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A NOTE ON THE EXISTENCE, UNIQUENESS AND SYMMETRY OF PAR-BALANCED REALIZATIONS

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We give a proof of the realization theorem of N.J. Young which states that analytic functions which are symbols of bounded Hankel operators admit par-balanced realizations. The main tool used in this proof is the induced Hilbert spaces and a lifting lemma of Kreĭn-Reid-Lax-Dieudonné. Alternatively one can use the Loewner inequality. A short proof of the uniqueness of par-balanced realizations is included. As an application, it is proved that par-balanced realizations of real symmetric transfer functions are J-self-adjoint.

1. Introduction

Consider a (discrete time, time invariant) *linear system* (A, B, C, D) with contractive *main operator* $A \in \mathcal{L}(\mathcal{H})$, *input operator* $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, *output operator* $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and *external operator* $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, where the *state space* $\mathcal{H}$, *input space* $\mathcal{U}$, and *output space* $\mathcal{Y}$ are Hilbert spaces. With every linear system (A, B, C, D) there is associated its *transfer function* $G: \rho(A) \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ as follows

$$G(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \rho(A). \tag{1.1}$$

Since the main operator A is assumed contractive, i.e. $\|A\| \leq 1$, the transfer function is defined and analytic for all $|\lambda| > 1$.

Following the general theory, e.g. see [10], the system (A, B, C, D) is called *observable* if for any $h \in \mathcal{H}$ we have $\sum_{k\geq 0} \|CA^k h\|^2 < \infty$ and the *observability* operator O defined by

$$Oh = (CA^k h)_{k\geq 0}, \quad h \in \mathcal{H},$$

is bounded and injective.

The system is called *reachable* if for any $(u_k)_{k\leq 0} \in \ell^2_{\mathcal{U}}$ the series $\sum_{k\leq 0} A^k B u_k$ converges strongly in $\mathcal{H}$ and the *reachability* operator R defined by

$$R((u_k)_{k\geq 0}) = \sum_{k\geq 0} A^k B u_k, \quad (u_k)_{k\leq 0} \in \ell^2_{\mathcal{U}},$$

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is bounded and has dense range.

Whenever the operators O and R are everywhere defined, one can introduce the operator $H = OR$. The operator H is called the *Hankel operator* associated to the system (A, B, C, D). From the definition of the operators O and R, the operator H has the following Hankel block matrix representation

$$H \sim \left(CA^{i+j}B\right)_{i,j\geq 0}, \tag{1.2}$$

more precisely, for all $u = (u_k)_{k\geq 0} \in \ell^2_{\mathcal{U}}$, $y = (y_k)_{k\geq 0} \in \ell^2_{\mathcal{Y}}$, we have

$$\langle Hu, y\rangle \quad \sum_{j\geq 0}\sum_{i\geq 0}\langle CA^{i+j}Bu_j, y_i\rangle. \tag{1.3}$$

If the system (A, B, C, D) has bounded and everywhere defined observability operator O, then one can define the *observability gramian* O^*O. Similarly, if the system has bounded reachability operator R then one can define the *reachability gramian* RR^*. Following N.J. Young [21], the system (A, B, C, D) is called *par-balanced* if the observability operator O and the reachability operator R are bounded everywhere defined operators and the observability gramian coincides with the reachability gramian, $O^*O = RR^*$. This is a generalization of the notion of balanced linear system introduced by B.C. Moore [16] for finite dimensional systems.

The above presentation corresponds to the *internal*, or, equivalently, the *state space* representation of a system. Sometimes a linear system is given only in its *external representation*, that is, given $\mathcal{U}$ and $\mathcal{Y}$ Hilbert spaces we consider $G\colon \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ an operator valued function which is analytic everywhere on its domain of definition and at infinity. One can define another operator valued analytic function $g\colon \mathbb{D} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ by

$$g(z) = \frac{1}{z}\Big(G\big(\frac{1}{z}\big) - G(\infty)\Big), \quad |z| < 1.$$

Then g has the Taylor expansion on $\mathbb{D}$

$$g(z) = \sum_{k\geq 0} S_k z^k, \quad |z| < 1,$$

where $S_k \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, $k \geq 0$. Associated with the function G one can consider the block-operator Hankel matrix

$$H = \begin{bmatrix} S_0 & S_1 & S_2 & & S_k & \\ S_1 & S_2 & S_3 & & S_{k+1} & \\ S_2 & S_3 & \cdots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ & & & & & \end{bmatrix} \tag{1.4}$$

The *realization problem* asks for the determination of a system (A, B, C, D), with some state space $\mathcal{H}$, input space $\mathcal{U}$, and output space $\mathcal{Y}$ such that (1.1) holds. Observable and reachable realizations of transfer functions can be obtained by the so-called reduced shift realization, e.g. see [10]. There was a problem for some time to prove that balanced realizations for nonrational transfer functions exist, e.g. see [12]. The main result of the paper of N.J. Young [21] is:

THEOREM 1.1 *Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces and let $G: \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U},\mathcal{Y})$ be an operator valued function which is analytic on its domain and at infinity. If the Hankel block-operator matrix in (1.4) defines a bounded operator $H : \ell^2_{\mathcal{U}} \to \ell^2_{\mathcal{Y}}$ then there exists a realization (A, B, C, D) of G, corresponding to some Hilbert state space $\mathcal{H}$, which is observable, reachable and par-balanced.*

The approach to the proof of this theorem in [21] is to use the restricted shift realization and then to perform a state space transformation which yields the desired par-balanced realization. The most difficult part of the proof is to show that the new main operator is bounded and even a contraction.

