

A NOTE ON THE EXISTENCE, UNIQUENESS AND SYMMETRY OF PAR-BALANCED REALIZATIONS

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We give a proof of the realization theorem of N.J. Young which states that analytic functions which are symbols of bounded Hankel operators admit par-balanced realizations. The main tool used in this proof is the induced Hilbert spaces and a lifting lemma of Kreĭn-Reid-Lax-Dieudonné. Alternatively one can use the Loewner inequality. A short proof of the uniqueness of par-balanced realizations is included. As an application, it is proved that par-balanced realizations of real symmetric transfer functions are J -self-adjoint.

1. Introduction

Consider a (discrete time, time invariant) *linear system* (A, B, C, D) with contractive *main operator* $A \in \mathcal{L}(\mathcal{H})$, *input operator* $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, *output operator* $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and *external operator* $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, where the *state space* \mathcal{H} , *input space* \mathcal{U} , and *output space* \mathcal{Y} are Hilbert spaces. With every linear system (A, B, C, D) there is associated its *transfer function* $G: \rho(A) \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ as follows

$$G(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \rho(A). \quad (1.1)$$

Since the main operator A is assumed contractive, i.e. $\|A\| \leq 1$, the transfer function is defined and analytic for all $|\lambda| > 1$.

Following the general theory, e.g. see [10], the system (A, B, C, D) is called *observable* if for any $h \in \mathcal{H}$ we have $\sum_{k \geq 0} \|CA^k h\|^2 < \infty$ and the *observability operator* O defined by

$$Oh = (CA^k h)_{k \geq 0}, \quad h \in \mathcal{H},$$

is bounded and injective.

The system is called *reachable* if for any $(u_k)_{k \leq 0} \in \ell_{\mathcal{U}}^2$ the series $\sum_{k \leq 0} A^k B u_k$ converges strongly in \mathcal{H} and the *reachability operator* R defined by

$$R((u_k)_{k \geq 0}) = \sum_{k \geq 0} A^k B u_k, \quad (u_k)_{k \leq 0} \in \ell_{\mathcal{U}}^2,$$

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is bounded and has dense range.

Whenever the operators O and R are everywhere defined, one can introduce the operator $H = OR$. The operator H is called the *Hankel operator* associated to the system (A, B, C, D) . From the definition of the operators O and R , the operator H has the following Hankel block matrix representation

$$H \sim \left(CA^{i+j}B \right)_{i,j \geq 0}, \tag{1.2}$$

more precisely, for all $u = (u_k)_{k \geq 0} \in \ell^2_{\mathcal{U}}$, $y = (y_k)_{k \geq 0} \in \ell^2_{\mathcal{Y}}$, we have

$$\langle Hu, y \rangle = \sum_{j \geq 0} \sum_{i \geq 0} \langle CA^{i+j}Bu_j, y_i \rangle. \tag{1.3}$$

If the system (A, B, C, D) has bounded and everywhere defined observability operator O , then one can define the *observability gramian* O^*O . Similarly, if the system has bounded reachability operator R then one can define the *reachability gramian* RR^* . Following N.J. Young [21], the system (A, B, C, D) is called *par-balanced* if the observability operator O and the reachability operator R are bounded everywhere defined operators and the observability gramian coincides with the reachability gramian, $O^*O = RR^*$. This is a generalization of the notion of balanced linear system introduced by B.C. Moore [16] for finite dimensional systems.

The above presentation corresponds to the *internal*, or, equivalently, the *state space* representation of a system. Sometimes a linear system is given only in its *external representation*, that is, given \mathcal{U} and \mathcal{Y} Hilbert spaces we consider $G: \{z \in \mathbb{C} \mid |z| > 1\} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ an operator valued function which is analytic everywhere on its domain of definition and at infinity. One can define another operator valued analytic function $g: \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ by

$$g(z) = \frac{1}{z} \left(G\left(\frac{1}{z}\right) - G(\infty) \right), \quad |z| < 1.$$

Then g has the Taylor expansion on \mathbb{D}

$$g(z) = \sum_{k \geq 0} S_k z^k, \quad |z| < 1,$$

where $S_k \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, $k \geq 0$. Associated with the function G one can consider the block-operator Hankel matrix

$$H = \begin{bmatrix} S_0 & S_1 & S_2 & \dots & S_k & \dots \\ S_1 & S_2 & S_3 & \dots & S_{k+1} & \dots \\ S_2 & S_3 & \dots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ \vdots & & & & & \end{bmatrix}. \tag{1.4}$$

The *realization problem* asks for the determination of a system (A, B, C, D) , with some state space \mathcal{H} , input space \mathcal{U} , and output space \mathcal{Y} such that (1.1) holds. Observable

and reachable realizations of transfer functions can be obtained by the so-called reduced shift realization, e.g. see [10]. There was a problem for some time to prove that balanced realizations for nonrational transfer functions exist, e.g. see [12]. The main result of the paper of N.J. Young [21] is:

