. Then for $t \geq 0$

$$(H_M L_t u)(\tau) = \int_{-\infty}^{0} M(\tau - s)(L_t u)(s)\, ds$$

$$= \int_{-\infty}^{0} M(\tau - s)u(t + s)\, ds$$

where we set $u(r) := 0$ for $r \geq 0$. But this last integral is equal to

$$\int_{-\infty}^{t} M(\tau + t - r)u(r)\, dr = \int_{-\infty}^{0} M(\tau + t - r)u(r)\, dr$$

On the other hand, for $\tau \geq 0$

$$(L_t H_M u)(\tau) = \left(L_t\left(\int_{-\infty}^{0} M(\cdot - s)u(s)\, ds\right)\right)(\tau)$$

$$= \int_{-\infty}^{0} M(\tau + t - s)u(s)\, ds$$

which shows that $H_M L_t = L_t H_M$. □

The following corollary shows that the range and kernel of a Hankel map are invariant under the left shift. This is an important property and will be crucial for the realization theory presented later.

Corollary 2: *Under the assumptions of the previous proposition we have*

(1) *range (H_M) is invariant under L_t, i.e. for each $t \geq 0$, L_t (range (H_N)) $\subseteq$ range (H_M).*

(2) *ker (H_M) is invariant under L_t, i.e. for each $t \geq 0$, L_t (ker (H_M)) $\subseteq$ ker (H_M).*

Proof:

(1) Let $y \in$ *range* (H_M), i.e. $y = H_M u$, for some $u \in PC_{tr}(\mathbb{R}_-)$. Then for $t \geq 0$, $L_t y = L_t H_M u = H_M L_t u$, and hence $L_t y \in$ *range* (H_M).

(2) Let $u \in$ *ker* (H_M). Then for all $t \geq 0$

$$H_M L_t u = L_t H_M u = L_t 0 = 0$$

i.e. $L_t u \in$ *ker* (H_M). □

The following proposition states the well-known result that a continuous semigroup of operators on a finite dimensional Euclidean space can be written as the exponential function of a linear map. For the sake of completeness of the presentation a proof of this proposition is given in the appendix.

Proposition 6: *Let X be a finite-dimensional Euclidean space and let $T(t): X \to X$, $t \geq 0$ be a semigroup of linear maps on X, i.e. $T(0) = I$, $T(t+s) = T(t)T(s)$, for $t, s \geq 0$, and $\|T(t) - I\| \to 0$ as $t \to 0+$. Then there exists a unique linear map $A: X \to X$ such that*

$$T(t) = \mathrm{e}^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

for $t \geq 0$.

In order to show that the input map B as discussed above is well-defined we need to show that Mu_0 is in the range of the finite rank Hankel map H_M for each $u_0 \in \mathbb{K}^m$. To prove this result we will first need to establish the following lemma whose rather technical proof is given in the appendix.

Lemma 1: *Let $g_1, g_2, \ldots, g_n$ be a basis of an n-dimensional subspace of the space of vector-valued functions $C(\mathbb{R}_+)$. Then there exist distinct points $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$ such that the matrix*

$$\mathcal{G} := \begin{pmatrix} g_1(t_1) & g_2(t_1) & \cdots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \cdots & g_n(t_2) \\ \vdots & \vdots & & \vdots \\ g_1(t_n) & g_2(t_n) & \cdots & g_n(t_n) \end{pmatrix}$$

has zero kernel. The left inverse $\mathcal{F}$ of $\mathcal{G}$, i.e. $\mathcal{F}\mathcal{G} = I$, is such that if $f \in C(\mathbb{R}_+)$ is a linear combination of $g_1, g_2, \ldots, g_n$, i.e.

$$f = \alpha^{(1)} g_1 + \alpha^{(2)} g_2 + \cdots + \alpha^{(n)} g_n$$

for some $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)} \in \mathbb{K}$ then

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix} = \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix}$$

Moreover the subspace of $C(\mathbb{R}_+)$ spanned by $g_1, g_2, \ldots, g_n$ is closed under pointwise convergence.

We can now prove the following lemma which guarantees that the map B is well defined, i.e. maps into the state space. In the single-input case this result also admits another interpretation. The symbol M of the Hankel map or the input–output map is also sometimes called the 'impulse response' of the system which is often heuristically introduced as being the output of the system if its input is a pulse or delta function at time 0. In fact in the proof of the lemma we construct a sequence of inputs that 'approximate' the delta function. The corresponding outputs then converge pointwise to the impulse response. With the help of the previous lemma we then conclude that the impulse response is in fact an element of the range of the Hankel map. The impulse response therefore lies in the state space of the realization that we are constructing.

Lemma 2: *Let H_M be a Hankel map of finite rank n. Then $Mu_0 \in$ range (H_M) for all $u_0 \in \mathbb{K}^m$.*

Proof: Since *range* (H_M) is an n-dimensional subspace of $C(\mathbb{R}_+)$ it is closed under pointwise convergence by Lemma 1.