The aim of this short note is to produce an alternate approach based on induced Hilbert spaces and a slightly generalized lifting Lemma of Kreĭn-Reid-Lax-Dieudonné. The approach with induced Hilbert spaces enables us to discuss the choice of the state space from a different perspective. We notice that a slightly different argument making use of the Loewner inequality, and inspired by the recent paper [15], can be used. Our interest in Theorem 1.1 is due to sign symmetric transfer functions that we considered in [11], where we used the induced Hilbert spaces and the Kreĭn-Reid-Lax-Dieudonné Lemma in the realization theory.

The material is organized as follows. In Section 2 we have a short discussion of induced Hilbert spaces (for the indefinite variant see [5]), the generalized version of the Kreĭn-Reid-Lax-Dieudonné Lemma (for an indefinite variant, see [7]) and two other representations of induced Hilbert spaces. Section 3 contains the proof of Theorem 1.1. In Section 4 we give a short proof of the uniqueness of par-balanced realizations, see Theorem 4.1. Finally, as an application, we prove that par-balanced realizations of real symmetric transfer functions are J-self-adjoint. An appendix containing a correction of N.J. Young to his original proof in [21], is also included.

The results discussed in this paper carry over to the continuous-time case using the bilinear transform method of [17].

2. Induced Hilbert Spaces

Let us consider a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and let A be a bounded positive operator on $\mathcal{H}$. A pair $(\mathcal{K}_A, \Pi_A)$ is called a *Hilbert space induced* by A if $\mathcal{K}_A$ is a Hilbert space and $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K}_A)$ has dense range and $\Pi_A^*\Pi_A = A$.

On $\mathcal{H}$ we consider the nonnegative inner product $\langle\cdot,\cdot\rangle_A$ defined by

$$\langle x, y\rangle_A = \langle Ax, y\rangle, \quad x, y \in \mathcal{H}.$$

Let $\widehat{\mathcal{H}} = \mathcal{H} \ominus \ker A$ and note that the restriction of the inner product $\langle\cdot,\cdot\rangle_A$ to $\widehat{\mathcal{H}}$ is nondegenerate. On $\widehat{\mathcal{H}}$ we consider the norm $\|A^{1/2}\cdot\|$ and let $\mathcal{K}_A$ be the completion of

$(\widehat{\mathcal{H}}, \|A^{1/2}\cdot\|)$ to a Hilbert space. In other words, the strong topology on the Hilbert space $\mathcal{K}_A$ is induced by the norm $\|A^{1/2}\cdot\|$. Define $\Pi_A\colon\mathcal{H}\to\mathcal{K}_A$ as the orthogonal projection $P_{\mathcal{H}\ominus\ker A}$ composed with the embedding of $\mathcal{H}\ominus\ker A$ into $\mathcal{K}_A$. It is easy to prove that $\Pi_A^*\Pi_A = A$. Since Π has dense range, $(\mathcal{K}_A, \Pi_A)$ is a Hilbert space induced by A.

Let now $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $A\in\mathcal{L}(\mathcal{H}_1)$, $A\geq 0$, and $B\in\mathcal{L}(\mathcal{H}_2)$, $B\geq 0$. Also, let $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ be given and consider the induced Hilbert spaces $(\mathcal{K}_A,\Pi_A)$ and $(\mathcal{K}_B,\Pi_B)$. We say that the operator T induces an operator $\tilde{T}\in\mathcal{L}(\mathcal{K}_A,\mathcal{K}_B)$ if $\tilde{T}\Pi_A = \Pi_B T$. Equivalently, $T\ker A\subseteq\ker B$ and denoting by $\hat{T}$ the corresponding quotient operator in $\mathcal{L}(\mathcal{H}_1\ominus\ker A, \mathcal{H}_2\ominus\ker B)$ the operator $\hat{T}$ is bounded with respect to the norms $\|A^{1/2}\cdot\|$ and, respectively, $\|B^{1/2}\cdot\|$. The operator $\tilde{T}$ is then the extension by continuity of the operator $\hat{T}$ and hence it is uniquely determined by T.

We now recall a result originally due to M.G. Kreĭn [13] and obtained independently by W.T. Reid [18], P.D. Lax [14], and J. Dieudonné [6]. An even more general indefinite variant was obtained by A. Dijksma, H. Langer, and H. de Snoo [7], see also [4]. The proof of the statement below follows by using the same iterative approach used in the original proof of M.G. Kreĭn and the others.

LEMMA 2.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $A\in\mathcal{L}(\mathcal{H}_1)$ and $B\in\mathcal{L}(\mathcal{H}_2)$ be positive operators and let $T_1\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and $T_2\in\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)$ be operators such that*

$$\langle T_1x, y\rangle_B = \langle x, T_2y\rangle_A,\quad x\in\mathcal{H}_1,\ y\in\mathcal{H}_2,$$

or, equivalently, $BT_1 = T_2^*A$. Then

$$\|B^{1/2}T_1x\| \leq \|T_2T_1\|^{1/2}\|A^{1/2}x\|,\quad x\in\mathcal{H}_1, \tag{2.1}$$

and similarly

$$\|A^{1/2}T_2y\| \leq \|T_1T_2\|^{1/2}\|B^{1/2}y\|,\quad y\in\mathcal{H}_2,$$

and hence, the operators T_1 and T_2 induce uniquely determined operators $\tilde{T}_1\in\mathcal{L}(\mathcal{K}_A,\mathcal{K}_B)$ and, respectively, $\tilde{T}_2\in\mathcal{L}(\mathcal{K}_B,\mathcal{K}_A)$, such that

$$\langle\tilde{T}_1x, y\rangle = \langle x, \tilde{T}_2y\rangle,\quad x\in\mathcal{K}_A,\ y\in\mathcal{K}_B.$$

Moreover, the norms of $\tilde{T}_1$ and $\tilde{T}_2$ are bounded by $\|T_2T_1\|^{1/2}$ and $\|T_1T_2\|^{1/2}$ respectively.