THEOREM 1.1 *Let \mathcal{U} and \mathcal{Y} be Hilbert spaces and let $G: \{z \in \mathbb{C} \mid |z| > 1\} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ be an operator valued function which is analytic on its domain and at infinity. If the Hankel block-operator matrix in (1.4) defines a bounded operator $H : \ell^2_{\mathcal{U}} \rightarrow \ell^2_{\mathcal{Y}}$ then there exists a realization (A, B, C, D) of G , corresponding to some Hilbert state space \mathcal{H} , which is observable, reachable and par-balanced.*

The approach to the proof of this theorem in [21] is to use the restricted shift realization and then to perform a state space transformation which yields the desired par-balanced realization. The most difficult part of the proof is to show that the new main operator is bounded and even a contraction.

The aim of this short note is to produce an alternate approach based on induced Hilbert spaces and a slightly generalized lifting Lemma of Kreĭn-Reid-Lax-Dieudonné. The approach with induced Hilbert spaces enables us to discuss the choice of the state space from a different perspective. We notice that a slightly different argument making use of the Loewner inequality, and inspired by the recent paper [15], can be used. Our interest in Theorem 1.1 is due to sign symmetric transfer functions that we considered in [11], where we used the induced Hilbert spaces and the Kreĭn-Reid-Lax-Dieudonné Lemma in the realization theory.

The material is organized as follows. In Section 2 we have a short discussion of induced Hilbert spaces (for the indefinite variant see [5]), the generalized version of the Kreĭn-Reid-Lax-Dieudonné Lemma (for an indefinite variant, see [7]) and two other representations of induced Hilbert spaces. Section 3 contains the proof of Theorem 1.1. In Section 4 we give a short proof of the uniqueness of par-balanced realizations, see Theorem 4.1. Finally, as an application, we prove that par-balanced realizations of real symmetric transfer functions are J -self-adjoint. An appendix containing a correction of N.J. Young to his original proof in [21], is also included.

The results discussed in this paper carry over to the continuous-time case using the bilinear transform method of [17].

2. Induced Hilbert Spaces

Let us consider a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let A be a bounded positive operator on \mathcal{H} . A pair (\mathcal{K}_A, Π_A) is called a *Hilbert space induced* by A if \mathcal{K}_A is a Hilbert space and $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K}_A)$ has dense range and $\Pi_A^* \Pi_A = A$.

On \mathcal{H} we consider the nonnegative inner product $\langle \cdot, \cdot \rangle_A$ defined by

$$\langle x, y \rangle_A = \langle Ax, y \rangle, \quad x, y \in \mathcal{H}.$$

Let $\widehat{\mathcal{H}} = \mathcal{H} \ominus \ker A$ and note that the restriction of the inner product $\langle \cdot, \cdot \rangle_A$ to $\widehat{\mathcal{H}}$ is nondegenerate. On $\widehat{\mathcal{H}}$ we consider the norm $\|A^{1/2} \cdot\|$ and let \mathcal{K}_A be the completion of

$(\widehat{\mathcal{H}}, \|A^{1/2} \cdot \|)$ to a Hilbert space. In other words, the strong topology on the Hilbert space \mathcal{K}_A is induced by the norm $\|A^{1/2} \cdot \|$. Define $\Pi_A: \mathcal{H} \rightarrow \mathcal{K}_A$ as the orthogonal projection $P_{\mathcal{H} \ominus \ker A}$ composed with the embedding of $\mathcal{H} \ominus \ker A$ into \mathcal{K}_A . It is easy to prove that $\Pi_A^* \Pi_A = A$. Since Π has dense range, (\mathcal{K}_A, Π_A) is a Hilbert space induced by A .

Let now \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and $A \in \mathcal{L}(\mathcal{H}_1)$, $A \geq 0$, and $B \in \mathcal{L}(\mathcal{H}_2)$, $B \geq 0$. Also, let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be given and consider the induced Hilbert spaces (\mathcal{K}_A, Π_A) and (\mathcal{K}_B, Π_B) . We say that the operator T induces an operator $\tilde{T} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ if $\tilde{T} \Pi_A = \Pi_B T$. Equivalently, $T \ker A \subseteq \ker B$ and denoting by \widehat{T} the corresponding quotient operator in $\mathcal{L}(\mathcal{H}_1 \ominus \ker A, \mathcal{H}_2 \ominus \ker B)$ the operator \widehat{T} is bounded with respect to the norms $\|A^{1/2} \cdot \|$ and, respectively, $\|B^{1/2} \cdot \|$. The operator \tilde{T} is then the extension by continuity of the operator \widehat{T} and hence it is uniquely determined by T .

We now recall a result originally due to M.G. Kreĭn [13] and obtained independently by W.T. Reid [18], P.D. Lax [14], and J. Dieudonné [6]. An even more general indefinite variant was obtained by A. Dijksma, H. Langer, and H. de Snoo [7], see also [4]. The proof of the statement below follows by using the same iterative approach used in the original proof of M.G. Kreĭn and the others.

