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A STATE-SPACE CALCULUS FOR RATIONAL PROBABILITY DENSITY FUNCTIONS AND APPLICATIONS TO NON-GAUSSIAN FILTERING*

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Abstract. We propose what we believe to be a novel approach to performing calculations for rational density functions using state-space representations of the densities. By standard results from realization theory, a rational probability density function is considered to be the transfer function of a linear system with generally complex entries. The stable part of this system is positive-real, which we call the density summand. The existence of moments is investigated using the Markov parameters of the density summand. Moreover, explicit formulae are given for the existing moments in terms of these Markov parameters. Some of the main contributions of the paper are explicit state-space descriptions for products and convolutions of rational densities.

As an application which is of interest in its own right, the filtering problem is investigated for a linear time-varying system whose noise inputs have rational probability density functions. In particular, state-space formulations are derived for the calculation of the prediction and update equations. The case of Cauchy noise is treated as an illustrative example.

Key words. probability theory, realization theory for linear systems, non-Gaussian filtering, rational functions, linear algebra

AMS subject classifications. 93, 60, 15, 62, 90

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1. Introduction.

We are going to consider the filtering problem for the first order system

$$\begin{aligned} x_{t+1} &= f_t x_t + \eta_t, \\ y_t &= h_t x_t + \epsilon_t, \end{aligned}$$

$t = 0, 1, 2, \dots,$ where f_t, h_t are assumed to be known real numbers and, for ease of exposition, are assumed to be such that $f_t \neq 0$ and $h_t > 0$, $t \geq 0$. The noise sequences $\{\eta_t\}_{t\geq 0}$ and $\{\epsilon_t\}_{t\geq 0}$ are assumed to be mutually independent sequences of independent random variables whose probability density functions are rational. The initial state x_0 is also assumed to be a random variable which is independent of the noise sequences and also has a rational density. No assumption is made that any of the random variables are identically distributed.

This filtering problem with non-Gaussian noise has applications in econometrics, for example in the analysis of financial time series. Studies have shown that the quantities that are encountered there often do not admit a Gaussian distribution ([7], [5], and see also [12]), since these distributions have "heavy tails." As one of the consequences, higher order moments may not exist. It has therefore been proposed (see, e.g., [11]) that these distributions be modelled by rational densities, both because

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they do have "heavy tails" and because of the richness of the class of distributions. Examples of rational probability densities which have been used in the literature are Cauchy densities and Student densities with odd number of degrees of freedom.

The state filtering problem is defined as the problem of finding the best estimate $\hat{x}_t$ of x_t for the quadratic loss function given knowledge of the distribution of x_0 and the values of $y_0, y_1, \dots, y_t$. Since

$$\hat{x}_t = \int_{-\infty}^{\infty} x p(x)_{x_t|y_t,y_{t-1},\dots,y_0} dx,$$

this estimate can be found if the conditional density $p_{x_t|y_t,y_{t-1},\dots,y_0}$ of x_t is known and the first moment exists, given the measured values of $y_t, y_{t-1}, \dots, y_0$ and knowledge of the distribution of x_0.

In principle, the calculation of the conditional densities is not difficult. The unnormalized conditional densities, denoted by ρ instead of p, are given by the following.

Update step. For $t = 0$,

$$\rho_{x_0|\mathcal{Y}_0}(x) = \rho_{x_0|y_0}(x) = \rho_{y_0|x}(y_0)\rho_{x_0}(x) = \rho_{\epsilon_0}(y_0 - h_0 x)\rho_{x_0}(x);$$

for $t \geq 1$,

$$\rho_{x_t|\mathcal{Y}_t}(x) = \rho_{y_t|x}(y_t)\rho_{x_t|\mathcal{Y}_{t-1}}(x) = \rho_{\epsilon_t}(y_t - h_t x)\rho_{x_t|\mathcal{Y}_{t-1}}(x),$$

$x \in \Re$.

Prediction step. For $t \geq 0$,

$$\rho_{x_{t+1}|\mathcal{Y}_t}(x) = (\rho_{f_t x_t|\mathcal{Y}_t} * \rho_{\eta_t})(x) = \int_{-\infty}^{\infty} \rho_{x_t|\mathcal{Y}_t}\left(\frac{\xi}{f_t}\right)\rho_{\eta_t}(x-\xi)\, d\xi, \quad x \in \Re.$$

Here we have set $\mathcal{Y}_t$ to be the collection of observations $y_t, y_{t-1}, \dots, y_0$.

In [11] it was noted that the various probability densities occurring in the filtering problem are all rational functions if the noise variables and the initial state have rational probability densities and if explicit formulas are given. The practical problem in doing these calculations for large numbers of observations is that the conditional densities are fairly complicated to calculate. To alleviate this problem we propose the use of state-space techniques for these calculations. Since by assumption the initial state and the noise sequences have rational densities, this is indeed possible. For this purpose we are going to develop a "state-space calculus" for rational probability density functions. We believe that the use of linear system theory to analyze rational probability densities is novel and may be of relevance beyond the application to non-Gaussian filtering as discussed here. Since the approach is valid in general, we develop the state-space approach for general probability density functions as well as for conditional probability density functions.

Let ρ be a not necessarily normalized rational probability density, i.e., $\rho(x)$ is a rational function in the independent variable x, such that $\rho(x) \geq 0$, $x \in \Re$, and $0 < \int_{-\infty}^{\infty} \rho(x) dx < \infty$. This implies that ρ is *strictly proper*, i.e., $\lim_{|x|\to\infty} \rho(x) = 0$. To speak of a not necessarily normalized or unnormalized probability density function is an abuse of the standard notion of a probability density function, since this term implies that its integral is 1. For ease of notation we use the notion of a not necessarily normalized or unnormalized density function to imply that all properties of a density function are given, with the possible exception of the normalization of its integral.

By standard realization theory there exists a minimal state-space realization such that

$$\rho(x) = c(ixI - A)^{-1}b, \qquad x \in \Re.$$

In particular, we will present here state-space formulae for the translation, scaling, product, and convolution of rational probability density functions. Most of our results will be formulated in terms of state-space realizations for the density summand, which is defined to be the "stable" part of the probability density function. One reason for doing this is that in this way the dimensions of the realizations are typically half of what they would be otherwise. For actual implementations of our results, this could lead to significant computational advantages, in particular when repeated applications are necessary such as can be expected for the filtering case. Moreover, we will investigate the existence of moments from the state-space point of view and give state-space formulae for the existing moments in terms of the Markov parameters of the density summand. A major part of the investigation will be built on a careful analysis of the connections between impulse responses, transfer functions, and characteristic functions of the various objects. In a result that may be of independent interest, a state-space formula is given for the system whose impulse response is the product of impulse responses of two systems.