In order to show that $Mu_0 \in$ *range* (H_M) for $u_0 \in \mathbb{K}^m$ we now construct a sequence of functions $u_k \in PC_{tr}(\mathbb{R}_-)$ such that $y_k = H_M u_k$ converges pointwise to Mu_0. Since *range* (H_M) is closed under pointwise convergence we therefore have that Mu_0 is in *range* (H_M). Define for $\tau \leq 0$ and $k \geq 1$

$$u_k(\tau) := \begin{cases} ku_0, & 0 \geq \tau \geq -1/k \\ 0, & \tau < -1/k \end{cases}$$

Let $t_0 \geq 0$. Then by the continuity of M

$$\begin{aligned} &\lim_{k\to\infty} \|[H_M(u_k)](t_0) - M(t_0)u_o\| \\ &= \lim_{k\to\infty} \left\| \int_{-\infty}^{0} M(t_0 - \tau) k u_0 \chi_{[-1/k,0]}(\tau)\, \mathrm{d}\tau - M(t_0)u_0 \right\| \\ &= \lim_{k\to\infty} \left\| \int_{-1/k}^{0} [kM(t_0 - \tau)u_0 - kM(t_0)u_0]\, \mathrm{d}\tau \right\| \\ &\leq \lim_{k\to\infty} k \int_{-1/k}^{0} \|M(t_0 - \tau)u_0 - M(t_0)u_0\|\, \mathrm{d}\tau \\ &\leq \lim_{k\to\infty} k \frac{1}{k} \max_{\tau\in[-1/k,0]} \|M(t_0 - \tau)u_0 - M(t_0)u_0\| \\ &= \lim_{\tau\to 0_+} \|M(t_0 - \tau)u_0 - M(t_0)u_0\| \\ &= 0 \end{aligned}$$

This implies the result as *range* (H_M) is closed under pointwise convergence. □

We are now in a position to prove the first realization theorem.

Theorem 1: *Let $M \in C(\mathbb{R}_+)$ such that H_M is a Hankel map of finite rank n. Then there exists a minimal n-dimensional state space system (A, B, C, D) such that*

$$M(t) = C\mathrm{e}^{tA}B, \qquad t \geq 0$$

Such a system is given by

$$X := \textit{range}\,(H_M) \subseteq C(\mathbb{R}_+)$$

$$B: \mathbb{K}^m \to X; \qquad u \mapsto Bu := Mu$$

$$C: X \to \mathbb{K}^p; \qquad x \mapsto Cx := x(0)$$

and $A: X \to X$, is such that for $t \geq 0$

$$e^{tA}: X \to X; \qquad x \mapsto L_t x$$

The map $D: \mathbb{K}^m \to \mathbb{K}^p$ is linear and arbitrary. The state space X has an arbitrary Euclidean structure.

Proof: We need to show that the maps A, B and C are well defined. Since $X \subseteq C(\mathbb{R}_+)$, the evaluation of an element of X at zero is defined and hence the map C is well defined. By Lemma 2 we have that the function Mu is in X for all $u \in \mathbb{K}^m$ which shows that B is well defined. Corollary 2 shows that $X = range\,(H_M)$ is invariant under L_t, $t \geq 0$. It is easily verified that $(L_t)_{t\geq 0}$ is such that $L_t(0) = I$, $L_t L_s = L_{t+s}$, for $t, s \geq 0$. Let $g_1, g_2, \ldots, g_n \in C(\mathbb{R}_+)$ be a basis of X. Let $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$ be the distinct points as in Lemma 1 and let $\mathcal{G}$ and $\mathcal{F}$ be the corresponding matrices. We now define an inner product structure on X by defining the Euclidean inner product on the coordinates of each $f \in X$ with respect to the basis $g_1, g_2, \ldots, g_n$, i.e. for $f,\ h \in X$ let $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)})^T \in \mathbb{K}^n$ and $\beta = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n)})^T \in \mathbb{K}^n$ be such that

$$f = \alpha^{(1)} g_1 + \alpha^{(2)} g_2 + \cdots + \alpha^{(n)} g_n$$

$$h = \beta^{(1)} g_1 + \beta^{(2)} g_2 + \cdots + \beta^{(n)} g_n$$

Then we define

$$\langle f, h \rangle_X := \sum_{i=1}^{n} \alpha^{(i)} \overline{\beta^{(i)}} = \langle \alpha, \beta \rangle$$

We now show that $\lim_{t\to 0+} \|L_t f - f\| = 0$ for each $f \in X$. This is the case since

$$\lim_{t\to 0+} \|L_t f - f\|$$

$$= \lim_{t\to 0+} \left\| \mathcal{F} \begin{pmatrix} (L_t f)(t_1) \\ (L_t f)(t_2) \\ \vdots \\ (L_t f)(t_n) \end{pmatrix} - \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix} \right\|$$

$$= \lim_{t\to 0+} \left\| \mathcal{F} \begin{pmatrix} f(t_1 + t) \\ f(t_2 + t) \\ \vdots \\ f(t_n + t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix} \right\|$$

$$= \left\| \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix} - \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix} \right\| = 0$$

Since X is finite dimensional the fact that $\lim_{t\to 0+} \|L_t f - f\| = 0$ for each $f \in X$ implies that $\lim_{t\to 0+} \|L_{t|X} - I\| = 0$. Hence all the conditions of Proposition 6 are satisfied and therefore there exists a linear map $A: X \to X$ such that for all $t \geq 0$

$$L_{t|X} = e^{tA}$$

Hence we have established that the maps A, B and C are well-defined.