LEMMA 2.1 *Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces, $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$ be positive operators and let $T_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be operators such that*

$$\langle T_1 x, y \rangle_B = \langle x, T_2 y \rangle_A, \quad x \in \mathcal{H}_1, \quad y \in \mathcal{H}_2,$$

or, equivalently, $BT_1 = T_2^* A$. Then

$$\|B^{1/2} T_1 x\| \leq \|T_2 T_1\|^{1/2} \|A^{1/2} x\|, \quad x \in \mathcal{H}_1, \tag{2.1}$$

and similarly

$$\|A^{1/2} T_2 y\| \leq \|T_1 T_2\|^{1/2} \|B^{1/2} y\|, \quad y \in \mathcal{H}_2,$$

and hence, the operators T_1 and T_2 induce uniquely determined operators $\tilde{T}_1 \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ and, respectively, $\tilde{T}_2 \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$, such that

$$\langle \tilde{T}_1 x, y \rangle = \langle x, \tilde{T}_2 y \rangle, \quad x \in \mathcal{K}_A, \quad y \in \mathcal{K}_B.$$

Moreover, the norms of \tilde{T}_1 and \tilde{T}_2 are bounded by $\|T_2 T_1\|^{1/2}$ and $\|T_1 T_2\|^{1/2}$ respectively.

Let $A \in \mathcal{L}(\mathcal{H})$ be a positive operator. Just from the definition it is easy to prove that any two Hilbert spaces (\mathcal{K}_i, Π_i) induced by A are *unitarily equivalent*, that is, there exists a (uniquely determined) unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Phi \Pi_1 = \Pi_2$. We will describe in the following two other representations of Hilbert spaces induced by a given bounded positive operator A . Of course, they will be unitarily equivalent with the induced Hilbert space (\mathcal{K}_A, Π_A) , but each one has some gain, as well as some limitations.

EXAMPLE 2.2 Given $A \in \mathcal{L}(\mathcal{H})$ positive, we consider $\mathcal{H}_A = \mathcal{H} \ominus \ker A$ with the scalar product induced from the scalar product of \mathcal{H} , and let $\pi_A \in \mathcal{L}(\mathcal{H}, \mathcal{H}_A)$ be $\pi_A = A^{1/2}$. Then (\mathcal{H}_A, π_A) is a Hilbert space induced by A . Moreover, it is easy to prove that the operator Φ defined by

$$\mathcal{K}_A \supset \mathcal{H} \ominus \ker A \ni x \mapsto \Phi x = A^{1/2} x \in \mathcal{H}_A$$

can be extended uniquely to a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_A, \mathcal{H}_A)$ such that $\Phi \Pi_A = \pi_A$. ■

EXAMPLE 2.3 Fix again $A \in \mathcal{L}(\mathcal{H})$ positive and define $\mathcal{B}_A = \mathcal{R}(A^{1/2})$ with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{B}_A}$ defined by

$$\langle A^{1/2}x, A^{1/2}y \rangle_{\mathcal{B}_A} = \langle x, y \rangle, \quad x, y \in \mathcal{H}.$$

Then $(\mathcal{B}_A, \langle \cdot, \cdot \rangle_{\mathcal{B}_A})$ is a Hilbert space. To see this, just note that we have made the operator $\Psi = A^{1/2}: \mathcal{H}_A \rightarrow \mathcal{B}_A$ unitary. Define the operator $\Pi_{\mathcal{B}_A}: \mathcal{H} \rightarrow \mathcal{B}_A$ by $\Pi_{\mathcal{B}_A}x = Ax, x \in \mathcal{H}$. Then $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ is a Hilbert space induced by A and, in addition, the unitary equivalence of \mathcal{H}_A and \mathcal{B}_A is given by $\Psi = A^{1/2} \in \mathcal{L}(\mathcal{H}_A, \mathcal{B}_A)$ such that $\Psi\pi_A = \Pi_{\mathcal{B}_A}$. ■

The Hilbert space \mathcal{B}_A as in Example 2.3 can be characterized in yet another way. Let \mathcal{K} be a Hilbert space continuously embedded in \mathcal{H} and $\iota: \mathcal{K} \hookrightarrow \mathcal{H}$ be the inclusion. Then $A = \iota\iota^* \in \mathcal{L}(\mathcal{H})$ is positive and (\mathcal{K}, ι) is a Hilbert space induced by A . Conversely, it is easy to see from Example 2.3 that the Hilbert space \mathcal{B}_A is continuously embedded in \mathcal{H} . These spaces were intensively used by L. de Branges [2], [3], and in a more general formulation they were studied by L. Schwartz [19]. As operator ranges they were studied by P.A. Fillmore and J.P. Williams [9] and it can be shown, cf. [19], that they are a particular type of reproducing kernel Hilbert spaces, e.g. see Aronszajn [1].

