Calculation of the Fisher Information Matrix for Multidimensional Data Sets

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***Abstract*—Data sets that are acquired in many practical systems can be described as the output of a multidimensional linear separable-denominator system with Gaussian measurement noise. An important example is nuclear magnetic resonance (NMR) spectroscopy. In NMR spectroscopy, high-accuracy parameter estimation is of central importance. A classical result on the Cramér–Rao lower bound states that the inverse of the Fisher information matrix (FIM) provides a lower bound for the covariance of any unbiased estimator of the parameter vector. The calculation of the FIM is therefore of central importance for an assessment of the accuracy with which parameters can be estimated. It is shown how the FIM can be expressed using the matrices that determine the system that generates the data set. For uniformly sampled data, it is shown how the FIM can be expressed through the solutions of Lyapunov equations. The novel techniques are demonstrated with an example arising from NMR spectroscopy.**

***Index Terms*—Cramér–Rao lower bound, Fisher information matrix, Lyapunov equation, multidimensional linear systems, NMR spectroscopy, parameter estimation.**

I. Introduction

DATA that can be considered to be generated by a multidimensional linear separable-denominator continuous system appears in many areas of applications. For example, in nuclear magnetic resonance (NMR) spectroscopy, it can be shown ([14] and see, e.g., [2] and [6] for general references) that the data of a so-called two-dimensional (2-D) NMR experiment typically has the form

$$y(t_1, t_2) = C_1 e^{A_{11} t_1} F + C_1 e^{A_{11} t_1} A_{12} e^{A_{22} t_2} B_2 \qquad t_1, t_2 \geq 0 \tag{1}$$

where A_{11}, A_{12}, A_{22}, C_1, B_2, and F are matrices of compatible sizes. The fundamental problem in NMR spectroscopy is that the system matrices are dependent on parameters (e.g., the resonant frequencies of the magnet spins) that need to be estimated through the experiment.

Estimation of parameters that determine dynamic data is a frequently encountered problem in many areas of applications. The question therefore naturally arises as to the accuracy with which these parameters can be estimated. The Cramér–Rao lower bound (CRLB) gives a lower bound for the covariance of the parameter estimates of an unbiased estimation procedure for a given data set [5], [9], [17]. The CRLB is in fact typically calculated as the inverse of a matrix called the Fisher information matrix (FIM). The relevance of this result is not only to evaluate a particular estimation procedure but can also give guidance for an appropriate design of an experiment to collect data (see, e.g., [13]). In many experimental situations, there is a limit on the number of data points that can be acquired. For example, in clinical trials of drugs, patients cannot be subjected to an arbitrary number of blood tests. It is therefore important to develop a strategy for experiment design that is likely to produce good quality parameter estimates while keeping the number of data samples low.

In [13], the FIM was derived for a concrete problem arising in NMR spectroscopy. The approach taken there was to derive the FIM from first principle using "hand calculations" to perform the derivatives that lead to the analytical expressions for each entry of the matrix. This is a time-consuming and cumbersome process that needs to be repeated for each, possibly minor, modification of the data model, which makes it impossible to use the technique in a routine manner in the applications at hand. The purpose of the current paper is to develop techniques that allow for a more systematic derivation of the FIM for data that arises as the output of a multidimensional separable-denominator continuous system.

Expressions for the FIM in system theoretic terms have appeared in the literature before in the context of the modeling of stationary time series [10], [15], [18]. Recently, a systematic investigation of the CRLB or FIM for the case of one-dimensional (1-D) deterministic dynamic systems corrupted by measurement noise is presented in [12]. The classes of 1-D data discussed in [12] include data of the form given by

$$y(t) = ce^{At}b, \qquad t \geq 0.$$

The calculation of the FIM for the above 1-D data plus measurement noise was done in terms of a derivative system and by using the solution of a Lyapunov equation. In this paper, we generalize the results of [12] to 2-D data described by (1), using 2-D system theoretic methods. This generalization is, however, not a straightforward extension of the results in [12] due to the significantly more intricate structure of 2-D systems.

This is not the first publication to address the derivation of the CRLB for 2-D signals. Various methods have been suggested for the computation of the CRLB for 2-D parameter estimation problem of undamped exponential signals with additive noise

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(see, e.g., [3], [7], and [16]). However, to our best knowledge, a closed-form expression for the CRLB for the parameter estimation problem for 2-D damped exponential signals is not available in the literature, although its 1-D counterpart was investigated two decades ago in [11]. Moreover, mixed signals consisting of 1-D and 2-D terms like those in (1) have seldom been studied in the signal processing community, although they do arise in many practical situations.

To derive a systematic approach to data sets described in (1), we consider a 2-D complex single-input single-output continuous system with a separable denominator using Roesser's model (RM)

$$\begin{bmatrix} \frac{\partial}{\partial t_1} x_\theta^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x_\theta^v(t_1, t_2) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_\theta^h(t_1, t_2) \\ x_\theta^v(t_1, t_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1, t_2) \tag{2}$$

$$y_\theta(t_1, t_2) = [C_1 \quad C_2] \begin{bmatrix} x_\theta^h(t_1, t_2) \\ x_\theta^v(t_1, t_2) \end{bmatrix} + Du(t_1, t_2), \quad t_1 \geq 0, t_2 \geq 0 \tag{3}$$

where $A_{11}, A_{12}, A_{22}, B_1, B_2, C_1, C_2$, and D are complex matrices of appropriate dimensions, depending on the unknown parameter vector $\Theta := [\theta_1 \quad \theta_2 \quad \cdots \quad \theta_K]^T$, $x_\theta^h(t_1, t_2)$, and $x_\theta^v(t_1, t_2)$ are horizontal and vertical state vectors, respectively, $u(t_1, t_2)$ is the input, and $\partial/\partial t_j$ denotes partial derivative with respect to t_j $(j = 1, 2)$. The boundary conditions are given by

$$x_\theta^h(0, t_2), \quad x_\theta^v(t_1, 0), \qquad t_1 \geq 0, t_2 \geq 0.$$

In the following lemma, we characterize the input–output description of such a system. See Appendix A for a proof.

Lemma 1.1: The output of the above 2-D separable-denominator continuous system is given by

$$y_\theta(t_1, t_2) = v_\theta(t_1, t_2) + q_\theta(t_1, t_2), \qquad t_1 \geq 0, t_2 \geq 0.$$

Here, $v_\theta(t_1, t_2)$ is the system response due to nonzero boundary conditions, and $q_\theta(t_1, t_2)$ is the system response due to system input, which are given by

$$v_\theta(t_1, t_2) = C_1 e^{A_{11}t_1} x_\theta^h(0, t_2) + C_2 e^{A_{22}t_2} x_\theta^v(t_1, 0) + \int_0^{t_1} C_1 e^{A_{11}(t_1-\tau_1)} A_{12} e^{A_{22}t_2} x_\theta^v(\tau_1, 0)\, d\tau_1$$

and

$$\begin{aligned} q_\theta(t_1, t_2) = &\int_0^{t_1} C_1 e^{A_{11}(t_1-\tau_1)} B_1 u(\tau_1, t_2)\, d\tau_1 \\ &+ \int_0^{t_2} C_2 e^{A_{22}(t_2-\tau_2)} B_2 u(t_1, \tau_2)\, d\tau_2 \\ &+ \int_0^{t_1}\int_0^{t_2} C_1 e^{A_{11}(t_1-\tau_1)} A_{12} e^{A_{22}(t_2-\tau_2)} \\ &\cdot B_2 u(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 + Du(t_1, t_2). \end{aligned}$$

Note that the data model that motivated our study in (1) can be seen to be the output of such a system if we set $u(t_1, t_2) = \delta(t_1, t_2)$, which is the 2-D unit impulse function, $x_\theta^v(t_1, 0) = 0$, $x_\theta^h(0, t_2) = F$, $B_1 = 0$, $C_2 = 0$, and $D = 0$.

Assume that we have acquired noise corrupted samples $s_\theta(n, m)$ $(n = 0, 1, \ldots, N-1; m = 0, 1, \ldots, M-1)$ of the measured output of a 2-D separable-denominator continuous system at various points (t_{1n}, t_{2m}), i.e.,

$$s_\theta(n, m) = y_\theta(t_{1n}, t_{2m}) + w(n, m)$$

where $y_\theta(t_{1n}, t_{2m})$ is the noise free data acquired at the sampling point (t_{1n}, t_{2m}) $(n = 0, 1, \ldots, N-1;\ m = 0, 1, \ldots, M-1)$, and $w(n, m)$ is the measurement noise component assumed to be complex Gaussian with zero mean. The real and imaginary parts of $w(n, m)$ are assumed to have variance $\sigma_{n,m}^2$ and to be independent/uncorrelated, i.e., var(Re$\{w(n, m)\}$) = var(Im$\{w(n, m)\}$) = $\sigma_{n,m}^2$, and E(Re$\{w(n, m)\}$Im$\{w(n, m)\}$) = 0. Note that one of the main results (Theorem 2.1) will deal with the general noise model introduced here in which the variance depends on the indices n and m. For some later results, we will, however, assume that the variance is uniform for all data points, i.e., $\sigma := \sigma_{n,m}$, $n, m \geq 0$. The general noise model was used in our earlier paper [13] on NMR spectroscopy, where a particular experiment could be interpreted as dealing with different variance levels (at least in one of the two dimensions).