2. Notation and preliminaries. The symbol $\mathcal{C}$ stands for the complex field, and the symbol $\Re$ stands for the real field. If (A, b, c) is a linear state-space system, we also often use the notation $\left(\begin{array}{c|c} A & b \\ \hline c & 0 \end{array}\right)$. If M is a complex matrix, M^* denotes the adjoint matrix. If G is a rational function, G^* is defined by $G^*(s) = \overline{(G(-\bar{s}))}$, $s \in \mathcal{C}$. If G has the realization (A, b, c) (i.e., $G(s) = c(sI - A)^{-1}b$ for $s \in \mathcal{C}\backslash\sigma(A)$, where $\sigma(A)$ is the spectrum of A), then G^* has the realization $(-A^*, c^*, -b^*)$. We call a system (A, b, c) *stable* if all eigenvalues of A are in the open left half plane. Note that such systems are often also called asymptotically stable. A rational function G is called strictly proper if $\lim_{|s|\to\infty} G(s) = 0$. An unnormalized probability density function ρ is a nonnegative integrable function on $\Re$ such that $\int_{-\infty}^{\infty} \rho(x)dx > 0$, but not necessarily 1. Then $p = \rho / \int_{-\infty}^{\infty} \rho(x)dx$ is a normalized density function. The set of functions $\mathcal{P}$ is defined in section 3.

3. State-space representations of rational densities. If ρ is not a necessarily normalized rational probability density function, then ρ is strictly proper, i.e., $\lim_{|x|\to\infty} \rho(x) = 0$. Therefore, by standard realization theory (see, e.g., [4, Section 2.1], [10, Sections 10–11]), there exists a minimal linear state-space system (A, b, c) such that

$$\rho(x) = c(ixI - A)^{-1}b, \qquad x \in \Re.$$

It should be noted that the system matrices A, b, c will be, in general, complex matrices. A rational probability density function which is symmetric with respect to 0, however, could be realized with real system matrices.

Note also that we have set up the realization in such a way that we consider the rational function to be defined on the imaginary axis. While in principle the choice of axis is arbitrary, it is convenient to choose the imaginary axis since then standard realization theoretic methods can be adopted without having to change the axis. In particular, we will be using the formal analogy of methods developed for spectral densities which are most naturally considered to be defined on the imaginary axis. To make this convention clear, set

$$\Phi(ix) := \rho(x), \qquad x \in \Re.$$

Since Φ is a rational function defined on the imaginary axis, it can be extended as a rational function to the whole complex plane. This rational function has the following properties.

1. $\Phi(s) = \Phi^*(s), s \in \mathcal{C}$.
2. Φ has no poles on the imaginary axis.
3. $\Phi(ix) \geq 0$, $x \in \Re$.
4. $\lim_{|s|\to\infty} \Phi(x) = 0$.

The set of rational functions that satisfies properties 1, 2, 3, and 4 is denoted by $\mathcal{P}$. Many of our calculations are going to be based on the following well-known additive decomposition (see Lemma 3.1) of Φ:

$$\Phi(s) = Z(s) + Z^*(s), \qquad s \in \mathcal{C},$$

where Z is a stable rational function, i.e., all poles of Z are in the open left half plane. This decomposition is unique if we assume that $Z(\infty) = 0$, which can be done since $\Phi(\infty) = 0$. The function Z is called the *spectral summand* or Φ. We will also call Z the *density summand* of ρ.

In the following lemma some elementary and standard state-space properties are collected concerning this additive decomposition of Φ. For the sake of completeness, a short proof is added for this standard result.

LEMMA 3.1. *Let $\Phi \in \mathcal{P}$. Then there exists a stable rational function Z such that*

$$\Phi = Z + Z^*.$$

Let (A, b, c) be a minimal realization of Φ, i.e., $\Phi(s) = c(sI - A)^{-1}b$, and (A, b, c) is minimal. There exists an equivalent realization

$$\left(\begin{array}{cc|c} A_1 & 0 & b_1 \\ 0 & A_2 & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right)$$

of (A, b, c) such that all eigenvalues of A_1 are in the open left half plane and all eigenvalues of A_2 are in the open right half plane. The state-space system (A_1, b_1, c_1) is a minimal realization of Z, and (A_2, b_2, c_2) is a minimal realization of Z^*.

Moreover, (A_2, b_2, c_2) is equivalent to $(-A_1^*, c_1^*, -b_1^*)$. In particular, there exists a minimal realization of Φ such that

$$\left(\begin{array}{cc|c} A_1 & 0 & b_1 \\ 0 & -A_1^* & c_1^* \\ \hline c_1 & -b_1^* & 0 \end{array}\right).$$

Proof. Let $\Phi = Z_s + Z_u$ be a stable-unstable partial fraction decomposition of Φ, i.e., the partial fraction decomposition of Φ such that Z_s is stable, meaning that all its poles are in the open left half plane, and Z_u is unstable, meaning that all its poles are in the open right half plane. Note that this decomposition is unique. Let

(A_1, b_1, c_1) be a minimal realization of Z_s, and let (A_2, b_2, c_2) be a minimal realization of Z_u. Then

$$\left(\begin{array}{cc|c} A_1 & 0 & b_1 \\ 0 & A_2 & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right)$$

is a minimal realization of Φ and hence equivalent to (A, b, c). Set $Z := Z_s$. We need to show that $Z_u = Z^*$. Now consider $Z_s + Z_u = \Phi = \Phi^* = (Z_s + Z_u)^* = Z_s^* + Z_s^*$. Note that Z_s^* has all its roots in the open right half plane, and Z_u^* has all its roots in the open left half plane. By the above-mentioned uniqueness of the stable-unstable partial fraction decomposition, we have that $Z_u = Z_s^* = Z^*$. The remaining parts of the lemma follow immediately. $\square$

Example. As a special case we are going to consider the Cauchy density, which was suggested, for example in [7], as a suitable density to study financial time series. The normalized Cauchy density is defined as

$$p(x) = \frac{1}{\pi} \frac{k}{(x - x_0)^2 + k^2}, \quad x \in \Re,$$

where $x_0 \in \Re$ and $k > 0$. A state-space realization of $\Phi(ix) := p(x)$, $x \in \Re$, is given by

$$\left[\begin{array}{c|c} A_\Phi & b_\Phi \\ \hline c_\Phi & 0 \end{array}\right] := \left[\begin{array}{cc|c} -k + ix_0 & 0 & \frac{1}{2\pi} \\ 0 & k + ix_0 & 1 \\ \hline 1 & -\frac{1}{2\pi} & 0 \end{array}\right].$$