3. Proof of Theorem 1.1

Proof: Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the scalar product and the corresponding norm on the Hilbert space \mathcal{U} . We consider the Hilbert space $\ell_{\mathcal{U}}^2$, of square summable sequences with entries in \mathcal{U} , endowed with scalar product also denoted by $\langle \cdot, \cdot \rangle$

$$\langle f, g \rangle = \sum_{k \geq 0} \langle f_k, g_k \rangle, \quad f = (f_k)_{k \geq 0}, \quad g = (g_k)_{k \geq 0} \in \ell_{\mathcal{U}}^2.$$

Let the Hilbert space $\mathcal{H} = \ell_{\mathcal{U}}^2$. According to our assumption, let $H: \ell_{\mathcal{U}}^2 \rightarrow \ell_{\mathcal{Y}}^2$ denote the bounded operator defined by the Hankel matrix with operator entries H as in (1.4). We consider (A_0, B_0, C_0, D_0) the right shift realization of G , that is,

$$G(z) = D_0 + C_0(zI - A_0)^{-1}B_0, \quad |z| > 1, \tag{3.1}$$

where the operator $A_0: \mathcal{H} \rightarrow \mathcal{H}$ is the right shift

$$(A_0f)_k = \begin{cases} f_{k-1}, & k \geq 1, \\ 0, & k = 0, \end{cases} \quad f = (f_k)_{k \geq 0} \in \mathcal{H} = \ell_{\mathcal{U}}^2, \tag{3.2}$$

the operator $B_0: \mathcal{U} \rightarrow \mathcal{H} = \ell_{\mathcal{U}}^2$ is defined by

$$B_0 = [I \ 0 \ \dots \ 0 \ \dots]^t, \tag{3.3}$$

where t denotes the matrix transpose, the operator $C_0: \mathcal{H} \rightarrow \mathcal{Y}$ is defined by

$$C_0 = [S_0 \ S_1 \ \dots \ S_k \ \dots], \tag{3.4}$$

and the external operator is $D_0 = G(\infty)$. Note that the operator C_0 is bounded due to the assumption on the boundedness of the Hankel operator H . Taking into account the definition of the shift realization as in (3.2), (3.3), and (3.4) it follows that the observability operator O_0 of the system (A_0, B_0, C_0, D_0) has the matrix representation

$$O_0 = \begin{bmatrix} C_0 & C_0 A_0 & C_0 A_0^2 & \dots \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_2 & \dots & S_k & \dots \\ S_1 & S_2 & S_3 & \dots & S_{k+1} & \dots \\ S_2 & S_3 & \dots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ \vdots & & & & & \end{bmatrix} = H. \tag{3.5}$$

Similarly, let R_0 be the reachability operator of the linear system (A_0, B_0, C_0, D_0) . From (3.2), (3.3), and (3.4) it follows that the matrix of R_0 is

$$R_0 = \begin{bmatrix} B_0 & A_0 B_0 & A_0^2 B_0 & \dots \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 & \dots \\ 0 & I & 0 & \dots & 0 & \dots \\ 0 & 0 & I & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \dots & I & \dots \\ \vdots & & & & & \end{bmatrix}. \tag{3.6}$$

Therefore, the reachability operator coincides with the operator of identification of $\ell_{\mathcal{U}}^2$ with \mathcal{H} .

Consider the modulus $|H| = (H^*H)^{1/2}$ of the Hankel operator H and let $(\mathcal{K}_{|H|}, \Pi_{|H|})$ be the induced Hilbert space. The main construction of the par-balanced realization lies in the ‘lifting’ of the system (A_0, B_0, C_0, D_0) with state space \mathcal{H} to the ‘induced’ system (A, B, C, D) with state space $\mathcal{K}_{|H|}$. To do this first consider the operator $B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and define $\tilde{B}_0 := |H|B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Hence, with $I_{\mathcal{U}}$ the identity operator on \mathcal{U} , we have that

$$B_0 I_{\mathcal{U}} = |H|B_0.$$

Therefore, by Lemma 2.1 B_0 induces a unique operator $B := \tilde{B}_0 \in \mathcal{L}(\mathcal{U}, \mathcal{K}_{|H|})$. Moreover $B = \Pi_{|H|}B_0$.

To deal with $C_0 \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, let $H = V|H|$ be the polar decomposition of the Hankel operator H and let

$$\tilde{C}_0 := \begin{bmatrix} I & 0 & \dots & 0 \dots \end{bmatrix} \in \mathcal{L}(\ell_{\mathcal{Y}}^2, \mathcal{Y}).$$

Then we have that

$$I_{\mathcal{Y}}C_0 = \tilde{C}_0 H = (\tilde{C}_0 V)|H|.$$

Again by Lemma 2.1 C_0 induces a unique operator $C := \tilde{C}_0 \in \mathcal{L}(\mathcal{K}_{|H|}, \mathcal{Y})$ and $C\Pi_{|H|} = C_0$.