Let $A\in\mathcal{L}(\mathcal{H})$ be a positive operator. Just from the definition it is easy to prove that any two Hilbert spaces $(\mathcal{K}_i,\Pi_i)$ induced by A are *unitarily equivalent*, that is, there exists a (uniquely determined) unitary operator $\Phi\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ such that $\Phi\Pi_1 = \Pi_2$. We will describe in the following two other representations of Hilbert spaces induced by a given bounded positive operator A. Of course, they will be unitarily equivalent with the induced Hilbert space $(\mathcal{K}_A,\Pi_A)$, but each one has some gain, as well as some limitations.

EXAMPLE 2.2 Given $A\in\mathcal{L}(\mathcal{H})$ positive, we consider $\mathcal{H}_A = \mathcal{H}\ominus\ker A$ with the scalar product induced from the scalar product of $\mathcal{H}$, and let $\pi_A\in\mathcal{L}(\mathcal{H},\mathcal{H}_A)$ be $\pi_A = A^{1/2}$. Then $(\mathcal{H}_A,\pi_A)$ is a Hilbert space induced by A. Moreover, it is easy to prove that the operator Φ defined by

$$\mathcal{K}_A\supset\mathcal{H}\ominus\ker A\ni x\mapsto\Phi x = A^{1/2}x\in\mathcal{H}_A$$

can be extended uniquely to a unitary operator $\Phi\in\mathcal{L}(\mathcal{K}_A,\mathcal{H}_A)$ such that $\Phi\Pi_A = \pi_A$. ∎

EXAMPLE 2.3 Fix again $A \in \mathcal{L}(\mathcal{H})$ positive and define $\mathcal{B}_A = \mathcal{R}(A^{1/2})$ with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{B}_A}$ defined by

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathcal{B}_A} = \langle x,y\rangle, \quad x,y \in \mathcal{H}.$$

Then $(\mathcal{B}_A, \langle\cdot,\cdot\rangle_{\mathcal{B}_A})$ is a Hilbert space. To see this, just note that we have made the operator $\Psi = A^{1/2}: \mathcal{H}_A \to \mathcal{B}_A$ unitary. Define the operator $\Pi_{\mathcal{B}_A}: \mathcal{H} \to \mathcal{B}_A$ by $\Pi_{\mathcal{B}_A}x = Ax$, $x \in \mathcal{H}$. Then $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ is a Hilbert space induced by A and, in addition, the unitary equivalence of $\mathcal{H}_A$ and $\mathcal{B}_A$ is given by $\Psi = A^{1/2} \in \mathcal{L}(\mathcal{H}_A, \mathcal{B}_A)$ such that $\Psi\pi_A = \Pi_{\mathcal{B}_A}$. ∎

The Hilbert space $\mathcal{B}_A$ as in Example 2.3 can be characterized in yet another way. Let $\mathcal{K}$ be a Hilbert space continuously embedded in $\mathcal{H}$ and $\iota: \mathcal{K} \hookrightarrow \mathcal{H}$ be the inclusion. Then $A = \iota\iota^* \in \mathcal{L}(\mathcal{H})$ is positive and $(\mathcal{K}, \iota)$ is a Hilbert space induced by A. Conversely, it is easy to see from Example 2.3 that the Hilbert space $\mathcal{B}_A$ is continuously embedded in $\mathcal{H}$. These spaces were intensively used by L. de Branges [2], [3], and in a more general formulation they were studied by L. Schwartz [19]. As operator ranges they were studied by P.A. Fillmore and J.P. Williams [9] and it can be shown, cf. [19], that they are a particular type of reproducing kernel Hilbert spaces, e.g. see Aronszajn [1].

3. Proof of Theorem 1.1

Proof: Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product and the corresponding norm on the Hilbert space $\mathcal{U}$. We consider the Hilbert space $\ell^2_{\mathcal{U}}$, of square summable sequences with entries in $\mathcal{U}$, endowed with scalar product also denoted by $\langle\cdot,\cdot\rangle$

$$\langle f,g\rangle = \sum_{k\geq 0}\langle f_k, g_k\rangle, \quad f = (f_k)_{k\geq 0},\ g = (g_k)_{k\geq 0} \in \ell^2_{\mathcal{U}}.$$