The density summand of p is

$$Z(s) = \frac{1}{2\pi} \frac{1}{s - (-k + ix_0)},$$

which has one pole at $-k + ix_0$. A state-space realization of Z is given by

$$\left[\begin{array}{c|c} A & b \\ \hline c & 0 \end{array}\right] := \left[\begin{array}{c|c} -k + ix_0 & \frac{1}{2\pi} \\ \hline 1 & 0 \end{array}\right].$$

4. Fourier transforms, moments, and Markov parameters.

In order to obtain state space formulae for the moments of probability density functions and for the convolution of such densities, we need to employ the Fourier transform. The main tool will be to interpret the density summand as the Fourier transform of the impulse response of a stable linear state-space system. Actually, we introduce the Fourier transform as the Laplace transform evaluated on the imaginary axis. For a general reference on Fourier transforms see, e.g., [9], [6]. This way of proceeding is of course closely related to the use of the characteristic function in statistics, but there are a few more minor technical differences.

For an integrable function f on $\Re$ define the Fourier transform as usual by

$$(\mathcal{F}(f))(iw) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt, \qquad iw \in i\Re.$$

If (A, b, c) is a stable system, let $m^+(t) := ce^{tA}b$ for $t \geq 0$, and $m^+(t) := 0$ for $t < 0$. Then the Fourier transform of m^+ is given by

$$\begin{aligned}(\mathcal{F}m^+)(iw) &= \int_0^\infty ce^{tA}be^{-itw}dt = c(-iwI + A)^{-1}e^{(-iwI+A)t}|_0^\infty b = c(iwI - A)^{-1}b \\ &=: G(iw), \qquad iw \in i\Re.\end{aligned}$$

If we set $m^-(t) := b^* e^{-tA^*}c^*$ for $t < 0$ and $m^-(t) := 0$ for $t \geq 0$, then the Fourier transform of m^- is given by

$$\begin{aligned}(\mathcal{F}m^-)(iw) &= \int_{-\infty}^0 b^* e^{-tA^*}c^* e^{-itw}dt = b^*(-iwI - A^*)^{-1}e^{(-iwI-A^*)t}|_{-\infty}^0 c^* \\ &= -b^*(iwI - (-A)^*)^{-1}c^* = G^*(iw), \quad iw \in i\Re.\end{aligned}$$

The lth derivative of m^+ at $t > 0$ is given by $(m^+)^{(l)}(t) = cA^l e^{tA}b$. Hence the right-hand side limit of the lth derivative at 0 is given by $(m^+)^{(l)}(0+) = cA^l b$. The lth derivative of m^- at $t < 0$ is given by $(m^-)^{(l)}(t) = b^*(-A^*)^l e^{-tA^*}c^*$. Hence the left-hand side limit of the lth derivative at 0 is given by $(m^-)^{(l)}(0-) := b^*(-A^*)^l c^* = (-1)^l(cA^l b)^* = (-1)^l((m^+)^{(l)}(0+))^*$, $l \geq 0$.

Assume now that (A, b, c) is a realization of the spectral summand Z of the function $\Phi \in \mathcal{P}$. Then $(\mathcal{F}m^+)(iw) = Z(iw)$, $(\mathcal{F}m^-)(iw) = Z^*(iw)$, and for $m := m^+ + m^-$ we have that $(\mathcal{F}m)(iw) = \Phi(iw)$, $iw \in i\Re$. Hence m is the inverse Fourier transform of Φ. Note that m is l times continuously differentiable at $t = 0$, $l \geq 0$, if and only if $cA^k b = (-1)^k(cA^k b)^*$, $k = 0, 1, \ldots, l$.

If G is a strictly proper rational function on $\mathcal{C}$, then G admits a Laurent expansion around ∞ such that

$$G(s) = \sum_{n=1}^\infty M(n)\frac{1}{s^n}$$

for $s \in \mathcal{C}$ with $|s|$ large enough. The parameters $M(n)$, $n = 1, 2, \ldots$, are the *Markov parameters* of G (see, e.g., [10, p. 194]). If (A, b, c) is a realization of G, then

$$G(s) = c(sI - A)^{-1}b = \frac{1}{s}c\left(I - \frac{A}{s}\right)^{-1}b = \frac{1}{s}c\sum_{k=0}^\infty\left(\frac{1}{s}A\right)^k b = \sum_{n=1}^\infty \frac{1}{s^n}cA^{n-1}b.$$

Hence the Markov parameters of G are given by

$$M(n) = cA^{n-1}b, \quad n = 1, 2, 3, \ldots.$$

The Markov parameters of a rational strictly proper function of $\mathcal{P}$ and its spectral summand are easily determined.

LEMMA 4.1. *Let Φ be a strictly proper rational function in $\mathcal{P}$ with spectral summand Z. If (A, b, c) is a realization of Z, then*

1. *the Markov parameters of Z are given by*

$$cA^{n-1}b, \qquad n = 1, 2, 3, \ldots,$$

2. *the Markov parameters of Z^* are given by*

$$(-1)^n b^*(A^*)^{n-1}c^* = (-1)^n(cA^{n-1}b)^*, \qquad n = 1, 2, 3, \ldots, \text{ and}$$

3. *the Markov parameters of* Φ *are given by*

$$cA^{n-1}b - (-1)^{n-1}(cA^{n-1}b)^*, \qquad n = 1, 2, 3, \ldots.$$

In the following lemma, a basic result on the integrability of rational functions is summarized.

LEMMA 4.2. *Let* $G = \frac{n}{d}$ *with* n *and* d *as a pair of coprime polynomials. Then*

$$\int_{-\infty}^{\infty} |G(x)| dx < \infty$$

if and only if $degree(n) \leq degree(d) - 2$ *and* $d(x) \neq 0$ *for all* $x \in \Re$.

If G is as defined in the lemma, then $degree(d) - degree(n)$ is called the *codegree* of the rational function G. Therefore, G is integrable if and only if the codegree of G is greater than or equal to 2. This lemma also implies that if the random variable X has the rational probability density function $p = \frac{n}{d}$, then the moments EX^k exist for $k = 0, 1, 2, \ldots$, codegree$(p) - 2$.

Let k be such that $M(n) = 0$ for $n = 1, 2, \ldots, k-1$ and $M(k) \neq 0$. Then the codegree of G is k [10, p. 254].