The central part of the proof is to show that $A_0 \in \mathcal{L}(\mathcal{H})$ induces a unique contractive operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. Indeed, considering A_0^* the left shift on $\ell_{\mathcal{Y}}^2$ then we have that

$$HA_0 = A_0^*H. \tag{3.7}$$

To see this, if $f = (f_k)_{k \geq 0}$ is an arbitrary sequence in \mathcal{H} and $g = (g_k)_{k \geq 0}$ is an arbitrary sequence in $l^2_{\mathcal{Y}}$ then

$$\begin{aligned} \langle HA_0f, g \rangle &= \sum_{j,k \geq 0} \langle S_{j+k}(A_0f)_j, g_k \rangle = \sum_{j \geq 1, k \geq 0} \langle S_{j+k}f_{j-1}, g_k \rangle \\ &= \sum_{j,k \geq 0} \langle S_{j+k+1}f_j, g_k \rangle = \langle A_0^*Hf, g \rangle. \end{aligned} \tag{3.8}$$

This proves (3.7). Passing to the adjoints in (3.7) we get $A_0^*H^* = H^*A_0$, and hence

$$H^*H - A_0^*H^*HA_0 = H^*H - H^*A_0A_0^*H = H^*(I - A_0A_0^*)H \geq 0,$$

where we used that $A_0A_0^* \leq I$ since A_0 is contractive. Since $H^*H = |H|^2$ this can be rewritten as

$$A_0^*|H|^2A_0 \leq |H|^2. \tag{3.9}$$

Therefore, there exists a contraction $Z \in \mathcal{L}(\mathcal{H})$ such that

$$|H|A_0 = Z^*|H|. \tag{3.10}$$

Applying Lemma 2.1 it follows that A_0 induces a unique operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. In addition,

$$\||H|^{1/2}A_0x\| \leq \|ZA_0\|^{1/2} \||H|^{1/2}x\|, \quad x \in \mathcal{H}. \tag{3.11}$$

Since the shift A_0 and the operator Z are contractions, from (3.11) we also get by Lemma 2.1 that $A \in \mathcal{L}(\mathcal{K}_{|H|})$ is contractive.

Having constructed the discrete-time system (A, B, C, D) , $D := D_0$, with state-space $\mathcal{K}_{|H|}$ and contractive A , we now need to show that this system has the required properties. Let R be its reachability operator. Note that for $n \geq 0$

$$\Pi_{|H|}A_0^nB_0 = A^n\Pi_{|H|}B_0 = A^nB.$$

Hence $\Pi_{|H|}R_0u = Ru$ for each finite sequence $u \in l^2_{\mathcal{U}}$. Therefore $R \in \mathcal{L}(l^2_{\mathcal{U}}, \mathcal{K}_{|H|})$ and by continuity

$$\Pi_{|H|}R_0 = R.$$

Let O be the observability operator of (A, B, C, D) . Since for $n \geq 0$

$$CA^n\Pi_{|H|} = C\Pi_{|H|}A_0^n = C_0A_0^n$$

we have for $x \in \mathcal{H}$ that $O\Pi_{|H|x} = O_0x$. This implies that $O \in \mathcal{L}(\mathcal{K}_{|H|}, l^2_{\mathcal{Y}})$ and that

$$O\Pi_{|H|} = O_0.$$

Note that we therefore also have for the Hankel operator H

$$H = O_0R_0 = O\Pi_{|H|R_0} = OR.$$

This shows that the system (A, B, C, D) realizes the same transfer function G as does (A_0, B_0, C_0, D_0) .

It remains to show that the system (A, B, C, D) is reachable, observable and par-balanced. We have that

$$RR^* = \Pi_{|H|}R_0R_0^*\Pi_{|H|}^* = \Pi_{|H|}\Pi_{|H|}^*$$

since $R_0 = I_{\mathcal{U}}$. Let $x_1, y_1 \in \mathcal{H}$ and let $x = \Pi_{|H|x_1}$, $y = \Pi_{|H|y_1}$. Then

$$\begin{aligned} \langle y, O^*Ox \rangle_{\mathcal{K}_{|H|}} &= \langle \Pi_{|H|y_1}, O^*O\Pi_{|H|x_1} \rangle_{\mathcal{K}_{|H|}} \\ &= \langle y_1, \Pi_{|H|}^*O^*O\Pi_{|H|x_1} \rangle_{\mathcal{H}} = \langle y_1, O_0^*O_0x_1 \rangle_{\mathcal{H}} = \langle y_1, |H|^2x_1 \rangle_{\mathcal{H}} \\ &= \langle y_1, \Pi_{|H|}^*\Pi_{|H|}\Pi_{|H|}^*\Pi_{|H|x_1} \rangle_{\mathcal{H}} = \langle y, \Pi_{|H|}\Pi_{|H|}^*x \rangle_{\mathcal{K}_{|H|}}. \end{aligned}$$