By the Cramér–Rao lower bound [9], [17], any unbiased estimator $\hat{\Theta}$ of Θ has a variance (provided certain regularity conditions hold) such that

$$\text{var}\left(\hat{\Theta}\right) \geq I^{-1}(\Theta)$$

where var$(\hat{\Theta}) \geq I^{-1}(\Theta)$ is interpreted as meaning that the matrix (var$(\hat{\Theta}) - I^{-1}(\Theta)$) is positive semidefinite [9]. Here, $I(\Theta)$ is the FIM given by

$$[I(\Theta)]_{st} = -E\left(\frac{\partial^2 \ln(p(S;\, \Theta))}{\partial\theta_s\, \partial\theta_t}\right), \qquad 1 \leq s, t \leq K$$

where Θ is the unknown parameter vector, S is the measured data set, $p(S;\, \Theta)$ is the probability density function of the measurements, and $E(\cdot)$ is the expected value with respect to the underlying probability measure.

In Section II, we discuss the derivation of the FIM for the data set generated from a 2-D separable-denominator continuous system model. For the special but important case of uniformly sampled data, we show in Section III that the computation of the FIM of a 2-D separable-denominator continuous system can be reduced to the computation of solutions to certain Lyapunov equations. The techniques introduced in this paper are then illustrated with an example that is motivated by NMR spectroscopy in Section IV. The results and approaches obtained here are compared with those in [13]. In particular, the differences between finite and infinite data sets are discussed. Finally, a conclusion is presented in Section V.

We denote by diag$(M_1, M_2, \ldots, M_r)$ the block diagonal matrix whose diagonal block entries are $M_1, M_2, \ldots, M_r$, and all other block entries are zero matrices. Throughout the paper, the phrase "2-D system" refers to the 2-D separable-denominator continuous system described by (2) and (3), and "noise" refers to the complex Gaussian noise with zero mean whose real and imaginary parts are assumed to have variance $\sigma_{n,m}^2$ and to be uncorrelated.

II. Fisher Information Matrix

In this section, we are going to derive an expression for the FIM $I(\Theta)$ for the parameter estimation problem for a 2-D data set with Gaussian measurement noise discussed in the previous section. The data sampling scheme employed for obtaining the data samples can be either uniform sampling or nonuniform sampling in this section. With the Gaussian noise model discussed in Section I, the probability density function is given by

$$p(S;\,\Theta) = \Pi_{n=0}^{N-1}\Pi_{m=0}^{M-1}\frac{1}{2\pi\sigma_{n,m}^2}\exp\Bigg(-\frac{1}{2\sigma_{n,m}^2}\cdot\Big[(\mathrm{Re}\{s_\theta(n,\,m)\}-\mathrm{Re}\{y_\theta(t_{1n},\,t_{2m})\})^2 + (\mathrm{Im}\{s_\theta(n,\,m)\}-\mathrm{Im}\{y_\theta(t_{1n},\,t_{2m})\})^2\Big]\Bigg).$$

In the following lemma, we are going to collect some basic results on the FIM adapted to the particular data model that we consider (see e.g., [9]).

Lemma 2.1:

1) For $1 \le s,\, t \le K$

$$[I(\Theta)]_{st} = -E\left(\frac{\partial^2 \ln(p(S;\,\Theta))}{\partial\theta_s\,\partial\theta_t}\right) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\frac{1}{\sigma_{n,m}^2}\cdot\mathrm{Re}\left\{\frac{\partial y_\theta(t_{1n},\,t_{2m})}{\partial\theta_s}\,\frac{\partial y_\theta^*(t_{1n},\,t_{2m})}{\partial\theta_t}\right\}$$

where $(\cdot)^*$ denotes conjugate.

2) Let

$$D_{y_\theta(t_{1n},\,t_{2m})} := \begin{bmatrix}\frac{\partial y_\theta(t_{1n},\,t_{2m})}{\partial\theta_1}\\ \frac{\partial y_\theta(t_{1n},\,t_{2m})}{\partial\theta_2}\\ \vdots \\ \frac{\partial y_\theta(t_{1n},\,t_{2m})}{\partial\theta_K}\end{bmatrix}.$$

Then

$$I(\Theta) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\frac{1}{\sigma_{n,m}^2}\,\mathrm{Re}\left\{D_{y_\theta(t_{1n},\,t_{2m})}D^H_{y_\theta(t_{1n},\,t_{2m})}\right\}$$

where $(\cdot)^H$ denotes complex conjugate transpose.

In order to calculate the FIM, it is necessary to compute the derivative $\frac{\partial y_\theta(t_{1n},\,t_{2m})}{\partial\theta_s}$ of the output with respect to the elements θ_s of the parameter vector Θ, $s = 1,\,\ldots,\,K$. This can be done either on $y_\theta(t_{1n},\,t_{2m})$ directly [13] or on the state-space realization of $y_\theta(t_{1n},\,t_{2m})$, as to be done in the following. We first quote a lemma from [12] that is essentially the continuous time equivalent of [5, Lemma 5.2-30].

Lemma 2.2: Let A be any complex square matrix depending on the parameter vector Θ, and denote for $s = 1,\,\ldots,\,K, t \ge 0$

$$\partial_s(e^{At}) := \begin{bmatrix} e^{At} & 0 \\ \frac{\partial e^{At}}{\partial\theta_s} & e^{At}\end{bmatrix},\quad \partial_s A := \begin{bmatrix} A & 0\\ \frac{\partial A}{\partial\theta_s} & A\end{bmatrix}.$$

We have for $s = 1,\,\ldots,\,K$ and $t \ge 0$,

$$\partial_s(e^{At}) = e^{\partial_s At}.$$

In the following lemma, we consider the derivative of a product of matrices depending on the parameter vector Θ, which can be regarded as an extension of a lemma for the calculation of the derivative of the 1-D function $y(t) = ce^{At}b$ in [12]. A proof is given in Appendix B.

Lemma 2.3: Consider the matrix product $H_1H_2\ldots H_l$, where $H_1, H_2, \ldots, H_l$ are matrices depending on the parameter vector Θ. Then

$$\frac{\partial(H_1H_2\cdots H_l)}{\partial\theta_s} = \begin{bmatrix}\frac{\partial H_1}{\partial\theta_s} & H_1\end{bmatrix}\begin{bmatrix} H_2 & 0\\ \frac{\partial H_2}{\partial\theta_s} & H_2\end{bmatrix}\cdots\begin{bmatrix} H_{l-1} & 0\\ \frac{\partial H_{l-1}}{\partial\theta_s} & H_{l-1}\end{bmatrix}\begin{bmatrix} H_l\\ \frac{\partial H_l}{\partial\theta_s}\end{bmatrix}.$$

In the following lemma, we can now give the desired system theoretic expression of the derivative of the output with respect to the elements of the parameter vector.

Lemma 2.4: With the notations in Section I, consider for $t_1 \ge 0,\, t_2 \ge 0$

$$y_\theta(t_1,\,t_2) = v_\theta(t_1,\,t_2) + q_\theta(t_1,\,t_2).$$

Then, for $1 \le s \le K$

$$\frac{\partial y_\theta(t_1,\,t_2)}{\partial\theta_s} = \frac{\partial v_\theta(t_1,\,t_2)}{\partial\theta_s} + \frac{\partial q_\theta(t_1,\,t_2)}{\partial\theta_s}$$

where

$$\frac{\partial v_\theta(t_1,\,t_2)}{\partial\theta_s} = \partial_s C_1 e^{\partial_s A_{11}t_1}\partial_s x_\theta^h(0,\,t_2) + \partial_s C_2 e^{\partial_s A_{22}t_2}\partial_s x_\theta^v(t_1,\,0) + \int_0^{t_1}\partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s A_{12} e^{\partial_s A_{22}t_2}\cdot\partial_s x_\theta^v(\tau_1,\,0)\,d\tau_1$$

and

1) for bounded piecewise continuous input $u(t_1,\,t_2)$

$$\frac{\partial q_\theta(t_1,\,t_2)}{\partial\theta_s} = \int_0^{t_1}\partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s B_1 u(\tau_1,\,t_2)\,d\tau_1 + \int_0^{t_2}\partial_s C_2 e^{\partial_s A_{22}(t_2-\tau_2)}\partial_s B_2 u(t_1,\,\tau_2)\,d\tau_2 + \int_0^{t_1}\int_0^{t_2}\partial_s C_1 e^{\partial_s A_{11}(t_1-\tau_1)}\partial_s A_{12} e^{\partial_s A_{22}(t_2-\tau_2)}\cdot\partial_s B_2 u(\tau_1,\tau_2)\,d\tau_1\,d\tau_2 + \partial_s D u(t_1,\,t_2);$$

2) for impulse response input $u(t_1,\,t_2) = \delta(t_1,\,t_2)$, with $B_1 = 0$, $C_2 = 0$ and $D = 0$

$$\frac{\partial q_\theta(t_1,\,t_2)}{\partial\theta_s} = \partial_s C_1 e^{\partial_s A_{11}t_1}\partial_s A_{12} e^{\partial_s A_{22}t_2}\partial_s B_2.$$

Here

$$\partial_s C_1 := \begin{bmatrix} \frac{\partial C_1}{\partial \theta_s} & C_1 \end{bmatrix}, \qquad \partial_s C_2 := \begin{bmatrix} \frac{\partial C_2}{\partial \theta_s} & C_2 \end{bmatrix}$$

$$\partial_s D := \frac{\partial D}{\partial \theta_s}, \qquad \partial_s A_{11} := \begin{bmatrix} A_{11} & 0 \\ \frac{\partial A_{11}}{\partial \theta_s} & A_{11} \end{bmatrix}$$

$$\partial_s A_{12} := \begin{bmatrix} A_{12} & 0 \\ \frac{\partial A_{12}}{\partial \theta_s} & A_{12} \end{bmatrix}, \quad \partial_s A_{22} := \begin{bmatrix} A_{22} & 0 \\ \frac{\partial A_{22}}{\partial \theta_s} & A_{22} \end{bmatrix}$$

$$\partial_s B_1 := \begin{bmatrix} B_1 \\ \frac{\partial B_1}{\partial \theta_s} \end{bmatrix}, \qquad \partial_s B_2 := \begin{bmatrix} B_2 \\ \frac{\partial B_2}{\partial \theta_s} \end{bmatrix}$$

$$\partial_s x_\theta^v(t_1, 0) := \begin{bmatrix} x_\theta^v(t_1, 0) \\ \frac{\partial x_\theta^v(t_1, 0)}{\partial \theta_s} \end{bmatrix}, \; \partial_s x_\theta^h(0, t_2) := \begin{bmatrix} x_\theta^h(0, t_2) \\ \frac{\partial x_\theta^h(0, t_2)}{\partial \theta_s} \end{bmatrix}.$$

Proof: The proof of part 1) is an application of Lemmas 2.2 and 2.3 and the fact that derivation and integration can be exchanged since the integrand is bounded and the integration is over a finite interval. The proof of part 2) is a direct consequence of the expression of the impulse response of the system and an application of Lemmas 2.2 and 2.3. □

In the following theorem, we summarize the previous results and state the general expression for the FIM for the data set corresponding to the output of a 2-D separable-denominator continuous system.