Let the Hilbert space $\mathcal{H} = \ell^2_{\mathcal{U}}$. According to our assumption, let $H: \ell^2_{\mathcal{U}} \to \ell^2_{\mathcal{Y}}$ denote the bounded operator defined by the Hankel matrix with operator entries H as in (1.4). We consider (A_0, B_0, C_0, D_0) the right shift realization of G, that is,

$$G(z) = D_0 + C_0(zI - A_0)^{-1}B_0, \quad |z| > 1, \tag{3.1}$$

where the operator $A_0: \mathcal{H} \to \mathcal{H}$ is the right shift

$$(A_0 f)_k = \begin{cases} f_{k-1}, & k \geq 1, \\ 0, & k = 0, \end{cases} \quad f = (f_k)_{k\geq 0} \in \mathcal{H} = \ell^2_{\mathcal{U}}, \tag{3.2}$$

the operator $B_0: \mathcal{U} \to \mathcal{H} = \ell^2_{\mathcal{U}}$ is defined by

$$B_0 = \begin{bmatrix} I & 0 & \dots & 0 & \dots \end{bmatrix}^t, \tag{3.3}$$

where t denotes the matrix transpose, the operator $C_0: \mathcal{H} \to \mathcal{Y}$ is defined by

$$C_0 = \begin{bmatrix} S_0 & S_1 & \dots & S_k & \dots \end{bmatrix}, \tag{3.4}$$

and the external operator is $D_0 = G(\infty)$. Note that the operator C_0 is bounded due to the assumption on the boundedness of the Hankel operator H. Taking into account the definition of the shift realization as in (3.2), (3.3), and (3.4) it follows that the observability operator O_0 of the system (A_0, B_0, C_0, D_0) has the matrix representation

$$O_0 = \begin{bmatrix} C_0 & C_0A_0 & C_0A_0^2 \dots \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_2 & \dots & S_k & \\ S_1 & S_2 & S_3 & \dots & S_{k+1} & \\ S_2 & S_3 & \dots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ & & & & & \end{bmatrix} = H. \tag{3.5}$$

Similarly, let R_0 be the reachability operator of the linear system (A_0, B_0, C_0, D_0). From (3.2), (3.3), and (3.4) it follows that the matrix of R_0 is

$$R_0 = \begin{bmatrix} B_0 & A_0B_0 & A_0^2B_0 \dots \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & .\,. & 0 & \dots \\ 0 & I & 0 & .\,. & 0 & \dots \\ 0 & 0 & I & .\,. & 0 & \dots \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & .\,. & I & \dots \\ \vdots & & & & & \end{bmatrix} \tag{3.6}$$

Therefore, the reachability operator coincides with the operator of identification of $\ell^2_{\mathcal{U}}$ with $\mathcal{H}$.

Consider the modulus $|H| = (H^*H)^{1/2}$ of the Hankel operator H and let $(\mathcal{K}_{|H|}, \Pi_{|H|})$ be the induced Hilbert space. The main construction of the par-balanced realization lies in the 'lifting' of the system (A_0, B_0, C_0, D_0) with state space $\mathcal{H}$ to the 'induced' system (A, B, C, D) with state space $\mathcal{K}_{|H|}$. To do this first consider the operator $B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and define $\hat{B}_0 := |H|B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Hence, with $I_{\mathcal{U}}$ the identity operator on $\mathcal{U}$, we have that

$$B_0 I_{\mathcal{U}} = |H|B_0.$$

Therefore, by Lemma 2.1 B_0 induces a unique operator $B := \tilde{B}_0 \in \mathcal{L}(\mathcal{U}, \mathcal{K}_{|H|})$. Moreover $B = \Pi_{|H|}B_0$.

To deal with $C_0 \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, let $H = V|H|$ be the polar decomposition of the Hankel operator H and let

$$\hat{C}_0 := \begin{bmatrix} I & 0 & \dots & 0 \dots \end{bmatrix} \in \mathcal{L}(l^2_{\mathcal{Y}}, \mathcal{Y}).$$

Then we have that

$$I_{\mathcal{Y}} C_0 = \hat{C}_0 H = (\hat{C}_0 V)|H|.$$

Again by Lemma 2.1 C_0 induces a unique operator $C := \tilde{C}_0 \in \mathcal{L}(\mathcal{K}_{|H|}, \mathcal{Y})$ and $C\Pi_{|H|} = C_0$.

The central part of the proof is to show that $A_0 \in \mathcal{L}(\mathcal{H})$ induces a unique contractive operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. Indeed, considering A_0^* the left shift on $l^2_{\mathcal{Y}}$ then we have that

$$HA_0 = A_0^*H. \tag{3.7}$$

To scc this, if $f = (f_k)_{k\geq 0}$ is an arbitrary sequence in $\mathcal{H}$ and $g = (g_k)_{k\geq 0}$ is an arbitrary sequence in $l^2_{\mathcal{Y}}$ then

$$\langle HA_0 f, g\rangle = \sum_{j,k\geq 0} \langle S_{j+k}(A_0 f)_j, g_k\rangle \qquad \sum_{j\geq 1, k\geq 0} \langle S_{j+k} f_{j-1}, g_k\rangle$$

$$= \sum_{j,k\geq 0} \langle S_{j+k+1} f_j, g_k\rangle = \langle A_0^* H f, g\rangle.$$

This proves (3.7). Passing to the adjoints in (3.7) we get $A_0^* H^* \quad H^* A_0$, and hence

$$H^*H \quad A_0^* H^* H A_0 \quad H^*H \quad H^* A_0 A_0^* H = H^*(I \quad A_0 A_0^*) H \geq 0,$$

where we used that $A_0 A_0^* \leq I$ since A_0 is contractive. Since $H^*H = |H|^2$ this can be rewritten as

$$A_0^* |H|^2 A_0 \leq |H|^2.$$

Therefore, there exists a contraction $Z \in \mathcal{L}(\mathcal{H})$ such that

$$|H| A_0 = Z^* |H|. \tag{3.10}$$

Applying Lemma 2.1 it follows that A_0 induces a unique operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. In addition,

$$\| |H|^{1/2} A_0 x\| \leq \|Z A_0\|^{1/2} \, \| |H|^{1/2} x\|, \quad x \in \mathcal{H}. \tag{3.11}$$

Since the shift A_0 and the operator Z are contractions, from (3.11) we also get by Lemma 2.1 that $A \in \mathcal{L}(\mathcal{K}_{|H|})$ is contractive.