Here we have used that $O_0 = H$ and that $|H| = \Pi_{|H|}^*\Pi_{|H|}$. Hence on the dense subset $\Pi_{|H|}\mathcal{H}$ of $\mathcal{K}_{|H|}$ we have that

$$O^*O = \Pi_{|H|}\Pi_{|H|}^*.$$

By continuity, this identity also holds on $\mathcal{K}_{|H|}$. Therefore the system is par-balanced with

$$RR^* = O^*O = \Pi_{|H|}\Pi_{|H|}^*.$$

Since $\Pi_{|H|}$ has dense range in $\mathcal{K}_{|H|}$, this also implies that R has dense range. Therefore $\Pi_{|H|}^*$ has zero kernel and hence O is injective. Thus (A, B, C, D) is reachable and observable. ■

REMARK 3.1 As pointed out in the introduction, our proof has a number of similarities with the approach taken by N.J. Young [21], e.g. we use the shift realization, etc. The main difference between the two proofs refers to the construction of the state space: N.J. Young uses the restricted shift realization and then a renorming while we construct the state space at once and then use Lemma 2.1. ■

REMARK 3.2 The fact that the system (A_0, B_0, C_0, D_0) can be lifted to the desired par-balanced realization can be also proved following an idea in Lemma 3.2 in Megretskii, Treil and Peller [15] which uses the Heinz’s Theorem. This can be done as follows:

We consider the shift realization (A_0, B_0, C_0, D_0) as in the previous proof, let $|H| = (H^*H)^{1/2}$ be the modulus of the Hankel operator H and let $(\mathcal{K}_{|H|}, \Pi_{|H|})$ be the induced Hilbert space. We now have to perform the ‘lifting’ of the system (A_0, B_0, C_0, D_0) with state space \mathcal{H} to the ‘induced’ system (A, B, C, D) with state space $\mathcal{K}_{|H|}$. For the operators $B, C,$ and D this is very similar with what is done in the first proof, with the difference that we do not apply Lemma 2.1. For example, consider the operator $B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and note that

$$\||H|^{1/2}B_0u\| \leq \|H^{1/2}\| \|u\|, \quad u \in \mathcal{U},$$

and hence B_0 can be lifted to a unique operator $B := \tilde{B}_0 \in \mathcal{L}(\mathcal{U}, \mathcal{K}_{|H|})$. Moreover $B = \Pi_{|H|}B_0$.

For the operator $C_0 \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, let $H = V|H|$ be the polar decomposition of the Hankel operator H and note that $C_0 = P_{\mathcal{Y}}H$, where \mathcal{Y} is naturally identified with the subspace $\mathcal{Y} \oplus 0 \oplus \dots \oplus 0 \oplus \dots$ of $\ell_{\mathcal{Y}}^2$. Therefore

$$\|C_0h\| = \|P_{\mathcal{Y}}Hh\| = \|P_{\mathcal{Y}}V|H|h\| \leq \||H|^{1/2}\| \||H|^{1/2}h\|, \quad h \in \mathcal{H} = \ell_{\mathcal{U}}^2,$$

and hence C_0 induces a unique operator $C := \tilde{C}_0 \in \mathcal{L}(\mathcal{K}_{|H|}, \mathcal{Y})$ and $C\Pi_{|H|} = C_0$.

As in the proof, the main concern is to show that $A_0 \in \mathcal{L}(\mathcal{H})$ induces a unique contractive operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. To see this, we first obtain the inequality (3.9) exactly as in the first proof. As $A_0 A_0^* \leq I$ we have from (3.9) that

$$(A_0^*|H|A_0)^2 \leq A_0^*|H|^2 A_0 \leq |H|^2.$$

By Loewner’s inequality (this can be obtained also as a consequence of the Heinz’s Theorem, see e.g. Theorem 9.4 in [20]) which expresses that the square root is operator monotonic, and in conjunction with the uniqueness of the square root, we therefore have that

$$A_0^*|H|A_0 \leq |H|.$$

But this immediately implies the claim since for $x \in \mathcal{K}_{|H|}$

$$\|A_0 x\|_{\mathcal{K}_{|H|}} = \||H|^{1/2} A_0 x\|_{\mathcal{H}} \leq \||H|^{1/2} x\|_{\mathcal{H}} = \|x\|_{\mathcal{K}_{|H|}}.$$

Moreover, $A\Pi_{|H|} = \Pi_{|H|}A_0$.