Theorem 2.1: Consider the augmented derivative system given by

$$\begin{aligned} D_{A_{11}} &:= \text{diag}(\partial_1 A_{11}, \partial_2 A_{11}, \ldots, \partial_K A_{11}) \\ D_{A_{22}} &:= \text{diag}(\partial_1 A_{22}, \partial_2 A_{22}, \ldots, \partial_K A_{22}) \\ D_{A_{12}} &:= \text{diag}(\partial_1 A_{12}, \partial_2 A_{12}, \ldots, \partial_K A_{12}) \\ D_{C_1} &:= \text{diag}(\partial_1 C_1, \partial_2 C_1, \ldots, \partial_K C_1) \\ D_{C_2} &:= \text{diag}(\partial_1 C_2, \partial_2 C_2, \ldots, \partial_K C_2) \end{aligned}$$

$$D_{B_1} := \begin{bmatrix} \partial_1 B_1 \\ \partial_2 B_1 \\ \vdots \\ \partial_K B_1 \end{bmatrix}, \; D_{B_2} := \begin{bmatrix} \partial_1 B_2 \\ \partial_2 B_2 \\ \vdots \\ \partial_K B_2 \end{bmatrix}$$

$$D_D := \begin{bmatrix} \partial_1 D \\ \partial_2 D \\ \vdots \\ \partial_K D \end{bmatrix}, \; D_{x_\theta^h(0, t_2)} := \begin{bmatrix} \partial_1 x_\theta^h(0, t_2) \\ \partial_2 x_\theta^h(0, t_2) \\ \vdots \\ \partial_K x_\theta^h(0, t_2) \end{bmatrix}$$

$$D_{x_\theta^v(t_1, 0)} := \begin{bmatrix} \partial_1 x_\theta^v(t_1, 0) \\ \partial_2 x_\theta^v(t_1, 0) \\ \vdots \\ \partial_K x_\theta^v(t_1, 0) \end{bmatrix}.$$

Then

1) for $t_1 \geq 0$, $t_2 \geq 0$

$$D_{y_\theta(t_1, t_2)} := \begin{bmatrix} \frac{\partial y_\theta(t_1, t_2)}{\partial \theta_1} \\ \frac{\partial y_\theta(t_1, t_2)}{\partial \theta_2} \\ \vdots \\ \frac{\partial y_\theta(t_1, t_2)}{\partial \theta_K} \end{bmatrix} = D_{v_\theta(t_1, t_2)} + D_{q_\theta(t_1, t_2)}$$

where

$$\begin{aligned} D_{v_\theta(t_1, t_2)} = {} & D_{C_1} e^{D_{A_{11}} t_1} D_{x_\theta^h(0, t_2)} + D_{C_2} e^{D_{A_{22}} t_2} D_{x_\theta^v(t_1, 0)} \\ & + \int_0^{t_1} D_{C_1} e^{D_{A_{11}}(t_1 - \tau_1)} D_{A_{12}} e^{D_{A_{22}} t_2} D_{x_\theta^v(\tau_1, 0)} \, d\tau_1 \end{aligned}$$

and

a) for bounded piecewise continuous input $u(t_1, t_2)$

$$\begin{aligned} D_{q_\theta(t_1, t_2)} = {} & \int_0^{t_1} D_{C_1} e^{D_{A_{11}}(t_1 - \tau_1)} D_{B_1} u(\tau_1, t_2) \, d\tau_1 \\ & + \int_0^{t_2} D_{C_2} e^{D_{A_{22}}(t_2 - \tau_2)} D_{B_2} u(t_1, \tau_2) \, d\tau_2 \\ & + \int_0^{t_1} \int_0^{t_2} D_{C_1} e^{D_{A_{11}}(t_1 - \tau_1)} D_{A_{12}} e^{D_{A_{22}}(t_2 - \tau_2)} \\ & \cdot D_{B_2} u(\tau_1, \tau_2) \, d\tau_1 \, d\tau_2 + D_D u(t_1, t_2); \end{aligned}$$

b) for impulse response input $u(t_1, t_2) = \delta(t_1, t_2)$, with $B_1 = 0$, $C_2 = 0$ and $D = 0$

$$D_{q_\theta(t_1, t_2)} = D_{C_1} e^{D_{A_{11}} t_1} D_{A_{12}} e^{D_{A_{22}} t_2} D_{B_2};$$

2) for the 2-D data set sampled at (t_{1n}, t_{2m}) ($n = 0, 1, \ldots, N-1$; $m = 0, 1, \ldots, M-1$), the FIM is

$$I(\Theta) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \frac{1}{\sigma_{n,m}^2} \text{Re}\left\{ D_{y_\theta(t_{1n}, t_{2m})} D_{y_\theta(t_{1n}, t_{2m})}^H \right\}.$$

Proof: Item 1) follows immediately from Lemma 2.4 by stacking up the variables for the system. Item 2) is the content of Lemma 2.1. □

In the next theorem, we are going to give a more explicit expression for the FIM for the type of 2-D signals introduced in (1) in Section I.

Theorem 2.2: Assume that the 2-D system is such that we have the following:

1) A finite number of samples of the output are acquired in both the t_1 and t_2 variables, i.e., at (t_{1n}, t_{2m}) ($n = 0, 1, \ldots, N-1$; $m = 0, 1, \ldots, M-1$).
2) The input $u(t_1, t_2)$ is a 2-D unit impulse function, i.e., $u(t_1, t_2) = \delta(t_1, t_2)$. The boundary conditions are given by $x_\theta^v(t_1, 0) = 0$ and $x_\theta^h(0, t_2) = F$, where F may depend on the parameter vector Θ. The matrices B_1, C_2, and D are all zero matrices. These assumptions

imply that the deterministic part of the measured signal is given by

$$y(t_{1n}, t_{2m}) = C_1 e^{A_{11}t_{1n}} F + C_1 e^{A_{11}t_{1n}} A_{12} e^{A_{22}t_{2m}} B_2$$
$$n = 0, 1, \ldots, N-1;\ m = 0, 1, \ldots, M-1. \quad (4)$$

Then, the FIM for the above data set with noise variance $\sigma^2_{n,m} =: \sigma^2, n \geq 0; m \geq 0$ is given by

$$\begin{aligned} I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re} \Bigg\{ D_{C_1} \Bigg[& M \sum_{n=0}^{N-1} e^{D_{A_{11}}t_{1n}} D_F D_F^H (e^{D_{A_{11}}t_{1n}})^H \\ & + \sum_{n=0}^{N-1} e^{D_{A_{11}}t_{1n}} \Bigg(\sum_{m=0}^{M-1} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} D_F^H \\ & + \sum_{m=0}^{M-1} D_F D_{B_2}^H (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H \Bigg) (e^{D_{A_{11}}t_{1n}})^H \\ & + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} \\ & \cdot D_{B_2}^H (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H (e^{D_{A_{11}}t_{1n}})^H \Bigg] D_{C_1}^H \Bigg\} \end{aligned} \quad (5)$$

where

$$\partial_s F := \begin{bmatrix} F \\ \dfrac{\partial F}{\partial \theta_s} \end{bmatrix}, \quad \text{and} \quad D_F := \begin{bmatrix} \partial_1 F \\ \partial_2 F \\ \vdots \\ \partial_K F \end{bmatrix}.$$