Having constructed the discrete-time system (A, B, C, D), $D := D_0$, with state-space $\mathcal{K}_{|H|}$ and contractive A, we now need to show that this system has the required properties. Let R be its reachability operator. Note that for $n \geq 0$

$$\Pi_{|H|} A_0^n B_0 = A^n \Pi_{|H|} B_0 \quad A^n B.$$

Hence $\Pi_{|H|} R_0 u \quad Ru$ for each finite sequence $u \in l^2_{\mathcal{U}}$. Therefore $R \in \mathcal{L}(l^2_{\mathcal{U}}, \mathcal{K}_{|H|})$ and by continuity

$$\Pi_{|H|} R_0 = R.$$

Let O be the observability operator of (A, B, C, D). Since for $n \geq 0$

$$C A^n \Pi_{|H|} = C \Pi_{|H|} A_0^n = C_0 A_0^n$$

we have for $x \in \mathcal{H}$ that $O \Pi_{|H|} x \quad O_0 x$. This implies that $O \in \mathcal{L}(\mathcal{K}_{|H|}, l^2_{\mathcal{Y}})$ and that

$$O \Pi_{|H|} \quad O_0.$$

Note that we therefore also have for the Hankel operator H

$$H \quad O_0 R_0 \quad O \Pi_{|H|} R_0 = OR.$$

This shows that the system (A, B, C, D) realizes the same transfer function G as does (A_0, B_0, C_0, D_0).

It remains to show that the system (A, B, C, D) is reachable, observable and par-balanced. We have that

$$RR^* = \Pi_{|H|} R_0 R_0^* \Pi_{|H|}^* = \Pi_{|H|} \Pi_{|H|}^*$$

since $R_0 = I_{\mathcal{U}}$. Let $x_1, y_1 \in \mathcal{H}$ and let $x = \Pi_{|H|} x_1$, $y = \Pi_{|H|} y_1$. Then

$$\begin{aligned}\langle y, O^* O x\rangle_{\mathcal{K}_{|H|}} &= \langle \Pi_{|H|} y_1, O^* O \Pi_{|H|} x_1\rangle_{\mathcal{K}_{|H|}} \\ &= \langle y_1, \Pi_{|H|}^* O^* O \Pi_{|H|} x_1\rangle_{\mathcal{H}} = \langle y_1, O_0^* O_0 x_1\rangle_{\mathcal{H}} = \langle y_1, |H|^2 x_1\rangle_{\mathcal{H}} \\ &= \langle y_1, \Pi_{|H|}^* \Pi_{|H|} \Pi_{|H|}^* \Pi_{|H|} x_1\rangle_{\mathcal{H}} = \langle y, \Pi_{|H|} \Pi_{|H|}^* x\rangle_{\mathcal{K}_{|H|}}.\end{aligned}$$

Here we have used that $O_0 = H$ and that $|H| = \Pi_{|H|}^* \Pi_{|H|}$. Hence on the dense subset $\Pi_{|H|}\mathcal{H}$ of $\mathcal{K}_{|H|}$ we have that

$$O^* O = \Pi_{|H|} \Pi_{|H|}^*.$$

By continuity, this identity also holds on $\mathcal{K}_{|H|}$. Therefore the system is par-balanced with

$$RR^* = O^* O = \Pi_{|H|} \Pi_{|H|}^*.$$

Since $\Pi_{|H|}$ has dense range in $\mathcal{K}_{|H|}$, this also implies that R has dense range. Therefore $\Pi_{|H|}^*$ has zero kernel and hence O is injective. Thus (A, B, C, D) is reachable and observable. ∎

REMARK 3.1 As pointed out in the introduction, our proof has a number of similarities with the approach taken by N.J. Young [21], e.g.we use the shift realization, etc. The main difference between the two proofs refers to the construction of the state space: N.J. Young uses the restricted shift realiazation and then a renorming while we construct the state space at once and then use Lemma 2.1. ∎

REMARK 3.2 The fact that the system (A_0, B_0, C_0, D_0) can be lifted to the desired par-balanced realization can be also proved following an idea in Lemma 3.2 in Megretskii, Treil and Peller [15] which uses the Heinz's Theorem. This can be done as follows:

We consider the shift realization (A_0, B_0, C_0, D_0) as in the previous proof, let $|H| = (H^* H)^{1/2}$ be the modulus of the Hankel operator H and let $(\mathcal{K}_{|H|}, \Pi_{|H|})$ be the induced Hilbert space. We now have to perform the 'lifting' of the system (A_0, B_0, C_0, D_0) with state space $\mathcal{H}$ to the 'induced' system (A, B, C, D) with state space $\mathcal{K}_{|H|}$. For the operators B, C, and D this is very similar with what is done in the first proof, with the difference that we do not apply Lemma 2.1. For example, consider the operator $B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and note that

$$\| |H|^{1/2} B_0 u \| \leq \| H^{1/2} \| \, \| u \|, \quad u \in \mathcal{U},$$

and hence B_0 can be lifted to a unique operator $B := \tilde{B}_0 \in \mathcal{L}(\mathcal{U}, \mathcal{K}_{|H|})$. Moreover $B = \Pi_{|H|} B_0$.