The fact that the system (A, B, C, D) is a par-balanced realization of the transfer function G follows exactly in the same way as in the proof. ■

REMARK 3.3 The fact that one can use the Loewner inequality instead of the Lemma Kreĭn-Reid-Lax-Dieudonné comes as no surprise since it can be proved that these tools are actually in the same circle of ideas, e.g. see [8]. ■

REMARK 3.4 The par-balanced realizations are unique, modulo a unitary equivalence, cf. [21] and the next section. In our proof of Theorem 1.1 we choose $\mathcal{K}_{|H|}$ for the state space. The state space transformation in [21] corresponds to the state space $\mathcal{H}_{|H|}$, see Example 2.2. But, as Example 2.3 shows, a third choice is possible, namely the state space $\mathcal{B}_{|H|}$. This has the advantage of working on the range of some operator, with no closure needed, with the cost, of course, of a more involved topology. ■

4. Uniqueness of Par-Balanced Realizations

In [21] N.J. Young proved that par-balanced realizations are unique, modulo unitary equivalence. His proof relied on relating this uniqueness question to the problem of the uniqueness of restricted shift realization and the closely related output-normal realizations. Here we present an alternative proof of his result.

In the previous sections of this paper we assumed that the main operator A of a system is a contraction. To allow for a somewhat greater generality of the following theorem we drop this assumption. All other definitions are as before.

Two systems (A_i, B_i, C_i, D_i) , with state spaces \mathcal{H}_i , $i = 1, 2$, are related by a *state space transformation* if there exists a bounded operator $T: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that

$$A_1 T = T A_2, \quad B_1 = T B_2, \quad C_1 T = C_2, \quad D_1 = D_2. \tag{4.1}$$

The two systems are called *similar* if there exists a *similarity*, that is, a bounded invertible state space transformation $T: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that (4.1) holds. If the similarity T is unitary the two systems are called *unitary equivalent*.

THEOREM 4.1 *Let (A_i, B_i, C_i, D_i) be discrete-time observable and reachable linear systems, with state spaces \mathcal{H}_i , bounded observability operators O_i and bounded reachability operators R_i , such that they are par-balanced, that is, $O_i^*O_i = R_iR_i^*$, $i = 1, 2$. If both systems are realizations of the same transfer function then the two systems are unitary equivalent.*

Moreover, this state-space transformation is unique, that is, if $U \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is the unitary state-space transformation between the two systems and $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is a state-space transformation between the two systems, then $T = U$.

Proof. Let H be the Hankel operator, associated as in (1.4) with the transfer function G , which is realized by the systems (A_i, B_i, C_i, D_i) , $i = 1, 2$. By assumption we have

$$H = O_1R_1 = O_2R_2,$$

and therefore, using the par-balanced condition we get

$$R_1^*R_1R_1^*R_1 = R_1^*O_1^*O_1R_1 = H^*H = R_2^*O_2^*O_2R_2 = R_2^*R_2R_2^*R_2.$$

Since $R_1^*R_1$ and $R_2^*R_2$ are positive, using the uniqueness of the square root it follows that

$$R_1^*R_1 = |H| = R_2^*R_2.$$

Therefore, there exists a uniquely determined isometric operator $U: \text{cl } \mathcal{R}(R_2) \rightarrow \text{cl } \mathcal{R}(R_1)$ such that

$$R_1 = UR_2. \tag{4.2}$$

The reachability of the systems and hence the fact that the reachability operators R_i have dense ranges imply that $U: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is unitary. Since

$$R_i = [B_i \ A_iB_i \ A_i^2B_i \ \dots \ A_i^k B_i \ \dots], \quad i = 1, 2,$$

this implies, by considering the first component in (4.2), that

$$B_1 = UB_2.$$

Note also that

$$A_iR_i = [A_iB_i \ A_i^2B_i \ A_i^3B_i \ \dots \ A_i^{k+1}B_i \ \dots], \quad i = 1, 2,$$

and hence

$$A_1UR_2 = A_1R_1 = UA_2R_2.$$

Since R_2 has dense range and the operators are bounded, from here we get

$$A_1U = UA_2.$$

Further, since

$$O_1UR_2 = O_1R_1 = H = O_2R_2,$$

in a similar way we get

$$O_1U = O_2.$$

As before, from here we obtain

$$C_1U = C_2.$$

That $D_1 = D_2$ follows from $D_1 = G(\infty) = D_2$. Hence the two systems are unitary equivalent.

To show the uniqueness of the state space transformation U assume that $T: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is a similarity state space transformation between the two systems. Note that

$$UR_2 = R_1 = \begin{bmatrix} TB_2 & A_1TB_2 & A_1^2TB_2 & \dots \end{bmatrix} = \begin{bmatrix} TB_2 & TA_2B_2 & TA_2^2B_2 & \dots \end{bmatrix} = TR_2.$$

Since R_2 has dense range and the operators T and U are bounded we get from here that $T = U$. ■

A consequence of this result in combination with the realization result is that the main operator of any par-balanced realization of a transfer function analytic outside the unit disk and with bounded Hankel operator is a contraction.