Proof: With the given assumptions and from Theorem 2.1, the FIM can be written as

$$\begin{aligned} I(\Theta) &= \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \frac{1}{\sigma^2_{n,m}} \operatorname{Re} \left\{ D_{y_\theta(t_{1n}, t_{2m})} D^H_{y_\theta(t_{1n}, t_{2m})} \right\} \\ &= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \operatorname{Re} \Big\{ \big(D_{C_1} e^{D_{A_{11}}t_{1n}} D_F \\ &\quad + D_{C_1} e^{D_{A_{11}}t_{1n}} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} \big) \\ &\quad \cdot \big(D_{C_1} e^{D_{A_{11}}t_{1n}} D_F + D_{C_1} e^{D_{A_{11}}t_{1n}} \\ &\quad \cdot D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} \big)^H \Big\} \\ &= \frac{1}{\sigma^2} \operatorname{Re} \Bigg\{ D_{C_1} \Bigg[\sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_F D_F^H (e^{D_{A_{11}}t_{1n}})^H \\ &\quad + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} D_F^H \\ &\quad \cdot (e^{D_{A_{11}}t_{1n}})^H + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_F D_{B_2}^H \\ &\quad \cdot (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H (e^{D_{A_{11}}t_{1n}})^H \\ &\quad + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} \\ &\quad \cdot D_{B_2}^H (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H (e^{D_{A_{11}}t_{1n}})^H \Bigg] D_{C_1}^H \Bigg\} \\ &= \frac{1}{\sigma^2} \operatorname{Re} \Bigg\{ D_{C_1} \Bigg[M \sum_{n=0}^{N-1} e^{D_{A_{11}}t_{1n}} D_F D_F^H (e^{D_{A_{11}}t_{1n}})^H \\ &\quad + \sum_{n=0}^{N-1} e^{D_{A_{11}}t_{1n}} \Bigg(\sum_{m=0}^{M-1} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} D_F^H \\ &\quad + \sum_{m=0}^{M-1} D_F D_{B_2}^H (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H \Bigg) (e^{D_{A_{11}}t_{1n}})^H \\ &\quad + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{D_{A_{11}}t_{1n}} D_{A_{12}} e^{D_{A_{22}}t_{2m}} D_{B_2} D_{B_2}^H \\ &\quad \cdot (e^{D_{A_{22}}t_{2m}})^H D_{A_{12}}^H (e^{D_{A_{11}}t_{1n}})^H \Bigg] D_{C_1}^H \Bigg\}. \end{aligned}$$

□

III. Fisher Information Matrix for Uniformly Sampled 2-D Data

Although Theorem 2.2 in the previous section is valid for both uniform and nonuniform sampling schemes, it is computationally rather inefficient to directly compute the 2-D summations in (5), particularly in the case when the number of samples is large in one or both of the variables. In this section, we develop an efficient method for calculating the FIM for 2-D data generated by uniformly sampling the output of a 2-D separable-denominator continuous system. To this end, it is assumed that all the eigenvalues of $e^{A_{11}T_1}$ and $e^{A_{22}T_2}$ are in the open unit disc or, equivalently, the eigenvalues of A_{11} and A_{22} are in the open half plane, where T_1 and T_2 are the sampling intervals for the variables t_1 and t_2, respectively. Theorem 2.2 can then be simplified significantly with the Lyapunov approach. For convenience of exposition, we denote $A_{d1} := e^{D_{A_{11}}T_1}$ and $A_{d2} := e^{D_{A_{22}}T_2}$. In the following lemma, a standard result on the Lyapunov equation is summarized [8].

Lemma 3.1: Let

$$P = \sum_{n=0}^{N-1} A^n Q (A^n)^H$$

where all the eigenvalues of A are in the open unit disc, and Q is a Hermitian matrix, i.e., $Q = Q^H$. Then, P is the unique solution to the following Lyapunov equation:

$$APA^H - P = -Q + A^N Q (A^N)^H. \quad (6)$$

Moreover

$$P = \sum_{n=0}^{\infty} A^n Q (A^n)^H$$

is the unique solution to the following Lyapunov equation:

$$APA^H - P = -Q.$$

There are standard methods for solving Lyapunov equations in the literature (see, e.g., [1] and [4]). In the following theorem, we are going to characterize the FIM for the data model in (1) through the solutions of Lyapunov equations.

Theorem 3.1: Consider the data model of Theorem 2.2, and assume that the signal is uniformly sampled with sampling interval T_1 for the variable t_1 and T_2 for t_2, respectively, i.e., at $t_{1n} = nT_1$, $n = 0, 1, \ldots, N-1$; $t_{2m} = mT_2$, $m =$

$0, 1, \ldots, M-1$. Moreover, assume that all the eigenvalues of A_{11} and A_{22} are in the open left half plane. Then, the FIM for the 2-D data set is given by

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{ D_{C_1}[MP_1 + P_2 + P_3]D_{C_1}^H \right\}$$

where P_1, P_2, and P_3 can be obtained as follows.

P_1 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_1A_{d1}^H - P_1 = -D_FD_F^H + A_{d1}^N D_F D_F^H (A_{d1}^N)^H.$$

P_2 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_2A_{d1}^H - P_2 = -R + A_{d1}^N R (A_{d1}^N)^H$$

where

$$R = D_{A_{12}}(I - A_{d2}^M)(I - A_{d2})^{-1} D_{B_2} D_F^H + D_F D_{B_2}^H ((I - A_{d2})^{-1})^H (I - A_{d2}^M)^H D_{A_{12}}^H.$$

P_4 is the unique solution to the following Lyapunov equation:

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2} D_{B_2}^H (A_{d2}^M)^H$$

and P_3 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H + A_{d1}^N D_{A_{12}} P_4 D_{A_{12}}^H (A_{d1}^N)^H.$$

Proof: In Theorem 2.2 and the assumption $t_{1n} = nT_1, n = 0, 1, \ldots, N-1; t_{2m} = mT_2, m = 0, 1, \ldots, M-1$, the FIM for the given 2-D data set can be written as

$$\begin{aligned} I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\Bigg\{ D_{C_1} \Bigg[& M \sum_{n=0}^{N-1} A_{d1}^n D_F D_F^H (A_{d1}^n)^H \\ & + \sum_{n=0}^{N-1} A_{d1}^n \left(\sum_{m=0}^{M-1} D_{A_{12}} A_{d2}^m D_{B_2} D_F^H \right. \\ & \left. + \sum_{m=0}^{M-1} D_F D_{B_2}^H (A_{d2}^m)^H D_{A_{12}}^H \right) (A_{d1}^n)^H \\ & + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} A_{d1}^n D_{A_{12}} A_{d2}^m D_{B_2} D_{B_2}^H \\ & \cdot (A_{d2}^m)^H D_{A_{12}}^H (A_{d1}^n)^H \Bigg] D_{C_1}^H \Bigg\} \end{aligned}$$

where $A_{d1} = e^{D_{A_{11}}T_1}$, and $A_{d2} = e^{D_{A_{22}}T_2}$. Let

$$\begin{aligned} P_1 &:= \sum_{n=0}^{N-1} A_{d1}^n D_F D_F^H (A_{d1}^n)^H \\ P_2 &:= \sum_{n=0}^{N-1} A_{d1}^n \left(\sum_{m=0}^{M-1} D_{A_{12}} A_{d2}^m D_{B_2} D_F^H \right. \\ &\quad \left. + \sum_{m=0}^{M-1} D_F D_{B_2}^H (A_{d2}^m)^H D_{A_{12}}^H \right) (A_{d1}^n)^H \end{aligned}$$

and

$$P_3 := \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} A_{d1}^n D_{A_{12}} A_{d2}^m D_{B_2} D_{B_2}^H (A_{d2}^m)^H D_{A_{12}}^H (A_{d1}^n)^H.$$

We then have

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{ D_{C_1}[MP_1 + P_2 + P_3]D_{C_1}^H \right\}.$$

Due to the way by which $D_{A_{11}}$ and $D_{A_{22}}$ are constructed from A_{11} and A_{22}, respectively, it is easy to show that the assumption that all the eigenvalues of A_{11} and A_{22} are in the open left half plane implies that all the eigenvalues of $A_{d1} = e^{D_{A_{11}}T_1}$ and $A_{d2} = e^{D_{A_{22}}T_2}$ are in the open unit disc. By Lemma 3.1, P_1 is therefore the unique solution to the following Lyapunov equation:

$$A_{d1}P_1A_{d1}^H - P_1 = -D_FD_F^H + A_{d1}^N D_F D_F^H (A_{d1}^N)^H.$$

As for P_2, since $\sum_{m=0}^{M-1} A_{d2}^m = (I - A_{d2}^M)(I - A_{d2})^{-1}$, P_2 can be rewritten as

$$\begin{aligned} P_2 &= \sum_{n=0}^{N-1} A_{d1}^n \left(D_{A_{12}} \sum_{m=0}^{M-1} A_{d2}^m D_{B_2} D_F^H \right. \\ &\quad \left. + D_F D_{B_2}^H \sum_{m=0}^{M-1} (A_{d2}^m)^H D_{A_{12}}^H \right) (A_{d1}^n)^H \\ &= \sum_{n=0}^{N-1} A_{d1}^n \left(D_{A_{12}} (I - A_{d2}^M)(I - A_{d2})^{-1} D_{B_2} D_F^H \right. \\ &\quad \left. + D_F D_{B_2}^H ((I - A_{d2})^{-1})^H (I - A_{d2}^M)^H D_{A_{12}}^H \right) (A_{d1}^n)^H. \end{aligned}$$

Let

$$R := D_{A_{12}}(I - A_{d2}^M)(I - A_{d2})^{-1} D_{B_2} D_F^H + D_F D_{B_2}^H ((I - A_{d2})^{-1})^H (I - A_{d2}^M)^H D_{A_{12}}^H.$$

It is then clear from Lemma 3.1 that P_2 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_2A_{d1}^H - P_2 = -R + A_{d1}^N R (A_{d1}^N)^H.$$

To calculate P_3, rewrite it as

$$\begin{aligned} P_3 &= \sum_{n=0}^{N-1} A_{d1}^n D_{A_{12}} \\ &\quad \cdot \left(\sum_{m=0}^{M-1} A_{d2}^m D_{B_2} D_{B_2}^H (A_{d2}^m)^H \right) D_{A_{12}}^H (A_{d1}^n)^H \\ &= \sum_{n=0}^{N-1} A_{d1}^n D_{A_{12}} P_4 D_{A_{12}}^H (A_{d1}^n)^H \end{aligned}$$

where

$$P_4 := \sum_{m=0}^{M-1} A_{d2}^m D_{B_2} D_{B_2}^H (A_{d2}^m)^H.$$

It is easy to see that P_4 is the unique solution to the following Lyapunov equation:

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2} D_{B_2}^H (A_{d2}^M)^H.$$

Hence, P_3 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H + A_{d1}^N D_{A_{12}} P_4 D_{A_{12}}^H (A_{d1}^N)^H.$$

The proof is thus completed. □

In the case that there are an infinite number of equidistant samples in the t_1 variable, i.e., $N \to \infty$ in Theorem 3.1, the previous theorem can be simplified with the help of Lemma 3.1.