It is easy to see that (A, B, C, D) is in fact a realization of M, i.e. that $M(t) = C e^{tA} B$, $t \geq 0$. Let $u \in \mathbb{K}^m$. Then for $t \geq 0$

$$C e^{tA} Bu = C e^{tA} Mu = CL_t Mu = (L_t Mu)(0) = M(t)u$$

which shows the claim. It remains to show that (A, B, C, D) is a minimal system. Since (A, B, C, D) is an n-dimensional system, Proposition 4 implies that (A, B, C, D) is minimal since H_M has rank n. Clearly, the result is independent of the particular Euclidean structure that is put on the state space X. □

The realization (A, B, C, D) introduced in the previous theorem is called the *(left) shift realization.* For different approaches to obtain the left-shift realization for infinite-dimensional continuous-time systems (see, e.g. Helton 1976, Salamon 1989).

In § 1 it was shown that each state space system gives rise to an input–output system. The following corollary states that an input–output system is the input–output system of a state space system if and only if the associated Hankel map has finite rank.

Corollary 3: *Let $M \in C(\mathbb{R}_+)$. Then, the following two statements are equivalent,*

1. *the input–output map with kernel function M is the input–output map of a state space system.*
2. *the Hankel map H_M has finite rank.*

In the following corollary we show that for the left shift realization the reachability and observability maps can be derived. It turns out that the observability map is the identity map on the state space, whereas the reachability map is the Hankel map with symbol M whose range is restricted to the state space.

Corollary 4: *We use the notation of the previous theorem. Then the observability map $\mathcal{O}$ of the system (A, B, C, D) is given by*

$$\mathcal{O}: X \to C(\mathbb{R}_+); \qquad x \mapsto x$$

and the reachability map $\mathcal{R}$ of the system is

$$\mathcal{R}: PC_{tr}(\mathbb{R}_-) \to X; \qquad u \mapsto H_M u$$

Proof: Let $x \in X$, then for $t \geq 0$

$$(\mathcal{O}x)(t) = C e^{tA} x = CL_t x = (L_t x)(0) = x(t + 0) = x(t)$$

which shows that $\mathcal{O}x = x$. Let $u \in PC_{tr}(\mathbb{R}_-)$, then for $t \geq 0$

$$\begin{aligned}(\mathcal{R}u)(t) &= \left(\int_{-\infty}^{0} e^{-\tau A} Bu(\tau)\, d\tau\right)(t) \\ &= \left(\int_{-\infty}^{0} L_{-\tau} Bu(\tau)\, d\tau\right)(t) \\ &= \left(\int_{-\infty}^{0} L_{-\tau} M(\cdot)u(\tau)\, d\tau\right)(t) \\ &= \int_{-\infty}^{0} M(t-\tau)u(\tau)\, d\tau = (H_M u)(t)\end{aligned}$$

which shows that $\mathcal{R}u = H_M u$, for $u \in PC_{tr}(\mathbb{R}_-)$. □

As an immediate consequence of the above realization result we have that the functions in the range of the Hankel map H_M are infinitely often continuously differentiable. Therefore the state space of the left shift realization is a space of infinitely often continuously differentiable functions. In fact the A map acts on the state space as a differentation operator.

Corollary 5: *Let $H_M : PC_{tr}(\mathbb{R}_-) \to C(\mathbb{R}_+)$ be a finite rank Hankel map. Then*

(1) *each $f \in$ range (H_M) is infinitely often continuously differentiable.*
(2) *M is infinitely often continuously differentiable.*
(3) *if (A, B, C, D) is the left shift realization of M we have for $f \in$ range (H_M)*

$$Af = f'$$

Proof:

(1) The result follows from realization theorem and Corollary 1.
(2) This follows from (1) and Lemma 2.
(3) On X we have that $L_t = e^{tA}$, $t \geq 0$. Therefore for $f \in X$, and $t \geq 0$

$$\begin{aligned}\lim_{\tau \to 0^+} \frac{1}{\tau}[f(t+\tau) - f(t)] &= \lim_{\tau \to 0^+} \frac{1}{\tau}[(L_\tau f)(t) - f(t)] \\ &= \lim_{\tau \to 0^+} \frac{1}{\tau}[(e^{\tau A} f)(t) - f(t)] \\ &= \lim_{\tau \to 0^+} \frac{1}{\tau}[(e^{\tau A} - I)f](t) \\ &= (Af)(t)\end{aligned}$$

Since f is infinitely often continuously differentiable, the right derivative equals the derivative of f and hence $f' = Af \in X$. □

4. The frequency domain realization problem

Having studied the realization problem in its time domain formulation in the previous section, we now consider the realization problem for rational functions. In fact we will show how Fuhrmann's shift realization for rational models (Fuhrmann 1976) can be derived from the shift realization that was studied in the previous section. The main tool to achieve this will be a state space transformation which is obtained by applying the Laplace transform to the elements in the state space of the left shift realization. Following the tutorial style of this paper we will recall without proof some facts on Laplace transforms which are important for our development. As general references to these results on Laplace transforms see, e.g. LePage (1961), Doetsch (1974) and Körner (1988).

Let $\mathcal{E}$ be the set of piecewise continuous functions $f : \mathbb{R}_+ \to \mathbb{K}^{r \times l}$ such that $f(t)\, e^{-at} \to 0$ as $t \to \infty$ for some $a \in \mathbb{R}$. A function $f \in \mathcal{E}$ is said to be in the set $\mathcal{E}_a$ if $f(t)\, e^{-bt} \to 0$ as $t \to \infty$ for all $b > a$. For $f \in \mathcal{E}_a$ the *(one-sided) Laplace transform* of f is given by

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

for all $s \in \mathbb{C}$ with $Re(s) > a$. Clearly the Laplace transform could be defined on a larger class of functions, but the presently chosen setup will be sufficient for our purposes.