For the operator $C_0 \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, let $H = V|H|$ be the polar decomposition of the Hankel operator H and note that $C_0 = P_{\mathcal{Y}} H$, where $\mathcal{Y}$ is naturally identified with the subspace $\mathcal{Y} \oplus 0 \oplus \cdots \oplus 0 \oplus \cdots$ of $\ell^2_{\mathcal{Y}}$. Therefore

$$\|C_0 h\| = \|P_{\mathcal{Y}} H h\| = \|P_{\mathcal{Y}} V |H| h\| \leq \| |H|^{1/2} \| \, \| |H|^{1/2} h \|, \quad h \in \mathcal{H} = \ell^2_{\mathcal{U}},$$

and hence C_0 induces a unique operator $C := \widetilde{C}_0 \in \mathcal{L}(\mathcal{K}_{|H|}, \mathcal{Y})$ and $C\Pi_{|H|} = C_0$.

As in the proof, the main concern is to show that $A_0 \in \mathcal{L}(\mathcal{H})$ induces a unique contractive operator $A \in \mathcal{L}(\mathcal{K}_{|H|})$. To see this, we first obtain the inequality (3.9) exactly as in the first proof. As $A_0 A_0^* \leq I$ we have from (3.9) that

$$(A_0^*|H|A_0)^2 \leq A_0^*|H|^2 A_0 \leq |H|^2.$$

By Loewner's inequality (this can be obtained also as a consequence of the Heinz's Theorem, see e.g. Theorem 9.4 in [20]) which expresses that the square root is operator monotonic, and in conjuction with the uniqueness of the square root, we therefore have that

$$A_0^*|H|A_0 \leq |H|.$$

But this immediately implies the claim since for $x \in \mathcal{K}_{|H|}$

$$\|A_0 x\|_{\mathcal{K}_{|H|}} = \||H|^{1/2} A_0 x\|_{\mathcal{H}} \leq \||H|^{1/2} x\|_{\mathcal{H}} = \|x\|_{\mathcal{K}_{|H|}}.$$

Moreover, $A\Pi_{|H|} = \Pi_{|H|} A_0$.

The fact that the system (A, B, C, D) is a par-balanced realization of the transfer function G follows exactly in the same way as in the proof. ∎

REMARK 3.3 The fact that one can use the Loewner inequality instead of the Lemma Kreĭn-Reid-Lax-Dieudonné comes as no surprise since it can be proved that these tools are actually in the same circle of ideas, e.g. see [8]. ∎

REMARK 3.4 The par-balanced realizations are unique, modulo a unitary equivalence, cf. [21] and the next section. In our proof of Theorem 1.1 we choose $\mathcal{K}_{|H|}$ for the state space. The state space transformation in [21] corresponds to the state space $\mathcal{H}_{|H|}$, see Example 2.2. But, as Example 2.3 shows, a third choice is possible, namely the state space $\mathcal{B}_{|H|}$. This has the advantage of working on the range of some operator, with no closure needed, with the cost, of course, of a more involved topology. ∎

4. Uniqueness of Par-Balanced Realizations

In [21] N.J. Young proved that par-balanced realizations are unique, modulo unitary equivalence. His proof relied on relating this uniqueness question to the problem of the uniqueness of restricted shift realization and the closely related output-normal realizations. Here we present an alternative proof of his result.

In the previous sections of this paper we assumed that the main operator A of a system is a contraction. To allow for a somewhat greater generality of the following theorem we drop this assumption. All other definitions are as before.

Two systems (A_i, B_i, C_i, D_i), with state spaces $\mathcal{H}_i$, $i = 1, 2$, are related by a *state space transformation* if there exists a bounded operator $T: \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$A_1 T = T A_2, \quad B_1 = T B_2, \quad C_1 T = C_2, \quad D_1 = D_2. \tag{4.1}$$

The two systems are called *similar* if there exists a *similarity*, that is, a bounded invertible state space transformation $T: \mathcal{H}_2 \to \mathcal{H}_1$ such that (4.1) holds. If the similarity T is unitary the two systems are called *unitary equivalent.*

THEOREM 4.1 *Let (A_i, B_i, C_i, D_i) be discrete-time observable and reachable linear systems, with state spaces $\mathcal{H}_i$, bounded observability operators O_i and bounded reachability operators R_i, such that they are par-balanced, that is, $O_i^*O_i = R_iR_i^*$, $i = 1, 2$. If both systems are realizations of the same transfer function then the two systems are unitary equivalent.*

Moreover, this state-space transformation is unique, that is, if $U \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is the unitary state-space transformation between the two systems and $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is a state-space transformation between the two systems, then $T = U$.

Proof. Let H be the Hankel operator, associated as in (1.4) with the transfer function G, which is realized by the systems (A_i, B_i, C_i, D_i), $i = 1, 2$. By assumption we have

$$H \quad O_1R_1 \quad O_2R_2,$$

and therefore, using the par-balanced condition we get

$$R_1^*R_1R_1^*R_1 \quad R_1^*O_1^*O_1R_1 \quad H^*H = R_2^*O_2^*O_2R_2 = R_2^*R_2R_2^*R_2.$$