Corollary 3.1: Assume that the 2-D system is the same as in Theorem 3.1, except that there are an infinite number of equidistant samples in the t_1 variable, i.e., $t_{1n} = nT_1,\ n = 0, 1, \ldots, \infty$. Then, the FIM is given by

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{D_{C_1}[MP_1 + P_2 + P_3]D_{C_1}^H\right\}$$

where P_1, P_2, and P_3 can be obtained as follows. P_1 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_1A_{d1}^H - P_1 = -D_F D_F^H.$$

P_2 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_2A_{d1}^H - P_2 = -\big(D_{A_{12}}(I - A_{d2}^M)(I - A_{d2})^{-1}D_{B_2}D_F^H + D_F D_{B_2}^H((I - A_{d2})^{-1})^H(I - A_{d2}^M)^H D_{A_{12}}^H\big).$$

P_4 is the unique solution to the following Lyapunov equation:

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2}D_{B_2}^H(A_{d2}^M)^H$$

and P_3 is the unique solution to the following Lyapunov equation:

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H.$$

Note that the expressions for the FIM given in Corollary 3.1 are not only simpler than those in Theorem 3.1 but also give the asymptotic FIM when an infinite number of samples are available in t_1. Note that when the number of samples increases, further positive terms are added to the FIM, and hence, the FIM becomes larger in the sense of positive definite matrices. This then implies the corresponding decrease in the inverse, i.e., the CRLB. Hence, the asymptotic FIM is useful in determining the lowest possible CRLB that can be achieved by increasing the number of acquired data points.

Further simplifications of the expression for the FIM can be achieved by assuming that the boundary conditions are all zero i.e., $x_\theta^h(0, t_2) = F = 0$ in Theorem 2.2. In this case, the 2-D noise free signal in (4) is given by

$$y(t_{1n}, t_{2m}) = C_1e^{A_{11}t_{1n}}A_{12}e^{A_{22}t_{2m}}B_2 \qquad n = 0, 1, \ldots, N-1;\ m = 0, 1, \ldots, M-1$$

and Theorem 3.1 and Corollary 3.1 reduce to Corollary 3.2 and Corollary 3.3, respectively, as given below.

Corollary 3.2: Assume that the 2-D system is the same as that in Theorem 3.1, except that $x_\theta^h(0, t_2) = F = 0$, i.e., assume that the deterministic part of the measured signal is given by

$$y(t_{1n}, t_{2m}) = C_1e^{A_{11}nT_1}A_{12}e^{A_{22}mT_2}B_2 \qquad n = 0, 1, \ldots, N-1;\ m = 0, 1, \ldots, M-1.$$

Then, the FIM is given by

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{D_{C_1}P_3D_{C_1}^H\right\}$$

where P_3 is solved as follows. Obtain P_4 as the unique solution to the following Lyapunov equation:

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2}D_{B_2}^H(A_{d2}^M)^H$$

and P_3 as the unique solution to the following Lyapunov equation:

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H + A_{d1}^N D_{A_{12}}P_4D_{A_{12}}^H(A_{d1}^N)^H.$$

Corollary 3.3: Assume that the 2-D system is the same as in Corollary 3.1, except that $x_\theta^h(0, t_2) = F = 0$, i.e., assume that the deterministic part of the measured signal is given by

$$y(t_{1n}, t_{2m}) = C_1e^{A_{11}nT_1}A_{12}e^{A_{22}mT_2}B_2 \qquad n = 0, 1, \ldots, \infty;\ m = 0, 1, \ldots, M-1.$$

Then, the FIM is given by

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{D_{C_1}P_3D_{C_1}^H\right\}$$

where P_3 is solved as follows. Obtain P_4 as the unique solution to the following Lyapunov equation:

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H + A_{d2}^M D_{B_2}D_{B_2}^H(A_{d2}^M)^H$$

and then get P_3 as the unique solution to the following Lyapunov equation:

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H.$$

In Corollary 3.3, as the 1-D term disappears due to the zero boundary condition, it is feasible to have an infinite number of equidistant samples for both the t_1 and t_2 variables. In this case, Corollary 3.3 can be further simplified as follows.

Corollary 3.4: Assume that the 2-D system is the same as that in Corollary 3.3, except that an infinite number of equidistant samples are acquired in both the t_1 and t_2 variables, i.e., $t_{1n} = nT_1,\ n = 0, 1, \ldots, \infty$; $t_{2m} = mT_2,\ m = 0, 1, \ldots, \infty$.. Then, the FIM is given by

$$I(\Theta) = \frac{1}{\sigma^2} \operatorname{Re}\left\{D_{C_1}P_3D_{C_1}^H\right\}$$

where P_3 can be solved as follows. First, obtain P_4 as the unique solution to the Lyapunov equation

$$A_{d2}P_4A_{d2}^H - P_4 = -D_{B_2}D_{B_2}^H$$

and then P_3 as the unique solution to the Lyapunov equation

$$A_{d1}P_3A_{d1}^H - P_3 = -D_{A_{12}}P_4D_{A_{12}}^H.$$

In the next section, we present an example to illustrate the novel methods developed in this section and compare the new results with those by an existing method of [13].

IV. Example

In this example, the methods that were introduced earlier will be illustrated with a concrete example that arose from our earlier work on the use of the CRLB in NMR spectroscopy [13]. Consider a simulated 2-D NMR data

$$y_\theta(t_1, t_2) = \sum_{k=1}^{2}\sum_{l=1}^{2} c_{kl}e^{(r_{1k}+iw_{1k})t_1+(r_{2l}+iw_{2l})t_2+i\phi_{kl}}$$

where the parameter vector is given by

$$\Theta = [c_{11} \quad c_{12} \quad c_{21} \quad c_{22} \quad r_{11} \quad r_{12} \quad r_{21} \quad r_{22} \quad \omega_{11} \quad \omega_{12} \quad \omega_{21} \quad \omega_{22} \quad \phi_{11} \quad \phi_{12} \quad \phi_{21} \quad \phi_{22}]^T.$$

The signal is assumed to be measured with additive complex Gaussian white noise, whose real and imaginary parts are uncorrelated and both have zero mean and fixed variance $\sigma_{m,n}^2 = \sigma^2$.

In [13], the FIM was calculated for the finite sample situation by calculating the partial derivatives term by term. Here, we will demonstrate that system theoretic methods allow a much more systematic approach that is easily adapted to other problems. In addition, the system theoretic approach also permits the calculation of the FIM and CRLB for the case when an infinite number of sampled data points are available. Having an infinite number of data points is, of course, not a situation that is encountered in a practical situation. It is, however, a limiting case that indicates to what extent the CRLB could be improved by increasing the number of data points, given that the other experimental parameters such as the sampling interval are kept constant.

A. Data as Output of a 2-D Separable-Denominator Continuous System

To apply the earlier results, it is important to note that the above simulated 2-D NMR data can be considered as the output of a 2-D separable-denominator continuous system with a state-space realization given by

$$A_{11} = \begin{bmatrix} r_{11}+i\omega_{11} & 0 \\ 0 & r_{12}+i\omega_{12} \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} c_{11}e^{i\phi_{11}} & c_{12}e^{i\phi_{12}} \\ c_{21}e^{i\phi_{21}} & c_{22}e^{i\phi_{22}} \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} r_{21}+i\omega_{21} & 0 \\ 0 & r_{22}+i\omega_{22} \end{bmatrix}$$

$$C_1 = [1 \quad 1], \; C_2 = 0, \; B_1 = 0, \; B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; D = 0$$

where the input and initial conditions are given by

$$u(t_1, t_2) = \delta(t_1, t_2), \quad x_\theta^h(0, t_2) = F = 0, \quad x_\theta^v(t_1, 0) = 0.$$

Hence, we describe the data as the uniformly sampled output of the 2-D signal

$$y_\theta(t_1, t_2) = C_1 e^{A_{11}t_1} A_{12} e^{A_{22}t_2} B_2, \qquad t_1 \geq 0, \; t_2 \geq 0$$

where A_{11}, A_{12}, A_{22}, C_1, and B_2 are given in the above.

B. Derivative System

A central step in our approach is based on the calculation of the derivative system. The expressions that are of relevance here will be determined now, i.e., $D_{A_{11}}$, $D_{A_{12}}$, $D_{A_{22}}$, D_{C_1}, and D_{B_2}.

Let $D_{A_{11}} := \text{diag}\{\partial_1 A_{11}, \partial_2 A_{11}, \ldots, \partial_{16} A_{11}\}$. It is then easy to show that

$$\partial_5 A_{11} = \begin{bmatrix} A_{11} & 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & A_{11} \end{bmatrix}, \; \partial_6 A_{11} = \begin{bmatrix} A_{11} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & A_{11} \end{bmatrix}$$

$$\partial_9 A_{11} = \begin{bmatrix} A_{11} & 0 \\ \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} & A_{11} \end{bmatrix}, \; \partial_{10} A_{11} = \begin{bmatrix} A_{11} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} & A_{11} \end{bmatrix}$$

and all the remaining diagonal entries of $D_{A_{11}}$ are identically equal to $\text{diag}\{A_{11}, A_{11}\}$.