Summarizing the previous remarks, we obtain the following proposition.

PROPOSITION 4.1. *Let* Φ *be a strictly proper rational function in* $\mathcal{P}$ *with spectral summand* Z. *Let* (A, b, c) *be a minimal realization of* Z. *Let* $m(t) := ce^{tA}b$ *for* $t \geq 0$ *and* $m(t) := b^* e^{-tA^*} c^*$ *for* $t < 0$. *Then the following hold.*

1. *The codegree of* Φ *is* k *if and only if* $M(n) = 0$ *for all* $n \in \{1, \ldots, k-1\}$ *and* $M(k) \neq 0$, *where* $M(n)$ *is the* nth *Markov parameter of* Φ.

2. *The codegree of* Φ *is* k *if and only if*

$$cA^{n-1}b = (-1)^{n-1}(cA^{n-1}b)^*$$

for all $n \in \{1, \ldots, k-1\}$ *and*

$$cA^{k-1}b \neq (-1)^{k-1}(cA^{k-1}b)^*.$$

3. m *is* $k-1$ *times continuously differentiable at* 0 *if and only if the first* k *Markov parameters of* Φ *are zero.*

4. Φ *has codegree* k *if and only if* m *is* $k-2$ *times continuously differentiable but not* $k-1$ *times continuously differentiable at* 0.

The following theorem provides important results concerning moments of a random variable with rational probability density.

THEOREM 4.1. *Let* X *be a random variable with unnormalized rational probability density function* ρ. *Let* (A, b, c) *be a realization of the density summand* Z *of* ρ. *Then the following hold.*

1. *The codegree of* ρ *is* k *if and only if*

$$cA^{n-1}b = (-1)^{n-1}(cA^{n-1}b)^*$$

for all $n \in \{1, \ldots, k-1\}$ *and* $cA^{k-1}b \neq (-1)^{k-1}(cA^{k-1}b)^*$.

2. *The* lth *moment* EX^l *of* X *with* l *a nonnegative integer exists if and only if* $l \in \{0, 1, \ldots, k-2\}$.

3. $EX^l = (-i)^l \frac{cA^l b}{cb}$ *for all* $l \in \{0, 1, \ldots, k-2\}$.

Proof. (1) The proof follows immediately from Proposition 4.1.

(2) Recall that the lth moment of X is given by

$$EX^l = \frac{1}{R}\int_{-\infty}^{\infty} x^l \rho(x)\,dx,$$

where $R := \int_{-\infty}^{\infty} \rho(x)\,dx$. The codegree of the integrand is $k-l$. By Lemma 4.2 the integrand is integrable if and only if its codegree is greater than or equal to 2. Hence the claim.

(3) Let $0 \le l \le k-2$. Set $\Phi(ix) := \rho(x)$, $x \in \Re$, and use the notation of Proposition 4.1. Then m is $k-2$ times continuously differentiable at 0 and therefore on $\Re$. Since the codegree of ρ is greater than or equal to 2, m is continuous on $\Re$. Since ρ and m are continuous and integrable, we have by the inversion theorem for Fourier transforms (see, e.g., [6, Theorem 60.1, p. 296]) that

$$m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(iw)e^{iwt}dw, \qquad t \in \Re.$$

Note that differentiation up to order $k-2$ under this integral is justified by the usual argument (see, e.g. [6, Theorem 53.5, p. 268]) as $|\omega^l \Phi(i\omega)e^{i\omega t}| = |\omega^l \Phi(i\omega)|$ is integrable for each $t \in \Re$ and $0 \le l \le k-2$. Hence for $t \in \Re$,

$$\frac{d^l}{dt^l}m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(iw)\frac{d^l}{dt^l}e^{iwt}dw = (i)^l\frac{1}{2\pi}\int_{-\infty}^{\infty} w^l\Phi(iw)e^{iwt}\,dw.$$

Evaluating at $t=0$, we have

$$\frac{d^l}{dt^l}m(t)|_{t=0} = \frac{1}{2\pi}(i)^l\int_{-\infty}^{\infty} w^l\Phi(iw)e^{iwt}dw|_{t=0} = R(i)^l\frac{1}{2\pi}EX^l.$$

Since $\frac{d^l}{dt^l}m(t)|_{t=0} = cA^lb$, $l = 0,\dots,k-2$, we have that

$$EX^l = \frac{2\pi}{R}(-i)^l cA^l b.$$

The constant R is determined by considering this equation for $l=0$. Since $EX^0 = 1$, we have that $R = 2\pi cb$. Hence $EX^l = (-i)^l\frac{cA^lb}{cb}$. ◻

In most of this paper we will be dealing with unnormalized rational probability densities ρ. If (A,b,c) is a state-space realization of the density summand of ρ, the normalized probability density function is given by $p := \rho/\int_{-\infty}^{\infty}\rho(x)dx$. By the above proposition $\int_{-\infty}^{\infty}\rho(x)dx = 2\pi cb$, which provides a state-space formula for the normalization constant.

If X is a random variable with rational probability density function ρ whose density summand has the state-space realization (A,b,c), then the first moment exists if the codegree of ρ is at least 3. This is the case if and only if

$$cb = (cb)^*$$

and

$$cAb = -(cAb)^*.$$

If the first moment, i.e., the mean, exists, then by the theorem it is given by

$$EX = -i\frac{cAb}{cb}.$$

In the above discussion we gave a state-space construction for the inverse Fourier transform m of a not necessarily normalized rational probability density function ρ, i.e.,

$$m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\omega t} d\omega, \qquad \omega \in \Re.$$

In the statistical literature an important object is the characteristic function of a random variable X which is defined by $E(e^{itX})$, $t \in \Re$. If X has the unnormalized probability density function ρ, then

$$E(e^{itX}) = \frac{1}{\int_{-\infty}^{\infty} \rho(x)dx} \int_{-\infty}^{\infty} e^{itx} \rho(x) dx = \frac{2\pi}{\int_{-\infty}^{\infty} \rho(x)dx} m(t), \qquad t \in \Re.$$

Hence up to a (known) scaling factor the function m is identical to the characteristic function.

Example continued. We continue the discussion of the Cauchy density from section 3. Note that for all $x_0 \in \Re$ and $k > 0$

$$cAb = \frac{1}{2\pi}(-k + ix_0) \neq -\frac{1}{2}(-k - ix_0) = -(cAb)^*.$$

Hence by the theorem the mean EX does not exist. This is of course also directly evident by consideration of the integral $\int_{-\infty}^{\infty} xp(x)dx$.