Since $R_1^*R_1$ and $R_2^*R_2$ are positive, using the uniqueness of the square root it follows that

$$R_1^*R_1 \quad |H| = R_2^*R_2.$$

Therefore, there exists a uniquely determined isometric operator $U\colon \mathrm{cl}\,\mathcal{R}(R_2) \to \mathrm{cl}\,\mathcal{R}(R_1)$ such that

$$R_1 = UR_2. \tag{4.2}$$

The reachability of the systems and hence the fact that the reachability operators R_i have dense ranges imply that $U\colon \mathcal{H}_2 \to \mathcal{H}_1$ is unitary. Since

$$R_i = \begin{bmatrix} B_i & A_iB_i & A_i^2B_i & \ldots & A_i^kB_i & \ldots \end{bmatrix}, \quad i \quad 1, 2,$$

this implies, by considering the first component in (4.2), that

$$B_1 = UB_2.$$

Note also that

$$A_iR_i \quad \begin{bmatrix} A_iB_i & A_i^2B_i & A_i^3B_i & \ldots & A_i^{k+1}B_i & \ldots \end{bmatrix}, \quad i \quad 1, 2,$$

and hence

$$A_1UR_2 = A_1R_1 = UA_2R_2.$$

Since R_2 has dense range and the operators are bounded, from here we get

$$A_1U \quad UA_2.$$

Further, since

$$O_1UR_2 = O_1R_1 \quad H = O_2R_2,$$

in a similar way we get

$$O_1U = O_2.$$

As before, from here we obtain

$$C_1U = C_2.$$

That $D_1 = D_2$ follows from $D_1 = G(\infty) = D_2$. Hence the two systems are unitary equivalent.

To show the uniqueness of the state space transformation U assume that $T: \mathcal{H}_2 \to \mathcal{H}_1$ is a similarity state space transformation between the two systems. Note that

$$UR_2 = R_1 \quad \begin{bmatrix} TB_2 & A_1TB_2 & A_1^2TB_2 & \dots \end{bmatrix} = \begin{bmatrix} TB_2 & TA_2B_2 & TA_2^2B_2 & \dots \end{bmatrix} \quad TR_2.$$

Since R_2 has dense range and the operators T and U are bounded we get from here that $T = U$. ∎

A consequence of this result in combination with the realization result is that the main operator of any par-balanced realization of a transfer function analytic outside the unit disk and with bounded Hankel operator is a contraction.

5. Real Symmetric Transfer Functions

A system (A, B, C, D), with state space $\mathcal{H}$, is called *completely J-symmetric* if there exists a *symmetry* $J \in \mathcal{L}(\mathcal{H})$, that is, J is unitary and selfadjoint, such that (A, B, C, D) and (A^*, B^*, C^*, D^*) are similar with similarity J, that is, $JA = A^*J$, $B = JC^*$, and $D = D^*$. In [11] we gave a direct proof of the existence of completely J-symmetric par-balanced realizations of real symmetric transfer functions with bounded Hankel operator. Here we give a proof of a sligthly stronger result than Theorem 6.2 in [11].

THEOREM 5.1 *Assume the assumptions and the notation of Theorem 1.1 and, in addition, that the transfer function G is real symmetric, that is, $G(\overline{z}) = G(z)^*$, $|z| > 1$. Let (A, B, C, D) be an observable and reachable par-balanced realization of G with state space $\mathcal{H}$. Then there exists a unique symmetry J on $\mathcal{H}$ such that (A, B, C, D) is completely J-symmetric. In addition, if O and R denote the observability and, respectively, the reachability operator of the system then $R = JO^*$ and hence, the Hankel operator H of the system admits the factorizations $H = OJO^* = R^*JR$.*

If $\Sigma := R^*R = OO^*$, then J commutes with Σ, i.e. $J\Sigma = \Sigma J$.

Proof. Let (A, B, C, D) be an observable and reachable par-balanced realization of G. Since by the real symmetry of G the dual system (A^*, C^*, B^*, D^*) is another realization of G

$$B^*(zI - A^*)^{-1}C^* + D^* = G(\overline{z})^* = G(z), \quad |z| > 1.$$

The standard duality results for linear systems imply that the dual system is also reachable, observable, and par-balanced.

By Theorem 4.1 there exists a unique unitary state space transformation $U \in \mathcal{L}(\mathcal{H})$ such that

$$(A, B, C, D) = (UA^*U^*, UC^*, B^*U^*, D^*).$$

Applying a state space transformation with U^* to both these systems we obtain

$$(U^*AU, U^*B, CU, D) = (A^*, C^*, B^*, D^*).$$

Taking the dual system we have

$$(U^*A^*U, U^*C, B^*U, D^*) = (A, B, C, D),$$

which shows that U^* is also a similarity for the systems (A, B, C, D) and (A^*, C^*, B^*, D^*). By the uniqueness of the similarity as in Theorem 4.1 we get $U = U^*$, that is, U is a symmetry and the system (A, B, C, D) is completely J-symmetric with $J = U$. The uniqueness of J follows again from Theorem 4.1 and the fact that both (A, B, C, D) and its dual system are reachable, observable and par-balanced realizations of G.

Clearly we have $R = JO^*$ and hence the Hankel operator H of the system admits the factorizations $H = OJO^* = R^*JR$. That J commutes with Σ follows since $\Sigma = RR^* = JO^*OJ^* = J\Sigma J^*$ and therefore $\Sigma J = J\Sigma$. ∎

6. Appendix

Our proof of Theorem 1.1 actually fills a gap in the proof provided in [21]. The proof in [21] relies on Lemma 1, p. 461 in [21]. The Lemma is, however, only valid if, in addition to the stated assumptions, the operator M is also assumed to be essentially selfadjoint. More precisely, the proof of this Lemma, as provided in [21], proves the following statement.

LEMMA 6.1 *Let M be a positive and essentially selfadjoint operator on a dense linear manifold $\mathcal{D}$ of a separable Hilbert space $\mathcal{H}$, let $P \in \mathcal{L}(\mathcal{H})$ have zero kernel and let PMP^{-1} be a contraction on $P\mathcal{D}$. Then M itself is a contraction.*

The additional assumption on M implies that the Lemma can no longer be used to give a complete proof of Theorem 1.