5. Real Symmetric Transfer Functions

A system (A, B, C, D) , with state space \mathcal{H} , is called *completely J -symmetric* if there exists a *symmetry* $J \in \mathcal{L}(\mathcal{H})$, that is, J is unitary and selfadjoint, such that (A, B, C, D) and (A^*, B^*, C^*, D^*) are similar with similarity J , that is, $JA = A^*J$, $B = JC^*$, and $D = D^*$. In [11] we gave a direct proof of the existence of completely J -symmetric par-balanced realizations of real symmetric transfer functions with bounded Hankel operator. Here we give a proof of a slightly stronger result than Theorem 6.2 in [11].

THEOREM 5.1 *Assume the assumptions and the notation of Theorem 1.1 and, in addition, that the transfer function G is real symmetric, that is, $G(\bar{z}) = G(z)^*$, $|z| > 1$. Let (A, B, C, D) be an observable and reachable par-balanced realization of G with state space \mathcal{H} . Then there exists a unique symmetry J on \mathcal{H} such that (A, B, C, D) is completely J -symmetric. In addition, if O and R denote the observability and, respectively, the reachability operator of the system then $R = JO^*$ and hence, the Hankel operator H of the system admits the factorizations $H = OJO^* = R^*JR$.*

*If $\Sigma := R^*R = OO^*$, then J commutes with Σ , i.e. $J\Sigma = \Sigma J$.*

Proof. Let (A, B, C, D) be an observable and reachable par-balanced realization of G . Since by the real symmetry of G the dual system (A^*, C^*, B^*, D^*) is another realization of G

$$B^*(zI - A^*)^{-1}C^* + D^* = G(\bar{z})^* = G(z), \quad |z| > 1.$$

The standard duality results for linear systems imply that the dual system is also reachable, observable, and par-balanced.

By Theorem 4.1 there exists a unique unitary state space transformation $U \in \mathcal{L}(\mathcal{H})$ such that

$$(A, B, C, D) = (UA^*U^*, UC^*, B^*U^*, D^*).$$

Applying a state space transformation with U^* to both these systems we obtain

$$(U^*AU, U^*B, CU, D) = (A^*, C^*, B^*, D^*).$$

Taking the dual system we have

$$(U^*A^*U, U^*C, B^*U, D^*) = (A, B, C, D),$$

which shows that U^* is also a similarity for the systems (A, B, C, D) and (A^*, C^*, B^*, D^*) . By the uniqueness of the similarity as in Theorem 4.1 we get $U = U^*$, that is, U is a symmetry and the system (A, B, C, D) is completely J -symmetric with $J = U$. The uniqueness of J follows again from Theorem 4.1 and the fact that both (A, B, C, D) and its dual system are reachable, observable and par-balanced realizations of G .

Clearly we have $R = JO^*$ and hence the Hankel operator H of the system admits the factorizations $H = OJO^* = R^*JR$. That J commutes with Σ follows since $\Sigma = RR^* = JO^*OJ^* = J\Sigma J^*$ and therefore $\Sigma J = J\Sigma$. ■

6. Appendix

Our proof of Theorem 1.1 actually fills a gap in the proof provided in [21]. The proof in [21] relies on Lemma 1, p. 461 in [21]. The Lemma is, however, only valid if, in addition to the stated assumptions, the operator M is also assumed to be essentially selfadjoint. More precisely, the proof of this Lemma, as provided in [21], proves the following statement.

LEMMA 6.1 *Let M be a positive and essentially selfadjoint operator on a dense linear manifold \mathcal{D} of a separable Hilbert space \mathcal{H} , let $P \in \mathcal{L}(\mathcal{H})$ have zero kernel and let PMP^{-1} be a contraction on $P\mathcal{D}$. Then M itself is a contraction.*

The additional assumption on M implies that the Lemma can no longer be used to give a complete proof of Theorem 1.

In this appendix we reproduce an argument communicated to us by N.J. Young [22] after receiving a preprint of this paper. In this argument the use of Lemma 1 in [21] is circumvented. We use the same notation as in [21]. Briefly, the idea is to use the closure of the operator \bar{A} instead of \bar{A} . To prove that this is possible, let T be the closure of the operator \bar{A} as defined in [21], p. 460. T exists since \bar{A}^* has domain $\supset W^{1/4}\mathcal{H}$ which is dense. Then it is straightforward to show

$$W^{1/4}T^*f = Z^*W^{1/4}f, \quad f \in \mathcal{D}(T^*).$$

Furthermore, extending the relation $W^{1/4}\bar{A} = AW^{1/4}$ using closure, we obtain $W^{1/4}Tf = AW^{1/4}f$ for all $f \in \mathcal{D}(T)$. For arbitrary $h \in \mathcal{D}(T^*T)$ we have $Th \in \mathcal{D}(T^*)$, and hence

$$W^{1/4}T^*Th = Z^*AW^{1/4}h.$$

Thus $W^{1/4}(T^*T)W^{-1/4}$ is a contraction on $W^{1/4}\mathcal{D}(T^*T)$. Since T is closed, T^*T is positive selfadjoint and hence the Lemma 6.1 applies to conclude that T is a contraction.

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