If $m^+(\tau) := \frac{1}{2\pi} e^{\tau(-k+ix_0)}$ for $t \geq 0$ and $m^+(\tau) := 0$ for $t < 0$, then $\mathcal{F}(m^+)(iw) = \frac{1}{2\pi} \frac{1}{iw-(-k+ix_0)}$, $iw \in i\Re$. If $m^-(\tau) := \frac{1}{2\pi} e^{-\tau(-k-ix_0)}$ for $t < 0$ and $m^-(\tau) := 0$ for $t \geq 0$, then $\mathcal{F}(m^-)(iw) = \frac{1}{2\pi} \frac{1}{iw+(k-ix_0)}$, $iw \in i\Re$. With $m := m^+ + m^-$, we have that m is continuous at 0. The derivative is given by

$$\frac{d}{dt} m(t) = \frac{1}{2\pi}(-k + ix_0) e^{\tau(-k+ix_0)}, \qquad \tau > 0,$$
$$\frac{d}{dt} m(t) = \frac{1}{2\pi}(k + ix_0) e^{-\tau(-k-ix_0)}, \qquad \tau < 0.$$

Note that the left-hand side limit and the right-hand side limit do not agree at 0. Hence m is not differentiable at 0. As the codegree of p is 2, this is in agreement with Proposition 4.1. The first two Markov parameters of Φ are

$$c_\Phi b_\Phi = 0, \qquad c_\Phi A_\Phi b_\Phi = \frac{-k}{\pi}.$$

Hence the second Markov parameter is nonzero, which is also in agreement with Proposition 4.1. $\square$

5. Operations on probability densities. In this section we are going to discuss state-space formulations of operations on rational probability densities. Given state-space realizations for the density summands of two probability densities, we will give state-space realizations for the density summand of the translation, scaling, product, and convolution of the densities.

5.1. Translation and scaling of a probability density. In the next straightforward lemma the effect of translation and scaling of a random variable on the state-space realization of the density is considered.

LEMMA 5.1. *Let X be a random variable with unnormalized rational density ρ. Let (A, b, c) be a minimal realization such that $\rho(x) = c(ixI - A)^{-1}b$, $x \in \Re$.*

Let $x_0 \in \Re$. Then the random variable $X + x_0$ has an unnormalized probability density function $q(x) = \rho(x - x_0)$, which has a realization $(A + ix_0 I, b, c)$, so

$$q(x) = c(ixI - (A + ix_0 I))^{-1}b, \qquad x \in \Re.$$

Let $a \in \Re$, $a \neq 0$; then the random variable aX has the unnormalized probability density function $q(x) = \frac{1}{a}\rho(\frac{x}{|a|})$ which has a realization $(aA, b, \frac{a}{|a|}c)$, so

$$q(x) = \frac{a}{|a|}c(ixI - aA)^{-1}b, \qquad x \in \Re.$$

In the following lemma, we are going to write down the analogous results for the case when a state-space realization is given for the density summand of the probability density. The proof is elementary.

LEMMA 5.2. *Let X be a random variable with unnormalized rational density ρ. Let (A, b, c) be a realization of the density summand Z of ρ.*

Let $x_0 \in \Re$; then the random variable $X + x_0$ has the unnormalized probability density function $q(x) = \rho(x - x_0)$, $x \in \Re$, whose density summand has a realization

$$\left(\begin{array}{c|c} A + ix_0 I & b \\ \hline c & 0 \end{array}\right).$$

Let $a \in \Re$, $a \neq 0$; then the random variable aX has the unnormalized probability density function $q(x) = \frac{1}{|a|}\rho(\frac{x}{a})$, whose density summand has a realization

$$\left(\begin{array}{c|c} aA & b \\ \hline c & 0 \end{array}\right)$$

if $a > 0$ and

$$\left(\begin{array}{c|c} -aA^* & c^* \\ \hline b^* & 0 \end{array}\right)$$

if $a < 0$.

5.2. Product of two rational probability densities. In the update step of the filtering problem, it is necessary to calculate the product of two density functions. We are going to do this also by state-space techniques using the decomposition into density summands. The following lemmas will be useful.

LEMMA 5.3. *Let G_1 and G_2 be two stable strictly proper rational functions with minimal state-space realizations (A_1, b_1, c_1) and (A_2, b_2, c_2). Then the product $G_1^* G_2$ can be decomposed as*

$$G_1^* G_2 = F + H^*,$$

where F, H are stable strictly proper rational functions such that F has the realizations given by

$$\left(\begin{array}{c|c} A_2 & b_2 \\ \hline b_1^* T_1 & 0 \end{array}\right), \qquad \left(\begin{array}{c|c} A_2 & T_2 c_1^* \\ \hline c_2 & 0 \end{array}\right),$$

and H^* has the realizations given by

$$\left(\begin{array}{c|c} -A_1^* & T_1 b_2 \\ \hline -b_1^* & 0 \end{array}\right), \qquad \left(\begin{array}{c|c} -A_1^* & -c_1^* \\ \hline c_2 T_2 & 0 \end{array}\right),$$

where T_1 is the unique solution to the Sylvester equation

$$A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0$$

and T_2 is the unique solution to the Sylvester equation

$$A_2 T_2 + T_2 A_1^* + b_2 b_1^* = 0.$$

Proof. Note that a realization of G_1^* is given by

$$(-A_1^*, c_1^*, -b_1^*),$$

and a realization of $G_1^* G_2$ is given by

$$\left(\begin{array}{cc|c} -A_1^* & c_1^* c_2 & 0 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & 0 & 0 \end{array}\right).$$

Performing a state-space basis transformation with transformation matrix $\left(\begin{smallmatrix} I & T_1 \\ 0 & I \end{smallmatrix}\right)$, we obtain the equivalent realization

$$\left(\begin{array}{cc|c} -A_1^* & A_1^* T_1 + T_1 A_2 + c_1^* c_2 & T_1 b_2 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & b_1^* T_1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -A_1^* & 0 & T_1 b_2 \\ 0 & A_2 & b_2 \\ \hline -b_1^* & b_1^* T_1 & 0 \end{array}\right)$$

since T_1 is such that $A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0$. Note that such a T_1 exists and is unique since both A_1^* and A_2 have all their eigenvalues in the open left half plane (see, e.g., [1, Vol. I, p. 225]). This representation implies the first set of realizations. The other set of realizations follows analogously by considering the state-space formula which corresponds to $G_2 G_1^*$. ∎

Remark. A method to generate explicit formulas for the solutions of Sylvester equations is presented in [3].

We can now derive the desired representation for the density summand of the product of two rational probability density functions.