Similarly, let $D_{A_{12}} := \text{diag}\{\partial_1 A_{12}, \partial_2 A_{12}, \ldots, \partial_{16} A_{12}\}$. We thus have

$$\partial_1 A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} e^{i\phi_{11}} & 0 \\ 0 & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_2 A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & e^{i\phi_{12}} \\ 0 & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_3 A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & 0 \\ e^{i\phi_{21}} & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_4 A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & e^{i\phi_{22}} \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_{13} A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} ic_{11}e^{i\phi_{11}} & 0 \\ 0 & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_{14} A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & ic_{12}e^{i\phi_{12}} \\ 0 & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_{15} A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & 0 \\ ic_{21}e^{i\phi_{21}} & 0 \end{bmatrix} & A_{12} \end{bmatrix}$$

$$\partial_{16} A_{12} = \begin{bmatrix} A_{12} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & ic_{22}e^{i\phi_{22}} \end{bmatrix} & A_{12} \end{bmatrix}$$

and all the remaining diagonal entries of $D_{A_{12}}$ are identically equal to $\text{diag}\{A_{12}, A_{12}\}$.

In addition, letting $D_{A_{22}} := \text{diag}\{\partial_1 A_{22}, \partial_2 A_{22}, \ldots, \partial_{16} A_{22}\}$, we have

$$\partial_7 A_{22} = \begin{bmatrix} A_{22} & 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & A_{22} \end{bmatrix}, \; \partial_8 A_{22} = \begin{bmatrix} A_{22} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & A_{22} \end{bmatrix}$$

$$\partial_{11} A_{22} = \begin{bmatrix} A_{22} & 0 \\ \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} & A_{22} \end{bmatrix}, \; \partial_{12} A_{22} = \begin{bmatrix} A_{22} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} & A_{22} \end{bmatrix}$$

and all the remaining diagonal entries of $D_{A_{22}}$ are identically equal to $\text{diag}\{A_{22}, A_{22}\}$.

Proceeding in the same manner, we have

$$\partial_1 B_2 = \partial_2 B_2 = \ldots = \partial_{16} B_2 = \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \\ 0 \end{array}\right]$$

and hence

$$D_{B_2} = [1 \;\; 1 \;\; 0 \;\; 0 \;|\; 1 \;\; 1 \;\; 0 \;\; 0 \;|\; \cdots \;|\; 1 \;\; 1 \;\; 0 \;\; 0]^T.$$

Similarly

$$\partial_1 C_1 = \partial_2 C_1 = \cdots = \partial_{16} C_1 = [0 \quad 0 \;|\; 1 \quad 1]$$
$$D_{C_1} = \text{diag}\{\partial_1 C_1, \partial_2 C_1, \ldots, \partial_{16} C_1\}.$$

Thus, A_{d1} is given by

$$A_{d1} = e^{D_{A_{11}} T_1} = \text{diag}\left\{e^{\partial_1 A_{11} T_1},\, e^{\partial_2 A_{11} T_1},\, \ldots,\, e^{\partial_{16} A_{11} T_1}\right\}$$

where T_1 is the sampling interval for the continuous variable t_1, and, as shown in the first set of equations at the bottom of the page, the remaining diagonal block entries of A_{d1} are all identically equal to

$$\text{diag}\{e^{(r_{11}+i\omega_{11})T_1},\, e^{(r_{12}+i\omega_{12})T_1},\, e^{(r_{11}+i\omega_{11})T_1},\, e^{(r_{12}+i\omega_{12})T_1}\}.$$

Similarly, A_{d2} is given by

$$A_{d2} = e^{D_{A_{22}} T_2} = \text{diag}\{e^{\partial_1 A_{22} T_2},\, e^{\partial_2 A_{22} T_2},\, \ldots,\, e^{\partial_{16} A_{22} T_2}\}$$

where T_2 is the sampling interval for the continuous variable t_2, and as shown in the second set of equations at the bottom of the page, the remaining diagonal block entries of A_{d2} are all identically equal to

$$\text{diag}\{e^{(r_{21}+i\omega_{21})T_2},\, e^{(r_{22}+i\omega_{22})T_2}\, e^{(r_{21}+i\omega_{21})T_2},\, e^{(r_{22}+i\omega_{22})T_2}\}.$$

C. CRLB

Having determined the components of the derivative system that are of importance for our data set, it is now possible to calculate the FIM once the remaining experimental parameters are set. We assume that the above simulated 2-D NMR data (with additive noise) is uniformly sampled in both the t_1 and t_2 variables. For the t_1 variable, the sampling interval is $T_1 = 0.015$ s, and the number of samples acquired is $N = 1024$. For t_2, the sampling interval is $T_2 = 1.54$ s, and the number of samples acquired is $M = 16$. It is further assumed that the noise

$$e^{\partial_5 A_{11} T_1} = \begin{bmatrix} e^{(r_{11}+i\omega_{11})T_1} & 0 & 0 & 0 \\ 0 & e^{(r_{12}+i\omega_{12})T_1} & 0 & 0 \\ T_1 e^{(r_{11}+i\omega_{11})T_1} & 0 & e^{(r_{11}+i\omega_{11})T_1} & 0 \\ 0 & 0 & 0 & e^{(r_{12}+i\omega_{12})T_1} \end{bmatrix}$$

$$e^{\partial_6 A_{11} T_1} = \begin{bmatrix} e^{(r_{11}+i\omega_{11})T_1} & 0 & 0 & 0 \\ 0 & e^{(r_{12}+i\omega_{12})T_1} & 0 & 0 \\ 0 & 0 & e^{(r_{11}+i\omega_{11})T_1} & 0 \\ 0 & T_1 e^{(r_{12}+i\omega_{12})T_1} & 0 & e^{(r_{12}+i\omega_{12})T_1} \end{bmatrix}$$

$$e^{\partial_9 A_{11} T_1} = \begin{bmatrix} e^{(r_{11}+i\omega_{11})T_1} & 0 & 0 & 0 \\ 0 & e^{(r_{12}+i\omega_{12})T_1} & 0 & 0 \\ iT_1 e^{(r_{11}+i\omega_{11})T_1} & 0 & e^{(r_{11}+i\omega_{11})T_1} & 0 \\ 0 & 0 & 0 & e^{(r_{12}+i\omega_{12})T_1} \end{bmatrix}$$

$$e^{\partial_{10} A_{11} T_1} = \begin{bmatrix} e^{(r_{11}+i\omega_{11})T_1} & 0 & 0 & 0 \\ 0 & e^{(r_{12}+i\omega_{12})T_1} & 0 & 0 \\ 0 & 0 & e^{(r_{11}+i\omega_{11})T_1} & 0 \\ 0 & iT_1 e^{(r_{12}+i\omega_{12})T_1} & 0 & e^{(r_{12}+i\omega_{12})T_1} \end{bmatrix}$$

$$e^{\partial_7 A_{22} T_2} = \begin{bmatrix} e^{(r_{21}+i\omega_{21})T_2} & 0 & 0 & 0 \\ 0 & e^{(r_{22}+i\omega_{22})T_2} & 0 & 0 \\ T_2 e^{(r_{21}+i\omega_{21})T_2} & 0 & e^{(r_{21}+i\omega_{21})T_2} & 0 \\ 0 & 0 & 0 & e^{(r_{22}+i\omega_{22})T_2} \end{bmatrix}$$

$$e^{\partial_8 A_{22} T_2} = \begin{bmatrix} e^{(r_{21}+i\omega_{21})T_2} & 0 & 0 & 0 \\ 0 & e^{(r_{22}+i\omega_{22})T_2} & 0 & 0 \\ 0 & 0 & e^{(r_{21}+i\omega_{21})T_2} & 0 \\ 0 & T_2 e^{(r_{22}+i\omega_{22})T_2} & 0 & e^{(r_{22}+i\omega_{22})T_2} \end{bmatrix}$$

$$e^{\partial_{11} A_{22} T_2} = \begin{bmatrix} e^{(r_{21}+i\omega_{21})T_2} & 0 & 0 & 0 \\ 0 & e^{(r_{22}+i\omega_{22})T_2} & 0 & 0 \\ iT_2 e^{(r_{21}+i\omega_{21})T_2} & 0 & e^{(r_{21}+i\omega_{21})T_2} & 0 \\ 0 & 0 & 0 & e^{(r_{22}+i\omega_{22})T_2} \end{bmatrix}$$

$$e^{\partial_{12} A_{22} T_2} = \begin{bmatrix} e^{(r_{21}+i\omega_{21})T_2} & 0 & 0 & 0 \\ 0 & e^{(r_{22}+i\omega_{22})T_2} & 0 & 0 \\ 0 & 0 & e^{(r_{21}+i\omega_{21})T_2} & 0 \\ 0 & iT_2 e^{(r_{22}+i\omega_{22})T_2} & 0 & e^{(r_{22}+i\omega_{22})T_2} \end{bmatrix}$$