In the following proposition some standard results are being stated concerning the Laplace transform of the impulse response of a system. The most important fact from our point of view is that the *transfer function* of a system, i.e. the Laplace transform of its impulse response is a proper rational (matrix-valued) function. Recall that a rational function G is called *proper* if the limit $G(\infty)$ of G at infinity is finite.

The frequency domain realization problem is, therefore, given a proper rational function G, to find a (minimal) system (A, B, C, D) such that G is that the transfer function of the system, i.e. $G(s) = C(sI - A)^{-1}B + D$, $s \in \mathbb{C}$.

Proposition 7: *Let (A, B, C, D) be a minimal system and let $M(t) := Ce^{tA}B$, $t \geq 0$. Then*

(1) *$M \in \mathcal{E}_a$, where $a = \max\{Re(\lambda) \mid \lambda \in \sigma(A)\}$.*
(2) *$(\mathcal{L}M)(s) = C(sI - A)^{-1}B$, for all $s \in \mathbb{C}$, $Re(s) > a$.*
(3) *$\mathcal{L}M$ is a proper rational matrix function which has no poles in the half plane $\{s \in \mathbb{C} \mid Re(s) > a\}$.*

In the next theorem a number of well-known properties of the one-sided Laplace transform are collected. In stating the results we use the following convention for the limit of a Laplace transform $(\mathcal{L}f)(s)$ as the independent variable s converges to infinity

$$\lim_{s\to\infty}(\mathcal{L}f)(s) := \lim_{\substack{x\in\mathbb{R}\\ x\to+\infty}}(\mathcal{L}f)(x+iy)$$

for some $y \in \mathbb{R}$. The limit is independent of the choice of y.

Theorem 2: *Let $f \in \mathcal{E}_a$.*

(1) **(Linearity)** *If $\lambda,\ \mu \in \mathbb{C},\ f \in \mathcal{E}_a,\ g \in \mathcal{E}_b$ and $c \geq a,\ c \geq b$, then $\lambda f + \mu g \in \mathcal{E}_c$ and for all $s \in \mathbb{C}$ with $Re(s) > c$*

$$[\mathcal{L}(\lambda f + \mu g)](s) = \lambda(\mathcal{L}f)(s) + \mu(\mathcal{L}g)(s)$$

(2) **(Differentiation)** *Assume that f is continuously differentiable on $]0,\infty[$, and $f' \in \mathcal{E}_a$, where we have set $f'(0) := \lim_{t\to 0+} f'(t)$. Set $f(0+) := \lim_{t\to 0+} f(t)$. Then for $s \in \mathbb{C}$, such that $Re(s) > a$*

$$(\mathcal{L}f')(s) = s(\mathcal{L}f)(s) - f(0+)$$

(3) **(Initial Value Theorem)** *Assume that f is continuously differentiable on $]0,\infty[$ and $f' \in \mathcal{E}_a$, where we have set $f'(0) := \lim_{t\to 0+} f'(t)$. Then*

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} s(\mathcal{L}f)(s)$$

(4) **(Final Value Theorem)** *Let $f \in \mathcal{E}_0$ and assume that f is continuously differentiable on $]0,\infty[$ and $f' \in \mathcal{E}_a$ for some $a < 0$, where we have set $f'(0) := \lim_{t\to 0+} f'(t)$. Then $\lim_{t\to\infty} f(t)$ exists and*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0+} s(\mathcal{L}f)(s)$$

$$:= \lim_{\substack{x>0,x\to 0\\ y\to 0}}(x+iy)(\mathcal{L}f)(x+iy)$$

(5) **(Injectivity)** *Let $f \in \mathcal{E}_a$. If $(\mathcal{L}f)(s) = 0$ for all $s \in \mathbb{C}$, $Re(s) > a$, then $f(t) = 0$ for $t \geq 0$.*

In the following well-known lemma we are going to give a realization of a strictly proper rational function, which is generally not minimal. A proper rational function G is called *strictly proper* if $G(\infty) = 0$.

Lemma 3: *Let $H(s)$ be a $p \times m$ strictly proper rational function. Write $H(s) = P(s)/q(s)$, where*

$$P(s) = P_{n-1}s^{n-1} + P_{n-2}s^{n-2} + \cdots + P_1 s + P_0$$

$$q(s) = s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + q_0$$

with $P_i \in \mathbb{K}^{p\times m}$, $q_i \in \mathbb{K}$, $i = 0, 1, \ldots, n-1$. Then

$$A = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & \cdots & 0_m \\ 0_m & 0_m & I_m & 0_m & \cdots & 0_m \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0_m & & & & I_m & 0_m \\ 0_m & \cdots & & \cdots & 0_m & I_m \\ -q_0 I_m & -q_1 I_m & \cdots & & -q_{n-2}I_m & -q_{n-1}I_m \end{pmatrix}$$

$$B = \begin{pmatrix} 0_m \\ 0_m \\ \vdots \\ \vdots \\ 0_m \\ I_m \end{pmatrix}$$

$$C = (P_0 \quad P_1 \quad \cdots \quad P_{n-2} \quad P_{n-1})$$

are such that for $s \in \mathbb{C}\backslash\sigma(A)$

$$H(s) = C(sI - A)^{-1}B$$

Proof: This identity is easily verified. □

The result of the previous lemma is of course a realization result, in that for a given strictly proper rational function a state space realization is constructed. One of the less satisfactory aspects of this realization is that it is not minimal if G is matrix-valued. A minimal realization can of course always be obtained from a non-minimal one, but then the structure of the realization will become obscured. Fuhrmann's shift realization in terms of rational models (Fuhrmann 1976) provides a minimal realization also for matrix-valued proper rational functions. In the following theorem we state this realization result in our language. While the result itself is of course not new, we believe that its proof is new and provides insights that are interesting in their own right. The strategy for the proof is as follows. We first obtain a symbol M whose Laplace transform is the rational function G for which we seek a realization. We then use the left shift realization (A, B, C, D) for M using Theorem 1. This realization is also a realization of G. Fuhrmann's shift realization is then obtained from the time domain realization by applying a state space transformation which is the Laplace transform restricted to the state space of (A, B, C, D). We therefore see that Fuhrmann's shift realization is nothing else but the frequency domain version of the time domain shift realization of Theorem 1.