PROPOSITION 5.1. *Let ρ_1 and ρ_2 be two unnormalized rational probability density functions with density summands Z_1 and Z_2. Let (A_i, b_i, c_i) be a minimal realization of Z_i, $i = 1, 2$. Then the density summand Z of the unnormalized rational probability density function $\rho = \rho_1 \rho_2$ has a realization given by*

$$\left(\begin{array}{cc|c} A_1 & b_1 c_2 & T_2^* c_2^* \\ 0 & A_2 & b_2 \\ \hline c_1 & b_1^* T_1 & 0 \end{array}\right),$$

where T_1, T_2 are the unique solutions to the Sylvester equations

$$A_1^* T_1 + T_1 A_2 + c_1^* c_2 = 0,$$
$$A_2 T_2 + T_2 A_1^* + b_2 b_1^* = 0.$$

Proof. We have that

$$\rho = \rho_1\rho_2 = (Z_1 + Z_1^*)(Z_2 + Z_2^*) = Z_1Z_2 + Z_1Z_2^* + (Z_1Z_2^*)^* + (Z_1Z_2)^*.$$

By Lemma 5.3 a state-space realization for the stable part of this expression is given by

$$\left(\begin{array}{cccc|c} A_1 & b_1c_2 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & b_2 \\ 0 & 0 & A_1 & 0 & T_2^*c_2^* \\ 0 & 0 & 0 & A_2 & b_2 \\ \hline c_1 & 0 & c_1 & b_1^*T_1 & 0 \end{array}\right),$$

where T_1 is the unique solution of the equation

$$A_1^*T_1 + T_1A_2 + c_1^*c_2 = 0$$

and T_2 is the unique solution of the equation

$$A_2T_2 + T_2A_1^* + b_2b_1^* = 0.$$

Performing a state-space basis transformation with transformation matrix

$$T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & I \end{pmatrix},$$

we obtain the equivalent realization

$$\left(\begin{array}{cccc|c} A_1 & b_1c_2 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & b_2 \\ 0 & 0 & A_1 & 0 & T_2^*c_2^* \\ 0 & 0 & 0 & A_2 & 0 \\ \hline c_1 & b_1^*T_1 & c_1 & b_1^*T_1 & 0 \end{array}\right),$$

which is equivalent to

$$\left(\begin{array}{ccc|c} A_1 & b_1c_2 & 0 & 0 \\ 0 & A_2 & 0 & b_2 \\ 0 & 0 & A_1 & T_2^*c_2^* \\ \hline c_1 & b_1^*T_1 & c_1 & 0 \end{array}\right).$$

On this realization perform another state-space basis transformation with transformation matrix

$$T = \begin{pmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

to obtain

$$\left(\begin{array}{ccc|c} A_1 & b_1c_2 & 0 & T_2^*c_2^* \\ 0 & A_2 & 0 & b_2 \\ 0 & 0 & A_1 & T_2^*c_2^* \\ \hline c_1 & b_1^*T_1 & 0 & 0 \end{array}\right),$$

which is equivalent to

$$\left(\begin{array}{cc|c} A_1 & b_1c_2 & T_2^*c_2^* \\ 0 & A_2 & b_2 \\ \hline c_1 & b_1^*T_1 & 0 \end{array}\right). \qquad \square$$

It was noted before that the codegree of a rational probability density function is at least 2. Therefore, the product of two such probability density functions has codegree at least 4. Hence for a random variable whose density is given by such a product, at least the first and second moments exist. This will be used in the next section to show the existence of a conditional mean and variance.

5.3. Convolution of probability densities. We now come to determine a state-space formulation for the convolution of two probability densities. Recall that if X and Y are two random variables with rational probability densities ρ_X and ρ_Y, then the probability density of $X+Y$ is given by the convolution $\rho_X * \rho_Y$.

Let ρ_1 and ρ_2 be two unnormalized rational probability functions with corresponding spectral summands Z_1 and Z_2. Let (A_i, b_i, c_i) be a realization of Z_i, $i=1,2$. Let, for $i=1,2$,

$$m_i^+(\tau) := \begin{cases} c_i e^{\tau A_i} b_i, & \tau \geq 0, \\ 0, & \tau < 0, \end{cases}$$

$$m_i^-(\tau) := \begin{cases} b_i^* e^{-\tau A_i^*} c_i^*, & \tau < 0, \\ 0, & \tau \geq 0. \end{cases}$$

Then $(\mathcal{F}m_i^+)(iw) = Z_i(iw)$, $(\mathcal{F}m_i^-)(iw) = Z_i^*(iw)$, $iw \in i\Re$, and

$$\begin{aligned} (\rho_1 * \rho_2)(w) &= \int_{-\infty}^{\infty} \Phi_1(iw - i\nu)\Phi_2(i\nu)d\nu = \mathcal{F}\left(\mathcal{F}^{-1}\int_{-\infty}^{\infty} \Phi_1(iw-i\nu)\Phi_2(i\nu)d\nu\right) \\ &= \mathcal{F}((\mathcal{F}^{-1}\Phi_1)(\mathcal{F}^{-1}\Phi_2))(iw) = \mathcal{F}((\mathcal{F}^{-1}(Z_1+Z_1^*))(\mathcal{F}^{-1}(Z_2+Z_2^*)))(iw) \\ &= \mathcal{F}((m_1^+ + m_1^-)(m_2^+ + m_2^-))(iw) = \mathcal{F}(m_1^+m_2^+ + m_1^-m_2^-)(iw) \\ &= \mathcal{F}(m_1^+m_2^+)(iw) + \mathcal{F}(m_1^-m_2^-)(iw). \end{aligned}$$

It follows that the spectral summand Z of $\rho_1 * \rho_2$ is given by $Z(iw) = \mathcal{F}(m_1^+m_2^+)(iw)$.

In the following proposition we are going to give the state-space formulae for the product of the impulse responses of two single-input single-output state-space systems. This will be the key step to determine a state-space realization for the convolution of two rational probability density functions.