TABLE I
CRLB FOR DIFFERENT METHODS WITH $T_1 = 0.015$, $T_2 = 1.54$, $N = 1024$, $M = 16$

Parameter	Method of [13]	Corollary 3.2	Corollary 3.3	Corollary 3.4
c_{11}	6.0825535e-005	6.0837388e-005	5.0550442e-005	4.9898934e-005
c_{12}	9.7595153e-005	9.7615157e-005	7.7776752e-005	7.7365093e-005
c_{21}	1.2564098e-004	1.2566020e-004	1.1645558e-004	1.1609027e-004
c_{22}	2.2393692e-004	2.2398477e-004	1.9971263e-004	1.9956275e-004
r_{11}	2.4572188e-005	2.4577153e-005	1.0231312e-005	1.0228864e-005
r_{12}	1.2490656e-003	1.2492219e-003	1.1568085e-003	1.1563498e-003
r_{21}	7.8325749e-005	7.8331453e-005	7.4050262e-005	7.0916432e-005
r_{22}	4.4494788e-004	4.4494467e-004	4.3022980e-004	4.2877092e-004
ω_{11}	2.4572188e-005	2.4577153e-005	1.0231312e-005	1.0228864e-005
ω_{12}	1.2490656e-003	1.2492219e-003	1.1568085e-003	1.1563498e-003
ω_{21}	7.8325749e-005	7.8331453e-005	7.4050262e-005	7.0916432e-005
ω_{22}	4.4494788e-004	4.4494467e-004	4.3022980e-004	4.2877092e-004
ϕ_{11}	2.7033571e-003	2.7038839e-003	2.2466863e-003	2.2177304e-003
ϕ_{12}	2.0164288e-003	2.0168421e-003	1.6069577e-003	1.5984523e-003
ϕ_{21}	8.7250678e-003	8.7264031e-003	8.0871927e-003	8.0618243e-003
ϕ_{22}	1.3250705e-002	1.3253537e-002	1.1817315e-002	1.1808447e-002

TABLE II
CRLB FOR DIFFERENT METHODS WITH $T_1 = 0.03$, $T_2 = 1.54$, $N = 1024$, $M = 16$

Parameter	Method of [13]	Corollary 3.2	Corollary 3.3	Corollary 3.4
c_{11}	1.0284806e-004	1.0286326e-004	1.0085283e-004	9.9553409e-005
c_{12}	1.5923391e-004	1.5925897e-004	1.5517096e-004	1.5435005e-004
c_{21}	2.3269368e-004	2.3272173e-004	2.3098609e-004	2.3025777e-004
c_{22}	4.0176069e-004	4.0182800e-004	3.9631455e-004	3.9601594e-004
r_{11}	2.2833559e-005	2.2836619e-005	2.0432256e-005	2.0427368e-005
r_{12}	2.3226337e-003	2.3228503e-003	2.3016239e-003	2.3007113e-003
r_{21}	1.4807943e-004	1.4808958e-004	1.4771751e-004	1.4146639e-004
r_{22}	8.5935800e-004	8.5935253e-004	8.5803951e-004	8.5513198e-004
ω_{11}	2.2833559e-005	2.2836619e-005	2.0432256e-005	2.0427368e-005
ω_{12}	2.3226337e-003	2.3228503e-003	2.3016239e-003	2.3007113e-003
ω_{21}	1.4807943e-004	1.4808958e-004	1.4771751e-004	1.4146639e-004
ω_{22}	8.5935800e-004	8.5935253e-004	8.5803951e-004	8.5513198e-004
ϕ_{11}	4.5710249e-003	4.5717007e-003	4.4823481e-003	4.4245960e-003
ϕ_{12}	3.2899568e-003	3.2904747e-003	3.2060115e-003	3.1890506e-003
ϕ_{21}	1.6159284e-002	1.6161231e-002	1.6040701e-002	1.5990123e-002
ϕ_{22}	2.3772822e-002	2.3776805e-002	2.3450565e-002	2.3432896e-002

variance $\sigma^2_{m,n} = \sigma^2 = 0.1$ in this example. For the purpose of illustration, we fix the value of the parameter vector as

$$[0.15,\, 0.22,\, 0.12,\, 0.13,\, -0.1,\, -0.35,\, -0.15,\, -0.45,\, 1.445\\ 2.136,\, 2.702,\, 0.88,\, 0.683,\, 1.366,\, 2.4167,\, 0.982]^T.$$

We then obtain the CRLB of the given data using the method presented in [13], as well as the new methods proposed in this paper. As the boundary conditions $x^h_\theta(0, t_2)$, $x^v_\theta(t_1, 0)$ of the associated 2-D system are both zero, Corollaries 3.2–3.4 can be applied. The resultant values for the CRLB are given in Table I. As the given simulated data are uniformly sampled with finite samples in both variables t_1 and t_2, the method of [13] and Corollary 3.2 would give the exact CRLB, whereas Corollaries 3.3 and 3.4 could give only the approximate CRLB. From Table I, it can be seen that the values in columns 1 and 2 are indeed very close (the differences are caused by numerical errors only), whereas there are some small differences between the values in columns 1 and 3 (or 4). Note that the value in column 3 is consistently smaller than the corresponding values in columns 1 and 2 but greater than that in column 4 for any row (parameter). In fact, it is easy to see that Corollary 3.3 gives expressions for the FIM associated with the asymptotic CRLB for an infinite number of samples for t_1, whereas Corollary 3.4 gives expressions for the FIM associated with the asymptotic CRLB for an infinite number of samples for both t_1 and t_2, as verified by this example.

When the lengths of the sampling intervals increase, while the number of samples remains unchanged, the relative differences between the values in columns 1 and 3 (or 4) decrease, as can be seen from Tables II and III. A heuristic explanation of this phenomenon rests on the fact that the data is exponentially decaying. Therefore, the data eventually decays into the noise and contributes little to the information content of the data set. Hence, a finite data set of suitable length can lead to a FIM that is close to what is achievable with an infinite data set with the same sampling interval. This argument applies similarly to the case when the number of samples increases while the lengths of the sampling intervals remain unchanged and, furthermore, to the case when both the lengths of the sampling intervals and the number of samples increase. Table II gives results on the CRLB with the same setting as in Table I, except that the sampling interval for t_1 is now equal to 0.03 s, whereas in Table III, the number of samples for t_1 is 2048 (the sampling interval for t_1 is still 0.015 for Table III).

TABLE III
CRLB FOR DIFFERENT METHODS WITH $T_1 = 0.015$, $T_2 = 1.54$, $N = 2048$, $M = 16$

Parameter	Method of [13]	Corollary 3.2	Corollary 3.3	Corollary 3.4
c_{11}	5.1551933e-005	5.1559553e-005	5.0550442e-005	4.9898934e-005
c_{12}	7.9816302e-005	7.9828867e-005	7.7776752e-005	7.7365093e-005
c_{21}	1.1731652e-004	1.1733065e-004	1.1645558e-004	1.1609027e-004
c_{22}	2.0245998e-004	2.0249390e-004	1.9971263e-004	1.9956275e-004
r_{11}	1.1433735e-005	1.1435266e-005	1.0231312e-005	1.0228864e-005
r_{12}	1.1673677e-003	1.1674765e-003	1.1568085e-003	1.1563498e-003
r_{21}	7.4231879e-005	7.4236965e-005	7.4050262e-005	7.0916432e-005
r_{22}	4.3089170e-004	4.3088896e-004	4.3022980e-004	4.2877092e-004
ω_{11}	1.1433735e-005	1.1435266e-005	1.0231312e-005	1.0228864e-005
ω_{12}	1.1673677e-003	1.1674765e-003	1.1568085e-003	1.1563498e-003
ω_{21}	7.4231879e-005	7.4236965e-005	7.4050262e-005	7.0916432e-005
ω_{22}	4.3089170e-004	4.3088896e-004	4.3022980e-004	4.2877092e-004
ϕ_{11}	2.2911970e-003	2.2915357e-003	2.2466863e-003	2.2177304e-003
ϕ_{12}	1.6490972e-003	1.6493568e-003	1.6069577e-003	1.5984523e-003
ϕ_{21}	8.1469805e-003	8.1479615e-003	8.0871927e-003	8.0618243e-003
ϕ_{22}	1.1979881e-002	1.1981888e-002	1.1817315e-002	1.1808447e-002

Despite the similar values between columns 1 and 2 in all three tables, there is a significant difference between the method of [13] and Corollary 3.2 in that Corollary 3.2 is much more efficient computationally than the method of [13] (at least 100 times more efficient based on our simulations carried out using the same PC). Moreover, as it takes almost the same effort and time to apply Corollaries 3.2–3.4 to the computation of the CRLB, we recommend that Corollary 3.2 should be adopted when one is interested in calculating the exact CRLB for parameter estimation for 2-D NMR data with zero boundary conditions for finite samples in both variables. On the other hand, if one is interested in knowing the asymptotic CRLB, i.e., for infinite samples in one or both variables, Corollary 3.3 or 3.4 should be adopted instead.

V. Conclusion

In this paper, we have developed an efficient method for the calculation of the Cramér–Rao lower bound for a wide class of 2-D signals that are samples of outputs of 2-D separable-denominator continuous systems. Explicit expression for the associated FIM is derived for the class of 2-D signals with a general data sampling scheme. For the special but important case of uniform sampling, the Lyapunov approach is exploited, which has speeded up considerably the calculation of the FIM. Although the results are derived for 2-D separable-denominator systems and the associated 2-D data sets, they can be easily generalized to multidimensional ($n > 2$) separable-denominator systems and the associated multidimensional ($n > 2$) data sets. However, it is nontrivial to generalize the results in this paper to the multidimensional ($n \geq 2$) nonseparable-denominator systems, and this is a challenging problem for further investigation.

We believe that the presented results will have a significant impact on applications dealing with a large number of data samples and a large number of parameters to be estimated, such as in multidimensional NMR spectroscopy. An illustrative example is also presented and compared with a recent result on the topic.