Theorem 3: *Let G be a proper rational matrix function. Then there exists a state space X and a minimal system (A, B, C, D) such that*

$$G(s) = C(sI - A)^{-1}B + D, \qquad s \in \mathbb{C}\backslash\sigma(A)$$

which is defined as follows. For a strictly proper rational function f let $[F(f)](s) := sf(s) - \lim_{s\to\infty} sf(s)$, $s \in \mathbb{C}$. Let $G_p(s) := G(s) - G(\infty)$, $s \in \mathbb{C}$. The state space X is defined to be the linear span of the spaces $F^k(G_p\mathbb{K}^m)$, $k = 0, 1, 2, \ldots$. The system maps are defined as

$$B : \mathbb{K}^m \to X; \qquad u \mapsto G_p u$$

$$C : X \to \mathbb{K}^p; \qquad x \mapsto \lim_{s\to\infty} sx(s)$$

$$A : X \to X; \qquad x \mapsto Ax,$$

where $(Ax)(s) = (Fx)(s) = sx(s) - \lim_{s\to\infty} sx(s)$, $s \in \mathbb{C}$

$$D = G(\infty)$$

Proof: First note that by Lemma 3 there exists a system (A_1, B_1, C_1, D_1) such that

$$G(s) = C_1(sI - A_1)^{-1}B_1 + D_1, \qquad s \in \mathbb{C}\backslash\sigma(A_1)$$

Now consider the Hankel map H_M with $M(t) = C_1 e^{tA_1} B_1$, $t \geq 0$. Clearly M is continuous and H_M is of finite rank. By the realization result of Theorem 1 we can therefore write

$$M(t) = C_2 e^{tA_2} B_2, \qquad t \geq 0$$

where (A_2, B_2, C_2, D_2), $D_2 = D_1$, is the minimal left shift realization of M. We now apply the Laplace transform to this realization. The state space X_2 of the left shift realization is $X_2 = range\,(H_M)$, whose dimension equals $rank\,(H_M)$. By Corollary 1, $X_2 = range\,(H_M) = \{(C_2 e^{tA_2} x)_{t\geq 0} \mid x \in X_2\}$. Hence by Proposition 7 each function in X_2 is in $\mathcal{E}_a$, where $a = \max\{Re(\lambda) \mid \lambda \in \sigma(A_2)\}$ and therefore admits a Laplace transform. Let

$$X := \{\mathcal{L}f \mid f \in X_2\}$$

The linearity and the injectivity of the Laplace transform as established in Theorem 2 shows that the Laplace transform induces a state space transformation $\mathcal{L}_{|X_2} : X_2 \to X$. Now define

$$(A, B, C, D) := (\mathcal{L}_{|X_2} A_2 \mathcal{L}^{-1}_{|X_2}, \mathcal{L}_{X_2} B_2, C_2 \mathcal{L}^{-1}_{|X_2}, D_2)$$

Clearly (A, B, C, D) is minimal. Let $u_0 \in \mathbb{K}^m$, then

$$Bu_0 = \mathcal{L}(B_2 u_0) = \mathcal{L}(M u_0) = G_p u_0$$

Let $x \in X$, and let $f \in X_2$ be the unique element such that $\mathcal{L}f = x$. Then for $s \in \mathbb{C}$, $Re(s) > a$

$$\begin{aligned}(Ax)(s) &= (\mathcal{L}_{|X_2} A_2 \mathcal{L}^{-1}_{|X_2} x)(s) = (\mathcal{L}_{|X_2} A_2 f)(s) \\ &= (\mathcal{L}_{|X_2} f')(s) = s(\mathcal{L}f)(s) - \lim_{t\to 0^+} f(t) \\ &= sx(s) - \lim_{s\to\infty} sx(s)\end{aligned}$$

by Corollary 5 and Theorem 2

$$Cx = C_2 \mathcal{L}^{-1}_{|X_2} x = C_2 f = f(0) = \lim_{s\to\infty} s(\mathcal{L}f)(s) = \lim_{s\to\infty} sx(s)$$

Since by Theorem 1 and Proposition 2, $X_2 = range\,(H_M) = span\,\{A_2^k B_2 \mathbb{K}^m, k \geq 0\}$, we have that

$$\begin{aligned}X = \mathcal{L}(X_2) &= span\,\{A^k G_p \mathbb{K}^m) \mid k \geq 0\} \\ &= span\,\{F^k(G_p \mathbb{K}^m) \mid k \geq 0\}\end{aligned} \qquad \square$$

5. Conclusions

A novel approach was presented to the time domain realization problem. Central to the derivation is the newly introduced notion of a Hankel map which is a generalization of the notion of a Hankel operator on L^2 spaces. This generalization allowed us to consider also unstable systems. The realization that was obtained is the left shift realization which has been extensively studied in the infinite dimensional continuous-time situation with the additional stability assumptions.