Proposition 5.2. *Let $m_i^+(\tau) := c_i e^{\tau A_i} b_i$ for $\tau \geq 0$, and $m_i^+(\tau) := 0$ for $\tau < 0$, where (A_i, b_i, c_i) is an n_i-dimensional single-input single-output system, $i=1,2$. Then*

$$m^+(\tau) := m_1^+(\tau)m_2^+(\tau), \quad \tau \geq 0,$$

has a realization $m^+(\tau) = ce^{\tau A}b$ *for* $\tau \geq 0$ *and* $m^+(\tau) = 0$ *for* $\tau < 0$*, where*

$$\begin{aligned} A &= A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2, \\ b &= b_1 \otimes b_2, \\ c &= c_1 \otimes c_2. \end{aligned}$$

(Here $\otimes$ *denotes the Kronecker product.)*

Proof. This follows immediately from basic rules on the Kronecker product (see, e.g., [8]), since for $\tau \geq 0$

$$\begin{aligned} m^+(\tau) = ce^{\tau A}b &= (c_1 \otimes c_2)e^{\tau(A_1\otimes I_{n_2} + I_{n_1}\otimes A_2)}(b_1 \otimes b_2) \\ &= (c_1 \otimes c_2)(e^{\tau A_1} \otimes e^{\tau A_2})(b_1 \otimes b_2) = c_1 e^{\tau A_1} b_1 \otimes c_2 e^{\tau A_2} b_2 = c_1 e^{\tau A_1} b_1 c_2 e^{\tau A_2} b_2 \\ &= m_1^+(\tau) m_2^+(\tau). \qquad \square \end{aligned}$$

The proposition is of interest in its own right, as it allows one to find state-space formulas for products of impulse response functions.

Summarizing, we have the following result.

PROPOSITION 5.3. *Let* ρ_1 *and* ρ_2 *be unnormalized rational probability densities whose spectral summands* Z_1 *and* Z_2 *have the* n_1*-dimensional and* n_2*-dimensional state-space realizations* (A_1, b_1, c_1) *and* (A_2, b_2, c_2)*. Then the density summand* Z *of the convolution* $\rho = \rho_1 * \rho_2$ *has the state-space realization*

$$\left(\begin{array}{c|c} A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 & b_1 \otimes b_2 \\ \hline c_1 \otimes c_2 & 0 \end{array}\right). \tag{1}$$

Proof. Suppose Z has the realization (1). Then the inverse Fourier transform of Z is $m_1^+ m_2^+$, showing that Z is the spectral summand of ρ. $\square$

Note that the state-space dimension of this realization is $n_1 n_2$, which implies that the McMillan degree of Z is at most $n_1 n_2$.

6. State-space expressions for the filtering equations. We are now in a position to derive state-space expressions for the unnormalized conditional densities in the filter equations which were discussed in the introduction.

THEOREM 6.1. *Assume the notation and assumptions for the filtering problem as presented in the introduction.*

Let $t \geq 0$*, and let* $(A_{x_{t|t-1}}, b_{x_{t|t-1}}, c_{x_{t|t-1}})$ *be a minimal* n_{x_t}*-dimensional state-space realization of the density summand of the unnormalized conditional density* $\rho_{x_t|\mathcal{Y}_{t-1}}$*. For* $t = 0$*, set* $\rho_{x_t|\mathcal{Y}_{t-1}} := \rho_{x_0}$*, the density of the initial state* x_0*. Let* $(A_{\eta_t}, b_{\eta_t}, c_{\eta_t})$ *be a minimal* n_{η_t}*-dimensional state-space realization of the density summand of the unnormalized rational density* ρ_{η_t} *of the noise random variable* η_t*, and let* $(A_{\epsilon_t}, b_{\epsilon_t}, c_{\epsilon_t})$ *be a minimal* η_{ϵ_t}*-dimensional state-space realization of the density summand of the unnormalized rational density* ρ_{ϵ_t} *of the noise random variable* ϵ_t*,* $t \geq 0$*.*

Let T_1 *be the unique solution to the equation*

$$\left(\frac{1}{h_t} A_{\epsilon_t} + i y_t I\right) T_1 + T_1 A_{x_{t|t-1}} + b_{\epsilon_t} c_{x_{t|t-1}} = 0,$$

and let T_2 *be the unique solution to the equation*

$$A_{x_{t|t-1}} T_2 + T_2 \left(\frac{1}{h_t} A_{\epsilon_t} + i y_t\right) + b_{x_{t|t-1}} c_{\epsilon_t} = 0.$$

Then the density summand of the unnormalized density $\rho_{x_t|\mathcal{Y}_t}$ has state-space realization

$$\left(\begin{array}{c|c} A_{x_{t|t}} & b_{x_{t|t}} \\ \hline c_{x_{t|t}} & 0 \end{array}\right) = \left(\begin{array}{cc|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t}c_{x_{t|t}} & T_2^*c^*_{x_{t|t}} \\ 0 & A_{x_{t|t}} & t_{x_{t|t}} \\ \hline b^*_{\epsilon_t} & c_{\epsilon_t}T_1 & 0 \end{array}\right).$$

The density summand of $\rho_{x_{t+1}|\mathcal{Y}_t}$ has state-space realization

$$\begin{aligned} &\left(\begin{array}{c|c} A_{x_{t+1|t}} & b_{x_{t+1|t}} \\ \hline c_{x_{t+1|t}} & 0 \end{array}\right) \\ &= \left(\begin{array}{c|c} f_tA_{x_{t|t}} \otimes I_{n_{\eta_t}} + I_{n_{\epsilon_t}+n_{x_t}} \otimes A_{\eta_t} & b_{x_{t|t}} \otimes b_{n_{\eta_t}} \\ \hline c_{x_{t|t}} \otimes c_{\eta_t} & 0 \end{array}\right) \quad \text{if } f_t > 0, \\ &= \left(\begin{array}{c|c} -f_tA^*_{x_{t|t}} \otimes I_{n_{\eta_t}} + I_{n_{\epsilon_t}+n_{x_t}} \otimes A_{\eta_t} & c^*_{x_{t|t}} \otimes b_{n_{\eta_t}} \\ \hline b^*_{x_{t|t}} \otimes c_{\eta_t} & 0 \end{array}\right) \quad \text{if } f_t < 0. \end{aligned}$$

Proof. Since by assumption $h_t > 0$, the density summand of the density $q(x) = \rho_{\epsilon_t}(y_t - h_tx)$, $x \in \Re$, has the realization

$$\left(\begin{array}{c|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t} \\ \hline b^*_{\epsilon_t} & 0 \end{array}\right).$$