Appendix A
Proof of Lemma 1.1

First, we represent the 2-D Laplace transform of $y_\theta(t_1, t_2)$, $x_\theta^h(t_1, t_2)$, $x_\theta^v(t_1, t_2)$, and $u(t_1, t_2)$ by $Y_\theta(s_1, s_2)$, $X_\theta^h(s_1, s_2)$, $X_\theta^v(s_1, s_2)$, and $U(s_1, s_2)$, respectively. Taking the 2-D Laplace transform of both sides of (2) and (3) and taking into account the initial conditions $x_\theta^h(0, t_2)$, $x_\theta^v(t_1, 0)$, we obtain

$$\begin{bmatrix} s_1 X_\theta^h(s_1, s_2) - X_0^h(s_2) \\ s_2 X_\theta^v(s_1, s_2) - X_0^v(s_1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_\theta^h(s_1, s_2) \\ X_\theta^v(s_1, s_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s_1, s_2)$$

$$Y_\theta(s_1, s_2) = [\, C_1 \quad C_2 \,] \begin{bmatrix} X_\theta^h(s_1, s_2) \\ X_\theta^v(s_1, s_2) \end{bmatrix} + DU(s_1, s_2)$$

where

$$X_0^h(s_2) = \int_0^\infty x_\theta^h(0, t_2) e^{-s_2 t_2}\, dt_2$$

and

$$X_0^v(s_1) = \int_0^\infty x_\theta^v(t_1, 0) e^{-s_1 t_1}\, dt_1.$$

With simple matrix algebra, we have

$$\begin{bmatrix} X_\theta^h(s_1, s_2) \\ X_\theta^v(s_1, s_2) \end{bmatrix} = \begin{bmatrix} s_1 I_{11} - A_{11} & -A_{12} \\ 0 & s_2 I_{22} - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_0^h(s_2) \\ X_0^v(s_1) \end{bmatrix} + \begin{bmatrix} s_1 I_{11} - A_{11} & -A_{12} \\ 0 & s_2 I_{22} - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s_1, s_2)$$

and

$$\begin{aligned} Y_\theta(s_1, s_2) &= [\, C_1 \quad C_2 \,] \begin{bmatrix} s_1 I_{11} - A_{11} & -A_{12} \\ 0 & s_2 I_{22} - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_0^h(s_2) \\ X_0^v(s_1) \end{bmatrix} \\ &\quad + [\, C_1 \quad C_2 \,] \begin{bmatrix} s_1 I_{11} - A_{11} & -A_{12} \\ 0 & s_2 I_{22} - A_{22} \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s_1, s_2) + DU(s_1, s_2). \end{aligned}$$

It is easy to verify that

$$\begin{bmatrix} s_1 I_{11} - A_{11} & -A_{12} \\ 0 & s_2 I_{22} - A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (s_1 I_{11} - A_{11})^{-1} & (s_1 I_{11} - A_{11})^{-1} A_{12} (s_2 I_{22} - A_{22})^{-1} \\ 0 & (s_2 I_{22} - A_{22})^{-1} \end{bmatrix}$$

when $(s_1 I_{11} - A_{11})^{-1}$ and $(s_2 I_{22} - A_{22})^{-1}$ exist. Therefore, the response due to nonzero boundary conditions in the (s_1, s_2)-domain is given by

$$\begin{aligned} V_\theta(s_1, s_2) &= [\, C_1 \quad C_2 \,] \\ &\quad \cdot \begin{bmatrix} (s_1 I_{11} - A_{11})^{-1} & (s_1 I_{11} - A_{11})^{-1} A_{12} (s_2 I_{22} - A_{22})^{-1} \\ 0 & (s_2 I_{22} - A_{22})^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} X_0^h(s_2) \\ X_0^v(s_1) \end{bmatrix} \\ &= C_1 (s_1 I_{11} - A_{11})^{-1} X_0^h(s_2) + C_2 (s_2 I_{22} - A_{22})^{-1} X_0^v(s_1) \\ &\quad + C_1 (s_1 I_{11} - A_{11})^{-1} A_{12} (s_2 I_{22} - A_{22})^{-1} X_0^v(s_1). \end{aligned}$$

Thus, in the (t_1, t_2)-domain, we have

$$v_\theta(t_1, t_2) = C_1 e^{A_{11} t_1} x^h(0, t_2) + C_2 e^{A_{22} t_2} x^v(t_1, 0) + \int_0^{t_1} C_1 e^{A_{11}(t_1 - \tau_1)} A_{12} e^{A_{22} t_2} x^v(\tau_1, 0)\, d\tau_1.$$

The system response to the input in the (s_1, s_2)-domain is given by

$$\begin{aligned} Q_\theta(s_1, s_2) &= [\, C_1 \quad C_2 \,] \\ &\quad \cdot \begin{bmatrix} (s_1 I_{11} - A_{11})^{-1} & (s_1 I_{11} - A_{11})^{-1} A_{12} (s_2 I_{22} - A_{22})^{-1} \\ 0 & (s_2 I_{22} - A_{22})^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s_1, s_2) + DU(s_1, s_2) \\ &= C_1 (s_1 I_{11} - A_{11})^{-1} B_1 U(s_1, s_2) \\ &\quad + C_2 (s_2 I_{22} - A_{22})^{-1} B_2 U(s_1, s_2) \\ &\quad + C_1 (s_1 I_{11} - A_{11})^{-1} A_{12} (s_2 I_{22} - A_{22})^{-1} B_2 U(s_1, s_2) \\ &\quad + DU(s_1, s_2). \end{aligned}$$

Thus

$$q_\theta(t_1, t_2) = \int_0^{t_1} C_1 e^{A_{11}(t_1-\tau_1)} B_1 u(\tau_1, t_2)\, d\tau_1 + \int_0^{t_2} C_2 e^{A_{22}(t_2-\tau_2)} B_2 u(t_1, \tau_2)\, d\tau_2 + \int_0^{t_1}\int_0^{t_2} C_1 e^{A_{11}(t_1-\tau_1)} A_{12} e^{A_{22}(t_2-\tau_2)} \cdot B_2 u(\tau_1, \tau_2)\, d\tau_1\, d\tau_2 + Du(t_1, t_2).$$

Therefore

$$y_\theta(t_1, t_2) = v_\theta(t_1, t_2) + q_\theta(t_1, t_2), \qquad t_1 \geq 0,\, t_2 \geq 0.$$

APPENDIX B
PROOF OF LEMMA 2.3

We prove this lemma by induction. When $l = 2$

$$\frac{\partial(H_1 H_2)}{\partial\theta_s} = \frac{\partial H_1}{\partial\theta_s} H_2 + H_1 \frac{\partial H_2}{\partial\theta_s} = \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 \\ \frac{\partial H_2}{\partial\theta_s} \end{bmatrix}.$$

When $l = 3$

$$\frac{\partial(H_1 H_2 H_3)}{\partial\theta_s} = \frac{\partial H_1}{\partial\theta_s} H_2 H_3 + H_1 \frac{\partial H_2}{\partial\theta_s} H_3 + H_1 H_2 \frac{\partial H_3}{\partial\theta_s} = \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \begin{bmatrix} H_3 \\ \frac{\partial H_3}{\partial\theta_s} \end{bmatrix}.$$

Assume this lemma is true for $l-1$ $(l > 2)$, i.e.,

$$\frac{\partial(H_1 H_2 \cdots H_{l-1})}{\partial\theta_s} = \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-2} & 0 \\ \frac{\partial H_{l-2}}{\partial\theta_s} & H_{l-2} \end{bmatrix} \begin{bmatrix} H_{l-1} \\ \frac{\partial H_{l-1}}{\partial\theta_s} \end{bmatrix}.$$

Then

$$\begin{aligned} &\frac{\partial(H_1 H_2 \cdots H_l)}{\partial\theta_s} \\ &= \frac{\partial(H_1 H_2 \cdots H_{l-1})}{\partial\theta_s} H_l + H_1 H_2 \cdots H_{l-1} \frac{\partial H_l}{\partial\theta_s} \\ &= \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-2} & 0 \\ \frac{\partial H_{l-2}}{\partial\theta_s} & H_{l-2} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} H_{l-1} \\ \frac{\partial H_{l-1}}{\partial\theta_s} \end{bmatrix} H_l + \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-2} & 0 \\ \frac{\partial H_{l-2}}{\partial\theta_s} & H_{l-2} \end{bmatrix} \begin{bmatrix} 0 \\ H_{l-1} \end{bmatrix} \frac{\partial H_l}{\partial\theta_s} \\ &= \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-2} & 0 \\ \frac{\partial H_{l-2}}{\partial\theta_s} & H_{l-2} \end{bmatrix} \\ &\quad \cdot \left(\begin{bmatrix} H_{l-1} \\ \frac{\partial H_{l-1}}{\partial\theta_s} \end{bmatrix} H_l + \begin{bmatrix} 0 \\ H_{l-1} \end{bmatrix} \frac{\partial H_l}{\partial\theta_s} \right) \\ &= \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-2} & 0 \\ \frac{\partial H_{l-2}}{\partial\theta_s} & H_{l-2} \end{bmatrix} \\ &\quad \cdot \left(\begin{bmatrix} H_{l-1} & 0 \\ \frac{\partial H_{l-1}}{\partial\theta_s} & H_{l-1} \end{bmatrix} \begin{bmatrix} H_l \\ \frac{\partial H_l}{\partial\theta_s} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{\partial H_1}{\partial\theta_s} & H_1 \end{bmatrix} \begin{bmatrix} H_2 & 0 \\ \frac{\partial H_2}{\partial\theta_s} & H_2 \end{bmatrix} \cdots \begin{bmatrix} H_{l-1} & 0 \\ \frac{\partial H_{l-1}}{\partial\theta_s} & H_{l-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} H_l \\ \frac{\partial H_l}{\partial\theta_s} \end{bmatrix}. \end{aligned}$$

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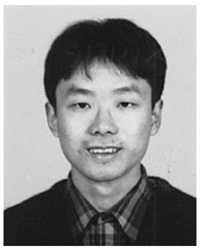

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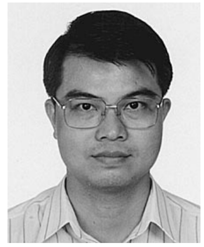

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