Using the Laplace transform as a state space transformation, Fuhrmann's rational model for a transfer function was derived from the continuous-time time domain shift realization.

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Appendix

Proof of Proposition 1: Let M be a continuous function on $\mathbb{R}_+$ and set $M(t) = 0$ for $t < 0$. Let $u \in PC_{tr}(\mathbb{R})$. We need to show that $y = IO_{M,D}(u)$ is a continuous function on $\mathbb{R}$, i.e. for $t_1 \in \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then $\|y(t_1) - y(t_2)\| < \varepsilon$. Let $t_2 \in \mathbb{R}$ be an arbitrary point such that $|t_1 - t_2| < 1$. Set $t_M := \max\{t_1, t_2\}$ and $t_m := \min\{t_1, t_2\}$. Since $u \in PC_{tr}(\mathbb{R})$ there exists $T \in \mathbb{R}$ such that $T < \min\{t_1, t_2\}$ and $u(t) = 0$ for $t \leq T$. Now

$$
\begin{aligned}
&\|y(t_1) - y(t_2)\| \\
&= \left\| \int_{-\infty}^{t_1} M(t_1 - \tau)u(\tau)\,\mathrm{d}\tau \right. \\
&\quad \left. - \int_{-\infty}^{t_2} M(t_2 - \tau)u(\tau)\,\mathrm{d}\tau \right\| \\
&= \left\| \int_{-\infty}^{t_M} [M(t_1 - \tau) - M(t_2 - \tau)]u(\tau)\,\mathrm{d}\tau \right\| \\
&\leq \int_{-\infty}^{t_M} \|M(t_M - \tau) - M(t_m - \tau)\|\,\|u(\tau)\|\,d\tau \\
&= \int_{t_m}^{t_M} \|M(t_M - \tau)\|\,\|u(\tau)\|\,\mathrm{d}\tau \\
&\quad + \int_{T}^{t_m} \|M(t_M - \tau) - M(t_m - \tau)\|\,\|u(\tau)\|\,\mathrm{d}\tau \\
&\leq \left(\sup_{\tau \in [t_m, t_M]} \|M(t_M - \tau)\| \right) \left(\sup_{\tau \in [t_m, t_M]} \|u(\tau)\| \right) (t_M - t_m) \\
&\quad + \left(\sup_{\tau \in [T, t_m]} \|M(t_M - \tau) - M(t_m - \tau)\| \right) \int_{T}^{t_m} \|u(\tau)\|\,\mathrm{d}\tau \\
&\leq K_1 K_2 |t_1 - t_2| + K_3 \sup_{\tau \in [T, t_m]} \|M(t_1 - \tau) - M(t_2 - \tau)\|
\end{aligned}
$$

where

$$K_1 := 1 + \sup_{\tau \in [0,1]} \|M(\tau)\|$$

$$K_2 := 1 + \sup_{\tau \in [t_1 - 1, t_1 + 1]} \|u(\tau)\|$$

$$K_3 := 1 + \int_{T}^{t_1} \|u(\tau)\|\,\mathrm{d}\tau$$

Note that M is continuous on $[0, \infty[$ and hence $0 < K_1 < \infty$. As u is piecewise continuous and hence bounded, we have that $0 < K_2,\ K_3 < \infty$. The function M is continuous on the compact interval $[0, t_1 + 1 - T]$ and is hence uniformly continuous on this interval. Therefore for $\varepsilon > 0$ there exists $\hat{\delta} > 0$ such that

$$\|M(\tau_1) - M(\tau_2)\| < \frac{\varepsilon}{2K_3}$$

for all $\tau_1, \tau_2 \in [0, t_1 + 1 - T]$ with $|\tau_1 - \tau_2| < \hat{\delta}$. If t_2 is chosen such that

$$|t_2 - t_1| < \delta := \min\left\{ \hat{\delta}, \frac{\varepsilon}{2K_1 K_2}, 1 \right\}$$

then for $\tau \in [T, \min\{t_1, t_2\}]$

$$|(t_2 - \tau) - (t_2 - \tau)| = |t_1 - t_2| < \delta \leq \hat{\delta}$$

and $t_1 - \tau$, $t_2 - \tau \in [0, \max\{t_1, t_2\} - T] \subseteq [0, t_1 + 1 - T]$. Hence

$$\|M(t_1 - \tau) - M(t_2 - \tau)\| < \frac{\varepsilon}{2K_3}$$

for all $\tau \in [T, \min\{t_1, t_2\}]$. With such a choice of t_2 we therefore have by the above inequalities that

$$
\begin{aligned}
\|y(t_1) - y(t_2)\| &< K_1 K_2 |t_1 - t_2| \\
&\quad + K_3 \sup_{\tau \in [T, t_m]} \|M(t_1 - \tau) - M(t_2 - \tau)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

Hence y is a continuous function on $\mathbb{R}$. □

A slightly different proof of a related result concerning the convolution of two integrable functions can be found in Appendix C of Körner (1988).

The proof of Proposition 6 is a minor adaption of the proof given in Rudin (1991) showing that the generator of a norm continuous semigroup is bounded. Another proof of this proposition is given in Bellman (1995), Theorem 4, page 173.