In this appendix we reproduce an argument communicated to us by N.J. Young [22] after receiving a preprint of this paper. In this argument the use of Lemma 1 in [21] is circumvented. We use the same notation as in [21]. Briefly, the idea is to use the closure of the operator $\bar{A}$ instead of $\bar{A}$. To prove that this is possible, let T be the closure of the operator $\bar{A}$ as defined in [21], p. 460. T exists since $\bar{A}^*$ has domain $\supset W^{1/4}\mathcal{H}$ which is dense. Then it is straightforward to show

$$W^{1/4}T^*f = Z^*W^{1/4}f, \quad f \in \mathcal{D}(T^*).$$

Furthermore, extending the relation $W^{1/4}\bar{A} = AW^{1/4}$ using closure, we obtain $W^{1/4}Tf = AW^{1/4}f$ for all $f \in \mathcal{D}(T)$. For arbitrary $h \in \mathcal{D}(T^*T)$ we have $Th \in \mathcal{D}(T^*)$, and hence

$$W^{1/4}T^*Th = Z^*AW^{1/4}h.$$

Thus $W^{1/4}(T^*T)W^{-1/4}$ is a contraction on $W^{1/4}\mathcal{D}(T^*T)$. Since T is closed, T^*T is positive selfadjoint and hence the Lemma 6.1 applies to conclude that T is a contraction.

References

[1] N. ARONSZAJN: Theory of reproducing kernels, *Trans. Amer. Math. Soc.*, **68**(1950), 337–404.

[2] L. DE BRANGES: *Hilbert Spaces of Entire Analytic Functions*, Prentice-Hall, Englewood Cliffs, N.J. 1968.

[3] L. DE BRANGES, J. ROVNYAK: Canonical models in quantum mechanics, in *Perturbation Theory and its Applications in Quantum Mechanics*, Proc. Adv. Sem. Math. Res. Center, Madison WS 1965, pp. 295–392.

[4] T. CONSTANTINESCU, A. GHEONDEA: Elementary rotations of operators in Kreĭn spaces, *J. Operator Theory*, **29**(1993), 167–203.

[5] T. CONSTANTINESCU, A. GHEONDEA: Representations of Hermitian Kernels by means of Kreĭn spaces, preprint no. 15, Institutul de Matematică al Academiei Române, Bucureşti 1996.

[6] J. DIEUDONNÉ: Quasi-hermitian operators, in *Proceedings of International Symposium on Linear Spaces*, Jerusalem 1961, pp. 115–122.

[7] A. DIJKSMA, H. LANGER, H. DE SNOO: Unitary colligations in Kreĭn spaces and their role in extension theory of isometric and symmetric linear relations in Hilbert spaces, in *Functional Analysis II*, Lecture Notes in Mathematics, no **1242**, Springer Verlag, Berlin - Heidelberg - New York 1987, pp. 123–143.

[8] M.A. DRITSCHEL: A method for constructing invariant subspaces for some operators on Kreĭn spaces, in *Operator Theory: Advances and Applications*, Vol. 61, Birkhäuser Verlag, Basel 1993.

[9] P.A. FILLMORE, J.P. WILLIAMS: On operator ranges, *Adv. in Math.*, **7**(1971), 254–281.

[10] P.A. FUHRMANN: *Linear Systems and Operators in Hilbert Space*, McGraw-Hill, 1981.

[11] A. GHEONDEA, R.J. OBER: Completely J-positive linear systems of finite order, *Math. Nachr.*, **203**(1999), 75–101.

[12] K. GLOVER, R.F. CURTAIN, J.R. PARTINGTON: Realisation and approximation of linear infinite dimensional systems with error bounds, *SIAM J. Control and Optimization*, **26**(1988), 863–898.

[13] M.G. KREĬN: On linear completely continuous operators in functional spaces with two norms, (Ukrainian), *Zbirnik Prac. Inst. Mat. Akad. Nauk USSR*, **9**(1947), 104–129.

[14] P.D. LAX: Symmetrizable linear transformations, *Comm. Pure Appl. Math.* **7**(1954). 633–647.

[15] A.V. MEGRETSKII, V.V. PELLER, S.R. TREIL: The inverse spectral problem for self-adjoint Hankel operators, *Acta Mathematica*, **174**(1995), 241-309.

[16] B.C. MOORE: Principal component analysis in linear systems: controllability, observability and model reduction, *IEEE Transactions on Automatic Control*, **26**(1981), 17–32.

[17] R.J. OBER, S. MONTGOMERY-SMITH: Bilinear transformation of infinite-dimensional state-space systems and balanced realizations of nonrational transfer functions, *SIAM Journal on Control and Optimization*, **28**(1990), 438–465.

[18] W.T. REID: Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Math. J.* **18**(1951), 41–56.

[19] L. SCHWARTZ: Sous espaces Hilbertiens d'espaces vectoriel topologiques et noyaux associés (noyaux reproduisants), *J. d'Analyse Math.*, **13**(1964), 115–256.

[20] J. WEIDMANN: *Linear Operators in Hilbert space*, Springer Verlag, Berlin - Heidelberg - New-York 1980.

[21] N.J. YOUNG: Balanced realizations in infinite dimensions, in *Operator Theory: Advances and Applications*, Vol. **19**, Birkhäuser Verlag, Basel 1986, pp. 449–471.

[22] N.J. YOUNG: Private communication.

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