As

$$\rho_{x_t|\mathcal{Y}_t}(x) = \rho_{\epsilon_t}(y_t - h_tx)\rho_{x_t|\mathcal{Y}_{t-1}}(x), \quad x \in \Re,$$

by Proposition 5.1 the density summand of ρ has the realization

$$\left(\begin{array}{cc|c} \frac{1}{h_t}A^*_{\epsilon_t} - iy_tI & c^*_{\epsilon_t}c_{x_{t|t-1}} & T_2^*c^*_{x_{t|t-1}} \\ 0 & A_{x_{t|t-1}} & b_{x_{t|t-1}} \\ \hline b^*_{\epsilon_t} & c_{\epsilon_t}T_1 & 0 \end{array}\right),$$

where T_1 is the unique solution to the equation

$$\begin{aligned} &\left(\frac{1}{h_t}A^*_{\epsilon_t} - iy_tI\right)^* T_1 + T_1A_{x_{t|t-1}} + b_{\epsilon_t}c_{x_{t|t-1}} \\ &\quad = \left(\frac{1}{h_t}A_{\epsilon_t} + iy_tI\right)T_1 + T_1A_{x_{t|t-1}} + b_{\epsilon_t}c_{x_{t|t-1}} = 0, \end{aligned}$$

and T_2 is the unique solution to the equation

$$\begin{aligned} &A_{x_{t|t-1}}T_2 + T_2\left(\frac{1}{h_t}A^*_{\epsilon_t} - iy_tI\right)^* + b_{x_{t|t-1}}c_{\epsilon_t} \\ &\quad = A_{x_{t|t-1}}T_2 + T_2\left(\frac{1}{h_t}A_{\epsilon_t} + iy_tI\right) + b_{x_{t|t-1}}c_{\epsilon_t} = 0. \end{aligned}$$

To obtain a state-space formula for the prediction step

$$\rho_{x_{t+1}|\mathcal{Y}_t} = \rho_{f_tx_t|\mathcal{Y}_t} * \rho_{\eta_t},$$

we use Proposition 5.3. We need to introduce two cases depending on the sign of f_t. If $f_t > 0$, the density summand of $\rho_{f_t x_t|\mathcal{Y}_t}$ has the realization

$$\left(\begin{array}{c|c} f_t A_{x_{t|t}} & b_{x_{t|t}} \\ \hline c_{x_{t|t}} & 0 \end{array}\right).$$

If $f_t < 0$, the density summand of $\rho_{f_t x_t|\mathcal{Y}_t}$ has the realization

$$\left(\begin{array}{c|c} -f_t A^*_{x_{t|t}} & c^*_{x_{t|t}} \\ \hline b^*_{x_{t|t}} & 0 \end{array}\right).$$

The remaining parts of the result now follow by Proposition 5.3. $\square$

It should be noted that the presented state-space realizations are data dependent and, in particular, dependent on $\mathcal{Y}_t$. As the formulae that use Kronecker products show, the dimensions of the state-space representation can potentially grow very quickly as the number of data points increases. It should be pointed out, however, that the growth in complexity is inherent in the use of random variables with rational densities (see also [11]). If, however, the density summand corresponding to η_t has only McMillan degree 1, i.e., η_t has Cauchy distribution, then the Kronecker products reduce to standard multiplication and the prediction step does not lead to an increase in dimension. Also, if the density summand corresponding to ϵ_t has McMillan degree 1, i.e., ϵ_t has Cauchy distribution, then the matrix equations can be solved explicitly to give

$$T_1 = -b_{\epsilon_t} c_{x_{t|t-1}} \left(\left(\frac{1}{h_t} A_{\epsilon_t} + i y_t \right) I + A_{x_{t|t-1}} \right)^{-1},$$

$$T_2 = -\left(\left(\frac{1}{h_t} A_{\epsilon_t} + i y_t \right) I + A_{x_{t|t-1}} \right)^{-1} b_{x_{t|t-1}} c_{\epsilon_t}.$$

Note that the inverse exists, since $A_{x_{t|t-1}}$ has all eigenvalues in the open left half plane and $\frac{1}{h_t} A_{\epsilon_t} + i y_t$ has negative real part, because of the stability of A_{ϵ_t} and since $h_t > 0$. From the remark after Proposition 5.1, it follows that the conditional mean $E(x_t|\mathcal{Y}_t)$ and the corresponding conditional variance $E(x_t - E(x_t|\mathcal{Y}_t)^2|\mathcal{Y}_t)$ exist and can be calculated from the density summand realization $(A_{x_{t|t}}, b_{x_{t|t}}, c_{x_{t|t}})$ using the formulas given in Theorem 4.1.

Note that prediction is also possible using the formulas presented here. For example, the unnormalized rational conditional probability density of the output variable at time $t+1$, given the observations of the output until time t, is equal to $\rho_{y_{t+1|t}}(y) = \rho_{h_{t+1} x_{t+1|t}} * \rho_{\epsilon_{t+1}}$, and the spectral summand of this density can be calculated using the formulas of section 5.

7. Conclusions.

State-space formulae have been developed for various operations on rational density functions, and it is shown how this can be used to treat the filtering problem in the case of a first order linear stochastic model with stochastically independent noise variables with rational probability densities and stochastically independent initial state with rational probability density. This makes such filters easy to program on present-day computers, using, e.g., a linear algebra package. If the number of observations is not very small, however, the order of the conditional rational densities will tend to grow quickly. Therefore, various schemes of order reduction for positive real functions may be of relevance in practical applications (see, e.g.,

[2]). The formulae presented can also be used for further theoretical research in the behavior of the optimal filter. It follows, for example, that the conditional mean of the present state, given present and past observations, is a rational function of the present and past observations, which could be further investigated. The formula that is presented for the realization of the product of impulse response functions appears to be important in its own right.

REFERENCES

[1] F. R. Gantmacher, *Matrix Theory*, Chelsea, New York, 1959.
[2] B. Hanzon and R. J. Ober, *Overlapping balanced canonical forms for various classes of linear systems*, Linear Algebra Appl., 281 (1998), pp. 171–225.
[3] B. Hanzon and R. L. M. Peeters, *A Faddeev sequence method for solving Lyapunov and Sylvester equations*, Linear Algebra Appl., 241–243 (1996), pp. 401–430.
[4] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.
[5] K. Koedijk, M. Schafgans, and C. de Vries, *The tail index of exchange rate returns*, J. International Economics, 29 (1990), pp. 93–108.
[6] T. Körner, *Fourier Analysis*, Cambridge University Press, Cambridge, UK, 1989.
[7] B. Mandelbrot, *The variation of certain speculative prices*, J. Business, 36 (1963), pp. 394–419.
[8] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed., Academic Press, New York, 1985.
[9] W. Rudin, *Real and Complex Analysis*, McGraw Hill, New York, 1987.
[10] W. J. Rugh, *Linear System Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 1996.
[11] I. J. Steyn, *State Space Models in Econometrics*, Ph.D. thesis, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands, 1996.
[12] S. J. Taylor, *Modelling Financial Time Series*, Wiley, New York, 1986.