Proof of Proposition 6: Note that $\rho \to \int_0^\rho T(s)\,\mathrm{d}s$ defines a continuously differentiable function on $[0, \infty[$ which has a continuous right derivative at 0 given by $\lim_{\rho \to 0} (1/\rho) \int_0^\rho T(s)\,\mathrm{d}s = I$. Hence for ρ small enough

$$\left\| I - \frac{1}{\rho} \int_0^\rho T(s)\,\mathrm{d}s \right\| < 1$$

This implies that $(1/\rho) \int_0^\rho T(s)\,\mathrm{d}s$ is invertible and therefore $\int_0^\rho T(s)\,\mathrm{d}s$ is invertible. Now for $h > 0$

$$
\begin{aligned}
\frac{1}{h}[T(h) - I] \int_0^\rho T(s)\,\mathrm{d}s &= \frac{1}{h} \left(\int_0^\rho T(s+h)\,\mathrm{d}s - \int_0^\rho T(s)\,\mathrm{d}s \right) \\
&= \frac{1}{h} \left(\int_\rho^{\rho+h} T(s)\,\mathrm{d}s - \int_0^h T(s)\,\mathrm{d}s \right)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{1}{h}[T(h) - I] = &\left(\frac{1}{h} \int_\rho^{\rho+h} T(s)\,\mathrm{d}s - \frac{1}{h} \int_0^h T(s)\,\mathrm{d}s \right) \\
&\times \left(\int_0^\rho T(s)\,\mathrm{d}s \right)^{-1}
\end{aligned}
$$

Letting $h \to 0^+$ in the above identity shows that $(1/h)[(T(h) - I]$ converges in norm to $A : [T(\rho) - I] (\int_0^\rho T(s)\,\mathrm{d}s)^{-1}$.

It remains to show that $T(t) = \mathrm{e}^{tA}$, for $t \geq 0$. We will show that given $T > 0$, we have that $T(t) = \mathrm{e}^{tA}$ for $0 \leq t \leq T$. Let $T > 0$ be fixed. Since $t \to \|T(t)\|$ and $t \to \|\mathrm{e}^{tA}\|$ are continuous there is a constant C such that $\|T(s)\|\,\|\mathrm{e}^{tA}\| \leq C$ for $0 \leq s, t \leq T$. Given $\varepsilon > 0$ it follows since

$$A = \lim_{t \to 0^+} \frac{T(t) - I}{t} = \lim_{t \to 0^+} \frac{\mathrm{e}^{tA} - I}{t}$$

that there exists a $\delta > 0$ such that

$$\frac{1}{h}\|T(h) - e^{hA}\| < \frac{\varepsilon}{TC} \quad \text{for } 0 < h \leq \delta$$

Let $0 \leq t \leq T$ and choose $n \geq 1$ such that $t/n < \delta$. From the semigroup property and the above inequality it then follows that

$$\begin{aligned}
\|T(t) - e^{tA}\| &= \left\|T\left(n\frac{t}{n}\right) - e^{n(t/n)A}\right\| \\
&\leq \sum_{k=0}^{n-1} \left\|T\left((n-k)\frac{t}{n}\right) e^{(kt/n)A} - T\left((n-k-1)\frac{t}{n}\right) e^{[(k+1)t/n]A}\right\| \\
&\leq \sum_{k=0}^{n-1} \left\|T\left((n-k-1)\frac{t}{n}\right)\right\| \times \left\|T\left(\frac{t}{n}\right) - e^{(t/n)A}\right\| \|e^{(kt/n)A}\| \\
&\leq Cn\frac{\varepsilon}{TC}\frac{t}{n} \leq \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $T(t) = e^{tA}$ for $0 \leq t \leq T$.

To show that A is unique, assume that A_1 is such that $e^{tA_1} = e^{tA}$ for $t \geq 0$. Then differentiating this identity and evaluating at $t = 0$, it follows that

$$A_1 = A_1 e^{0A_1} = A e^{0A} = A \qquad \square$$

Proof of Lemma 1: We first show inductively that there exist $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$ such that $\mathcal{G}$ has zero kernel. Let

$$\hat{\mathcal{G}}_1(t) := [g_1(t) \quad g_2(t) \quad \cdots \quad g_n(t)]$$

$t \geq 0$. Since $g_1, g_2, \ldots, g_n$ are independent there exists $t_1 \in \mathbb{R}_+$ such that the dimension of $ker(\mathcal{G}_1)$ is less than or equal to $n-1$, where $\mathcal{G}_1 := \hat{\mathcal{G}}_1(t_1)$. This is the case since otherwise there exists a non-zero vector $\alpha(\alpha_1 \; \alpha_2 \; \cdots \; \alpha_n)^T \in \mathbb{K}^n$ such that

$$\hat{\mathcal{G}}_1(t)\alpha = \alpha_1 g_1(t) + \alpha_2 g_2(t) + \cdots + \alpha_n g_n(t) = 0$$

for $t \geq 0$, which is a contradiction to the linear independence of $g_1, \ldots, g_n$. If the dimension of $ker(\mathcal{G}_1)$ is zero then $\mathcal{G}$ has zero kernel for $t_2, \ldots, t_n$ chosen arbitrarily and the existence of the required points $t_1, t_2, \ldots, t_n$ is established.

Now assume that there are k distinct points $t_1, \ldots, t_k \in \mathbb{R}_+$, $1 \leq k < n$, such that the kernel of

$$\mathcal{G}_k := \begin{pmatrix} g_1(t_1) & g_2(t_1) & \cdots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \cdots & g_n(t_2) \\ \vdots & \vdots & & \vdots \\ g_1(t_k) & g_2(t_k) & \cdots & g_n(t_k) \end{pmatrix}$$

has dimension less than or equal to $n-k$. If the kernel of $\mathcal{G}_k$ is zero, define $\mathcal{G}$ by adding rows to $\mathcal{G}_k$ for arbitrary values of $t_{k+1}, \ldots, t_n$. Then $\mathcal{G}$ also has zero kernel and the claim is established. We therefore assume that the kernel of $\mathcal{G}_k$ is not zero, Now let

$$\hat{\mathcal{G}}_{k+1}(t) := \begin{pmatrix} g_1(t_1) & g_2(t_1) & \cdots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \cdots & g_n(t_2) \\ \vdots & \vdots & & \vdots \\ g_1(t_k) & g_2(t_k) & \cdots & g_n(t_k) \\ g_1(t) & g_2(t) & \cdots & g_n(t) \end{pmatrix}$$

for $t \geq 0$. Clearly for each $t \geq 0$ we have that

$$dim\,(ker\,(\hat{G}_{k+1}(t))) \geq dim\,(ker\,(G_k)).$$

But if

$$dim\,(ker\,(\hat{G}_{k+1}(t))) = dim\,(ker\,(G_k))$$

for each $t \geq 0$, then

$$ker\,(\hat{G}_{k+1}(t)) = ker\,(\mathcal{G}_k)$$

for each $t \geq 0$. Since the dimension of the kernel of $\mathcal{G}_k$ is not zero there exists $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{K}^n$, $\alpha \neq 0$, such that for $t \geq 0$

$$\hat{\mathcal{G}}_{k+1}(t)\alpha = \mathcal{G}_k\alpha = 0$$

Considering the last row of $\hat{G}_{k+1}(t)$ we therefore have that $\alpha_1 g_1(t) + \alpha_2 g_2(t) + \cdots + \alpha_n g_n(t) = 0$ for $t \geq 0$ which is a contradiction to the linear independence of $g_1, \ldots, g_n$. Therefore there exists $t_{k+1} \geq 0$ such that $dim\,(ker\,(\hat{\mathcal{G}}_{k+1}(t_{k+1}))) > dim\,(ker\,(\mathcal{G}_k))$. Hence there exists $t_{k+1} \in \mathbb{R}_+$, $t_{k+1} \neq t_i$, $i = 1, \ldots, k$, such that the dimension of the kernel of $\mathcal{G}_{k+1} := \hat{\mathcal{G}}_{k+1}(t_{k+1})$ is less than the dimension of the kernel of $\mathcal{G}_k$, i.e. less than $n-k-1$. If $k = n$ setting $\mathcal{G} := \mathcal{G}_n$ this completes the proof of the claim. Also if the dimension of $ker\,(\mathcal{G}_{k+1})$ is zero then the kernel of $\mathcal{G}$ is zero, where we have defined $\mathcal{G}$ by adding rows to $\mathcal{G}_{k+1}$ for arbitrary values of $t_{n+2}, \ldots, t_n$. In all other cases continue inductively until $k = n$ or until $\mathcal{G}_{k+1}$ has zero kernel.

Since $ker\,(\mathcal{G}) = \{0\}$ there exists a matrix $\mathcal{F}$ such that $\mathcal{F}\mathcal{G} = I$. Let f be an element of the space spanned by the functions $g_1, g_2, \ldots, g_n$. Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)} \in \mathbb{K}$ be such that

$$f(t) = \alpha^{(1)}g_1(t) + \alpha^{(2)}g_2(t) + \cdots + \alpha^{(n)}g_n(t)$$

for $k \geq 1$, $t \geq 0$. Therefore

$$\begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix} = \mathcal{G} \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix}$$

and hence

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix} = \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix}$$

Let now $(f_k)_{k\geq 1}$ be a sequence in the subspace spanned by $g_1, g_2, \dots, g_n$ that converges pointwise to a function f. Therefore there exist constants $\alpha_k^{(1)}, \alpha_k^{(2)}, \dots, \alpha_k^{(n)}$ such that

$$f_k(t) = \alpha_k^{(1)} g_1(t) + \alpha_k^{(2)} g_2(t) + \cdots + \alpha_k^{(n)} g_n(t)$$

for $k \geq 1$, $t \geq 0$. Hence

$$\begin{pmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ \vdots \\ \alpha_k^{(n)} \end{pmatrix} = \mathcal{F} \begin{pmatrix} f_k(t_1) \\ f_k(t_2) \\ \vdots \\ f_n(t_n) \end{pmatrix}$$

As $k \to \infty$ we therefore have that the limit

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix} := \lim_{k\to\infty} \begin{pmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ \vdots \\ \alpha_k^{(n)} \end{pmatrix}$$

$$= \mathcal{F} \lim_{k\to\infty} \begin{pmatrix} f_k(t_1) \\ f_k(t_2) \\ \vdots \\ f_n(t_n) \end{pmatrix} = \mathcal{F} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{pmatrix}$$

exists. But this implies that for $t \geq 0$

$$\begin{aligned} f(t) &= \lim_{k\to\infty} f_k(t) \\ &= \lim_{k\to\infty} \alpha_k^{(1)} g_1(t) + \alpha_2^{(2)} g_2(t) + \cdots + \alpha_k^{(n)} g_n(t) \\ &= \alpha^{(1)} g_1(t) + \alpha^{(2)} g_2(t) + \cdots + \alpha^{(n)} g_n(t) \end{aligned}$$

thereby proving the claim. □

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